IMMERSION EXTENSION-LIFT OVER A MORSE FUNCTION

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Abstract. Let $V$ be a compact connected oriented surface with boundary and $f : \partial V \times [0, 1) \to \mathbb{R}$ a non-singular function such that $f|\partial V \times \{0\}$ is a Morse function. Let $\iota : \partial V \times [0, 1) \to V$ be a collaring of $\partial V$ and $\pi : \mathbb{R}^2 \to \mathbb{R}$ an orthogonal projection. In this paper, we study existence of an orientation preserving immersion $F : V \to \mathbb{R}^2$ such that $\pi \circ F \circ \iota = f$. We also study image homotopy classes of $F$ when we fix $f$ and study relation between two image homotopy classes when $f$ is deformed under a Morse homotopy.

1. Introduction

Throughout the paper, all manifolds and maps are differentiable of class $C^\infty$. Let $V$ be a compact connected oriented surface with boundary. For a smooth function $\bar{f} : \partial V \to \mathbb{R}$, the set of singular points of $\bar{f}$ is $S(\bar{f}) = \{ p \in \partial V | d\bar{f}|_p = 0 \}$. We call any point $p$ of the set $S(\bar{f})$ a singularity and $\bar{f}(p)$ a singular value of $\bar{f}$, respectively. A function $\bar{f} : \partial V \to \mathbb{R}$ is called a Morse function if there exist local coordinates $x$ and $y$ around every $p \in S(\bar{f})$ and $\bar{f}(p) \in \mathbb{R}$ respectively, such that $y = \pm x^2$.

Let $\iota : \partial V \times [0, 1) \to V$ be a collaring of $\partial V$, $f : \partial V \times [0, 1) \to \mathbb{R}$ a non-singular function such that $f|\partial V \times \{0\}$ is a Morse function and $\pi : \mathbb{R}^2 \to \mathbb{R}$ an orthogonal projection such that $\pi(y_1, y_2) = y_1$. Note that the orientation of $\partial V \times [0, 1)$ is induced from that of $V$ by $\iota$. In the following, we abuse that $f : \partial V \times [0, 1) \to \mathbb{R}$ is a Morse function and that $S(f) = S(f|\partial V \times \{0\})$ for the simplicity. In this paper, we study existence of an orientation preserving immersion $F : V \to \mathbb{R}^2$ such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{F} & \mathbb{R}^2 \\
\iota \downarrow & & \downarrow \pi \\
\partial V \times [0, 1) & \xrightarrow{f} & \mathbb{R}
\end{array}
$$

is commutative when $f$ is a Morse function. We call $F$ an immersion extension-lift over $f$. When we fix a Morse function $f : \partial V \times [0, 1) \to \mathbb{R}$, we denote by $\text{Imm}_f(V, \mathbb{R}^2)$ the set of immersion extension-lifts $F : V \to \mathbb{R}^2$ over $f$. We also study $f$-image homotopy classes of $\text{Imm}_f(V, \mathbb{R}^2)$. Here, $F, F' \in \text{Imm}_f(V, \mathbb{R}^2)$ are $f$-image homotopic means that there exist a family of immersion extension-lifts $F_t \in \text{Imm}_f(V, \mathbb{R}^2)$ ($t \in [0, 1]$) and an orientation preserving diffeomorphism $h : V \to V$ such that $h(\partial V \times [0, 1)) = \text{id}$, $F_0 = F$ and $F_1 = F' \circ h$.

Let $f_s : \partial V \times [0, 1) \to \mathbb{R}$ ($s \in [0, 1]$) be a family of non-singular functions such that $f_s|\partial V \times \{0\}$ is a Morse function for each $s$. We call $f_s$ a Morse homotopy.
between \( f_0 \) and \( f_1 \). Let \( f_s : \partial V \times [0, 1) \to \mathbb{R} \) be a Morse homotopy between Morse functions \( f_0 \) and \( f_1 \) \((s \in [0, 1])\) and \( F \in \text{Imm}_{f_0}(V, \mathbb{R}^2) \), \( G \in \text{Imm}_{f_1}(V, \mathbb{R}^2) \) two immersion extension-lifts. If there exists a family of immersion extension-lifts \( \Phi_s \in \text{Imm}_{f_s}(V, \mathbb{R}^2) \) and an orientation preserving diffeomorphism \( k : V \to V \) such that \( k\partial V \times [0, 1) = \text{id} \), \( \Phi_0 = F \) and \( \Phi_1 = G \circ k \), then we call that \( F \) and \( G \) are image homotopic over the Morse homotopy \( f_s \). We can see that the definition of an image homotopy over a Morse homotopy is well-defined. That is, we can check that if \( F \) and \( G \) are \( f_0 \)-image homotopic, \( G \) and \( G' \) are \( f_1 \)-image homotopic and \( F \) and \( G \) are image homotopic over a Morse homotopy between \( f_0 \) and \( f_1 \), then \( F' \) and \( G' \) are image homotopic over the same Morse homotopy. In this paper, we study the conditions when \( F \) and \( G \) are image homotopic over a Morse homotopy.

Remark 1.1. Blank and Laudenbach [1] studied a submersion extension \( F : V \to \mathbb{R} \) such that \( F \circ \iota = f \) and they gave a necessary and sufficient condition of existence of a submersion extension over \( f \).

This paper is organized as follows. In Section 2, we give a necessary and sufficient condition of existence of an immersion extension-lift \( F : V \to \mathbb{R}^2 \) over a Morse function \( f : \partial V \times [0, 1) \to \mathbb{R} \). In Section 3, we classify immersion extension-lifts up to \( f \)-image homotopy. In Section 4, we give invariants of image homotopy classes over a Morse homotopy.

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2. CONSTRUCTION OF AN IMMERSION EXTENSION-LIFT

In this section, we study an orientation preserving immersion extension-lift \( F : V \to \mathbb{R}^2 \) over a Morse function \( f : \partial V \times [0, 1) \to \mathbb{R} \).

Definition 2.1. Let \( f : \partial V \times [0, 1) \to \mathbb{R} \) be a Morse function and \( p \) a local maximum of the restriction \( f|\partial V \times \{0\} \). If \( p \) is also a local maximum for \( f \), then \( p \) is a true maximum, otherwise it is said to be a false maximum. We give similar definitions for true minimums and false minimums of \( f|\partial V \times \{0\} \). We denote by \( \mathcal{Z}^{\text{max}} \) the set of true maximums of \( f \), by \( \mathfrak{R}^{\text{max}} \) the set of false maximums of \( f \). Similarly, we denote by \( \mathcal{Z}^{\text{min}} \) the set of true minimums of \( f \) and by \( \mathfrak{R}^{\text{min}} \) the set of false minimums of \( f \). We put \( \mathcal{X} = \mathcal{Z}^{\text{max}} \cup \mathcal{Z}^{\text{min}} \) and \( \mathfrak{S} = \mathfrak{R}^{\text{max}} \cup \mathfrak{R}^{\text{min}} \).

It is easy to see that \( \#S(f) = \#\mathcal{X} + \#{\mathfrak{S}} \equiv 0 \pmod{2} \) and \( \#S(f)/2 = \#\mathcal{Z}^{\text{max}} + \#\mathfrak{R}^{\text{max}} = \#\mathcal{Z}^{\text{min}} + \#\mathfrak{R}^{\text{min}} \). Here, \( \#X \) is the number of points in \( X \). Let the orientation of \( \partial V \times \{0\} \) be induced by that of \( \partial V \times [0, 1) \). Then we denote by \( S^+ \) (resp. \( S^- \)) the closure of the set of regular points at which \( f|\partial V \times \{0\} \) preserves (resp. reverses) orientation.

Suppose that \( p_1, \ldots, p_l \in \mathfrak{S}^{\text{max}} \) and \( p_{l+1}, \ldots, p_{l+m} \in \mathfrak{S}^{\text{min}} \) satisfy \( f(p_1) = \cdots = f(p_l) = \cdots = f(p_{l+m}) \) and there is at least one connected component \( S_0^+ \) of \( S^+ \) such that \( \text{Int} S_0^+ \cap f^{-1}(f(p_1)) \neq \emptyset \). We denote by \( p_0 \) the regular point in \( \text{Int} S_0^+ \) which satisfies that \( f(p_1) = f(p_0) \). Let \( S_i^+ \) be a connected component of \( S^+ \) which contains \( p_i \) \((1 \leq i \leq l + m)\). Note that if \( p_i \neq p_j \), then \( S_i^+ \neq S_j^+ \). We define the following two operations.
Definition 2.2. Suppose that the line $\pi^{-1}(y_1) = \{y_1\} \times \mathbb{R}$ is canonically oriented for each $y_1$ and $\epsilon$ is a sufficiently small positive number.

(1) Let $\sigma$ and $\tau$ be elements of the permutation group of $l$-words and $m$-words respectively. We embed each $S^+_i$ in $\mathbb{R}^2$ which satisfies that $\pi_i S^+_i = f_i S^+_i (0 \leq i \leq l + m)$, $S^+_0 < S^+_{\tau(1)} < \cdots < S^+_{\tau(l)}$ holds on the band $\pi^{-1}([f(p_0) - \epsilon, f(p_0)])$ and $S^+_0 < S^+_{\tau(1)+l} < \cdots < S^+_{\tau(m)+l}$ holds on the band $\pi^{-1}((f(p_0), f(p_0) + \epsilon])$. Then we attach $S^+_1, \ldots, S^+_{l+m}$ to $S^+_0$ at $p_0$ keeping the above orders on the band $\pi^{-1}([f(p_0) - \epsilon, f(p_0) + \epsilon] \setminus \{f(p_0)\})$. We call this operation an attaching operation.

(2) Let $W$ be the CW complex which is obtained by the above attaching operation. By removing $p_0$, $W$ separates into $l + m + 2$ arcs and especially, $S^+_0$ falls apart into two arcs $S^+_{0-}$ and $S^+_{0+}$, where $\pi(S^+_{0-}) < f(p_0)$ and $\pi(S^+_{0+}) > f(p_0)$ hold. We slide each arc along the vertical direction so that $S^+_{0-} < S^+_{\tau(1)} < \cdots < S^+_{\tau(l)}$ holds on $\pi^{-1}([f(p_0) - \epsilon, f(p_0)])$, $S^+_{0+} < S^+_{\tau(1)+l} < \cdots < S^+_{\tau(m)+l}$ holds on $\pi^{-1}((f(p_0), f(p_0) + \epsilon])$ and $S^+_{\tau(l)}$ and $S^+_{\tau(m)+l}$ are in the same horizontal level. By taking closures for each arcs, we have $l+m+1$ embedded arcs with boundaries. Especially, $S^+_{\tau(l)} \cup S^+_{\tau(m)+l}$ becomes a connected arc with two boundary points. We call this operation a switching operation.

Figure 1 shows a typical example for these two operations.

Then, we have the following theorem.
Theorem 2.3. Let $f : \partial V \times [0, 1) \to \mathbb{R}$ be a Morse function and $V$ a compact connected oriented surface. Then there exists an immersion extension-lift $F : V \to \mathbb{R}^2$ over $f$ if and only if the following conditions are satisfied:

1. $\chi(V) = (\# S^+ - \# S^-)/2$, where $\chi(V)$ is the Euler characteristic of $V$.
2. We have a CW complex $W_{S^+}$ in $\mathbb{R}^2$ which is obtained by attaching operation on $S^+$ for each point in $V$.
3. The set of $\# S(f)/2$ arcs obtained by switching operation for each vertex of multi degree in $W_{S^+}$ is the same as the set of orientation reversing arcs $S^-.$

Proof. Suppose that there exists an immersion extension-lift $F : V \to \mathbb{R}^2$ over $f$. Because $F$ is an immersion, $\pi \circ F$ is a submersion. For any $y \in \mathbb{R}$, each fiber $(\pi \circ F)^{-1}(y)$ is either empty or diffeomorphic to a finite disjoint union of closed intervals and points. For $x_1, x_2 \in V$, we define $x_1 \sim x_2$ if $\pi \circ F(x_1) = \pi \circ F(x_2)$ and $x_1, x_2$ are in the same connected component of $(\pi \circ F)^{-1}(\pi \circ F(x_1))$. Denote by $W_F = V/\sim$ the quotient space of $V$ for this equivalence relation and by $q_F : V \to W_F$ the quotient map. We call $W_F$ the Reeb graph of $\pi \circ F$. In general, $W_F$ is not a manifold, however, it is homeomorphic to a 1-dimensional finite CW complex. Figure 2 is examples of local correspondences between $V$ and $W_F$. Note that for the type (a), $r$ is not a vertex of $W_F$. On the other hand, for the types (b) and (c), $r$ is a vertex of $W_F$. We do not have a degree 2 vertex in $W_F$.

Because $q_F^{-1}(r)$ is a closed interval or a point, $W_F$ and $V$ are homotopy equivalent. Thus, we have $\chi(V) = \chi(W_F)$. Suppose that each vertex $R_1, \ldots, R_u$ corresponds to a degree one vertex in $W_F$ and each vertex $r_1, \ldots, r_v$ corresponds to a degree $\alpha_1, \ldots, \alpha_v$ vertex, respectively ($\alpha_i > 2$). Each fiber $q_F(R_i)$ contains exactly

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Examples of local correspondences between $V$ and $W_F$.}
\end{figure}
one true singularity and each fiber \( q_F(r_i) \) contains \( \alpha_i - 2 \) false singularities (see Figure 2). That is, we have \( u = \#T \) and \( \alpha_1 + \cdots + \alpha_v = \#T. \) Therefore, we have the equation of (1).

For a vertex \( r \in W_F \) of the type Figure 2 (c), we define that \( q_F^{-1}(r) = [a, b] \) such that \( a \in S^+ \) and \( b \in S^- \). We suppose that \( p \in \partial S \) (resp. \( \partial S \)) is the nearest point to \( a \) (resp. \( b \)) among \( [a, b] \cap \partial S \) and that the direction from \( a \) to \( b \) is the positive direction of \([a, b]\). By cutting branches of \( V \) along the interval \([p, b]\), we see that around \( r, W_F \) is obtained by attaching operations on \( S^+ \) for each point \([a, b] \cap \partial S \).

The order of arcs which are connected to \( r \) is determined by the orientation of \([a, b]\). By using this order of arcs around each multi-degree vertex, \( W_F \) can be realized in \( \mathbb{R}^2 \). This \( W_F \subset \mathbb{R}^2 \) satisfies the condition (2). Similarly, by cutting branches of \( V \) along the interval \([a, \partial S]\), we see that \( S^+ \) is obtained by switching operations on \( W_F \subset \mathbb{R}^2 \) for the points \([a, b] \cap \partial S \). This satisfies the condition (3).

Conversely, suppose that the conditions of Theorem 2.3 are satisfied. Let \( W_{S^+} \subset \mathbb{R}^2 \) be a CW complex which is obtained by attaching operations on \( S^+ \) for each point of \( \partial S \). By thickening \( W_{S^+} \) in \( \mathbb{R}^2 \), we have and oriented immersed surface with boundary. That is, we have an orientation preserving immersion \( F: V \rightarrow \mathbb{R}^2 \) such that \( \chi(V) = \sharp(S(f)/2 - \sharp(S) = \sharp(T - \sharp(S))/2 \) and \( \pi \circ F \circ i(S(p \circ F \circ i)) = f(S) \). Let \( T^+ \) (resp. \( T^- \)) be the closure of the set of regular points at which \( \pi \circ F \circ i \) coincides with the given function \( f \). Therefore, the orientation preserving immersion \( F: V \rightarrow \mathbb{R}^2 \) is the desired immersion extension-lift over \( f \).

This completes the proof. \( \square \)

If a CW complex \( W_{S^+} \) satisfies the conditions (1) and (2) of Theorem 2.3, we call it a virtual Reeb graph of \( f \). If \( W_{S^+} \) satisfies all the three conditions of Theorem 2.3, we call it a real Reeb graph of \( f \).

We give two examples to clarify Theorem 2.3.

**Example 2.4.** Let \( f : \partial D^2 \times [0, 1) \rightarrow \mathbb{R} \) be a Morse function such that \( \mathbb{T}_{\min} = \{P_1\}, \mathbb{T}_{\min} = \{q_1\} \) and \( \mathbb{T}_{\max} = \{q_2\} \) and \( \mathbb{T}_{\max} = \{P_2, P_3\} \) and \( \mathbb{T}_{\max} = \{q_2\} \) and \( \mathbb{T}_{\max} = \{P_2, P_3\}, \). Suppose that \( f(P_1) < f(q_1) < f(P_2) < f(P_3) \), then there exists an immersion extension-lift \( F_1 : D^2 \rightarrow \mathbb{R}^2 \) over \( f \). See Figure 3(a). On the other hand, If \( f \) satisfies that \( f(q_1) < f(P_1) < f(P_2) < f(P_3) \), we cannot make a virtual Reeb graph from the arcs \( S^+, S^- \). Therefore, there does not exist an immersion extension-lift over \( f \). See Figure 3(b).

**Example 2.5.** Let \( f : \partial D^2 \times [0, 1) \rightarrow \mathbb{R} \) be a Morse function such that \( \mathbb{T}_{\min} = \{P_1, P_2\}, \mathbb{T}_{\min} = \{q_1\} \) and \( \mathbb{T}_{\max} = \{P_2, P_3\} \) and \( \mathbb{T}_{\max} = \{q_2\} \) and \( \mathbb{T}_{\max} = \{P_2, P_3\} \). Suppose that \( f(P_1) < f(P_2) < f(q_1) < f(q_2) < f(P_2) < f(P_3) \). We can construct three virtual Reeb graphs \( W_{S^+} \), \( W_{S^+} \), and \( W_{S^+} \). Because two graphs \( W_{S^+} \) and \( W_{S^+} \) satisfy the condition (3) of Theorem 2.3, we have immersion extension-lifts \( F_1 : D^2 \rightarrow \mathbb{R}^2 \) and \( F_2 : D^2 \rightarrow \mathbb{R}^2 \) over \( f \) such that \( W_{F_1} = W_{S^+} \) and \( W_{F_2} = W_{S^+} \). See Figure 4 (a) and (b). On the other hand, since \( W_{S^+} \) does not satisfy the condition (3) of Theorem 2.3, there does not exit an immersion extension-lift \( F : D^2 \rightarrow \mathbb{R}^2 \) over \( f \) such that \( W_F = W_{S^+} \). See Figure 4 (c).
Figure 3. (a) satisfies the conditions of Theorem 2.3, but (b) does not satisfy the condition (2) of Theorem 2.3.

3. $f$-image homotopy classes of immersion extension-lifts

Let $\text{Imm}_f(V,\mathbb{R}^2)$ be the set of immersion extension-lifts $F : V \to \mathbb{R}^2$ over a Morse function $f : \partial V \times [0,1) \to \mathbb{R}$. In this section, we determine $f$-image homotopy classes of $\text{Imm}_f(V,\mathbb{R}^2)$.

**Definition 3.1.** Let $f : \partial V \times [0,1) \to \mathbb{R}$ be a Morse function, $S^+$ the closure of orientation preserving regular points of $f|\partial V \times \{0\}$ and $\mathfrak{F}$ the set of false singularity of $f$. Let $W_{S^+}$ and $W_{S^+}'$ be virtual Reeb graphs of $f$. If both graphs are obtained by the same attaching operation on $S^+$ for each point in $\mathfrak{F}$, then we call these two virtual Reeb graphs are same.

**Theorem 3.2.** Let $f : \partial V \times [0,1) \to \mathbb{R}$ be a Morse function and $F$ and $F' : V \to \mathbb{R}^2$ two immersion extension-lifts over $f$. Suppose that both Reeb graphs $W_F$ and $W_{F'}$ are obtained by attaching operation on $S^+$ for each point in $\mathfrak{F}$ as described in the proof of Theorem 2.3. Then $F$ and $F'$ are $f$-image homotopic if and only if $W_F$ and $W_{F'}$ are same in the sense of Definition 3.1.
Proof. Suppose that $F$ and $F' : V → \mathbb{R}^2$ are $f$-image homotopic, $F_t : V → \mathbb{R}^2$ family of immersion extension-lifts over $f$ ($t ∈ [0, 1]$) and $h : V → V$ an orientation preserving diffeomorphism such that $h|\partial V × [0, 1] = id$, $F_0 = F$ and $F_1 = F' ∘ h$. For any $p ∈ \mathfrak{F}$, let $I_{p,t} ⊂ V$ be a connected component of $(π ∘ F_t)^{-1}(π ∘ F_t ∗ h(p))$ which contains $p$. Suppose that the degree of the vertex $q_F(p)$ of the Reeb graph $W_{F_t}$ equals $a$. Since $f$ is fixed, $a$ is constant for each $t$. Then we put $S^+ ∩ I_{p,t} = \{p_0', p_1, p_2, \ldots, p_{α-2}\}$ where $p_1 = p$, $p_0'$ is a regular point of $f$ and $\{p_1, p_2, \ldots, p_{α-2}\} ⊂ \mathfrak{F}$. Because the restricted map $F_t|S^+ : S^+ → \mathbb{R}^2$ is a regular homotopy, $p_0'$ belongs to the same connected component $S^+_0$ of $S^+$ during this homotopy. Note that if $p_i$ is a boundary of $S^+_i$, the order of $S^+_0, S^+_1, \ldots, S^+_{α-2}$ around $q_F(p_i)$ of $W_{F_t}$ corresponds to the order of singular points $p_1, \ldots, p_{α-2}$ on $∂V$. Therefore, $W_{F_0} = W_F$ and $W_{F_1} = W_{F' ∗ h}$ are obtained by the same attaching operation for each point in $\mathfrak{F}$. Because $h : V → V$ is a diffeomorphism such that $h|\partial V × [0, 1] = id$, we have that $W_{F'} = W_{F' ∗ h}$. Therefore, $W_F$ and $W_{F'}$ are same.

Conversely, suppose that $W_F$ and $W_{F'}$ are the same Reeb graphs. Because of the reconstruction of an immersion from the true Reeb graph, we have a family of diffeomorphisms $h_t : \mathbb{R}^2 → \mathbb{R}^2$ ($t ∈ [0, 1]$) such that $h_0 = id$, $h_1 (F(V)) = F'(V)$ and $π = π ∘ h_t$. Thus, there is an orientation preserving diffeomorphism $h : V → V$.
such that $\bar{h}_1 \circ F = F' \circ h$ and $h|\partial V \times [0, 1) = \text{id}$. If we put $F_t = \bar{h}_t \circ F$, then we have that $F$ and $F'$ are $f$-image homotopic. This completes the proof. ∎

By Theorem 3.2, two immersion extension-lifts $F_1$ and $F_2 : D^2 \to \mathbb{R}^2$ over $f$ which are given in Example 2.5 are not $f$-image homotopic.

Remark 3.3. Consider an immersion extension-lift of an oriented surface $V$ into $\mathbb{R}^3$ over a Morse function $f : \partial V \times [0, 1) \to \mathbb{R}$. If $F_1$ and $F_2 : V \to \mathbb{R}^3$ are $f$-image homotopic immersion extension-lifts over $f$, then the Reeb graphs $W_{F_1}$ and $W_{F_2}$ are same. On the other hand, there exist immersion extension-lifts $F_1$ and $F_2$ over $f$ such that their Reeb graphs $W_{F_1}$ and $W_{F_2}$ are same but they are not $f$-image homotopic. Figure 5 is one of such examples. In this example, there is an edge $e$ in the Reeb graph $W_{F_1}$ such that for a point $r \in e$, $q_{F_1}^{-1}(r) = q_{F_2}^{-1}(r) = S^1$ and $F_1|q_{F_1}^{-1}(r) : S^1 \to \mathbb{R}^2$ and $F_2|q_{F_2}^{-1}(r) : S^1 \to \mathbb{R}^2$ have different winding numbers.

As a corollary of Theorem 3.2, we have an upper bound of the number of $f$-image homotopy classes of $\text{Imm}_f(V, \mathbb{R}^2)$ for a stable Morse function $f$. Here, a stable Morse function means that every singular value of a Morse function is disjoint.

Corollary 3.4. Let $\{p_1, \ldots, p_{n_f}\}$ be the set of false singularity points of a stable Morse function $f : \partial V \times [0, 1) \to \mathbb{R}$. Let $E_f^{\text{min}} = \#\left(\mathbb{Z}_{\text{min}} \cup \mathbb{R}^{\text{min}}\right) \cap f^{-1}((-\infty, f(p_i)))$
and \( E_{i}^\text{max} = \#((\mathcal{Z}_{i}^\text{max} \cup \mathcal{Z}_{i}^\text{min}) \cap f^{-1}((-\infty, f(p_i)))) \). If we define

\[
E_i = \begin{cases} 
2(E_{i}^\text{min} - E_{i}^\text{max}) & \text{if } p_i \in \mathcal{Z}_{i}^\text{min}, \\
2(E_{i}^\text{min} - E_{i}^\text{max} - 1) & \text{if } p_i \in \mathcal{Z}_{i}^\text{max}.
\end{cases}
\]

Then the number of \( f \)-image homotopy classes of \( \text{Imm}_f(V, \mathbb{R}^2) \) is less than or equals to \( \prod \# F_i = 1 E_i \).

**Proof.** Because \( f \) is a stable Morse function, the degree of a vertex of a virtual Reeb graph is one or three. Thus, it is easy to see that \( \#(S^+ \cap f^{-1}(f(p_i))) = E_{i}^\text{min} - E_{i}^\text{max} \) holds for \( p_i \in \mathcal{Z} \). Since \( f \) is a stable Morse function, the degree of a virtual Reeb graph at a vertex of \( \mathcal{Z} \) must be three. This means that for each \( p_i \), there are \( E_i \) possibilities for attaching operation on \( S^+ \). Therefore, there are \( \prod \# F_i = 1 E_i \) virtual Reeb graphs which are not same in the sense of Definition 3.1. Then we have the desired inequality. \( \square \)

4. IMAGE HOMOTOPY OVER A MORSE HOMOTOPY

Let \( f_s : \partial V \to \mathbb{R} \ (s \in [0, 1]) \) be a Morse homotopy and \( F, G : V \to \mathbb{R}^2 \) immersion extension-lifts over \( f_0 \) and \( f_1 \), respectively. In this section, we classify immersion extension-lifts up to image homotopy over the Morse homotopy \( f_s \).

Let the genus of \( V \) be \( g \) and \( c_0, \ldots, c_n \) the connected components of \( \partial V \) \((n \geq 0)\). Let \( a_1, b_1, \ldots, a_g, b_g \) and \( d_1, \ldots, d_{g-1} \) be oriented simple closed curves as depicted in Figure 6. We take the orientations of \( a_i, b_i \) and \( d_i \) so that each intersection number \( b_i \cdot a_i \) and \( d_i \cdot b_i \) in \( V \) equals one, respectively.

![Figure 6. Simple closed curves on V.](image)

Let \( F : V \to \mathbb{R}^2 \) be an orientation preserving immersion, \( \alpha : S^1 \to V \) an oriented simple closed curve in \( V \). We denote by \( D_F(\alpha) \in \mathbb{Z} \) the rotation number of \( F \circ \alpha \).

Then we have the following theorem.

**Theorem 4.1.** Let \( f_s : \partial V \to \mathbb{R} \ (s \in [0, 1]) \) be a Morse homotopy such that the number of connected components of \( \partial V \) is \( n + 1 \) \((n \geq 0)\). Let \( F \) and \( G : V \to \mathbb{R}^2 \) be immersion extension-lifts over \( f_0 \) and \( f_1 \), respectively. then we have the following.

1. Suppose that the genus of \( V \) equals to zero. If \( F \) and \( G \) are image homotopic over the Morse homotopy \( f_s \), then

\[
(D_F(c_0), \ldots, D_F(c_n)) = (D_G(c_0), \ldots, D_G(c_n))
\]

holds.
(2) Suppose that the genus of $V$ equals to one. If $F$ and $G$ are image homotopic over the Morse homotopy $f_s$, then

\[
\gcd (D_F(a_1), D_F(b_1)) = \gcd (D_G(a_1), D_G(b_1)),
\]
and

\[
(D_F(c_0), \ldots, D_F(c_n)) = (D_G(c_0), \ldots, D_G(c_n))
\]
hold.

(3) Suppose that the genus of $V$ is greater than one. If $F$ and $G$ are image homotopic over the Morse homotopy $f_s$, then

\[
\sum_{i=1}^{g} (D_F(a_i) + 1)(D_F(b_i) + 1) \equiv \sum_{i=1}^{g} (D_G(a_i) + 1)(D_G(b_i) + 1) \pmod{2},
\]
and

\[
(D_F(c_0), \ldots, D_F(c_n)) = (D_G(c_0), \ldots, D_G(c_n))
\]
hold.

**Proof.** Let $F$ and $G : V^2 \to \mathbb{R}^2$ be orientation preserving immersions. It is known that $F$ and $G$ are regularly homotopic if and only if

\[
(D_F(a_1), D_F(b_1), \ldots, D_F(a_g), D_F(b_g)) = (D_G(a_1), D_G(b_1), \ldots, D_G(a_g), D_G(b_g))
\]
and

\[
(D_F(c_0), \ldots, D_F(c_n)) = (D_G(c_0), \ldots, D_G(c_n))
\]
are satisfied.

Let $\text{Diff}^+_\partial(V)$ be the space of orientation preserving diffeomorphisms $h : V \to V$ such that $h \partial V \times [0,1] = \text{id}$. It is known that the mapping class group of $\text{Diff}^+_\partial(V)$ is generated by the isotopy classes of Dehn twists $\tilde{a}_i, \tilde{b}_i$ and $\tilde{d}_j$ along $a_i, b_i$ and $d_j$ (1 \leq i \leq g, 1 \leq j \leq g - 1), respectively. By [2, 3], we have the following relations:

\[
\left(\sum_{i=1}^{g} D_F(a_i), \ldots, D_F(b_i)\right) = \left(\sum_{i=1}^{g} D_G(a_i), \ldots, D_G(b_i)\right),
\]
and

\[
\left(\sum_{i=1}^{g} D_F(a_i), \ldots, D_F(b_i)\right) = \left(\sum_{i=1}^{g} D_G(a_i), \ldots, D_G(b_i)\right),
\]

\[
\left(D_F(a_1), \ldots, D_F(a_i), D_F(b_1), \ldots, D_F(b_j)\right) = \left(D_G(a_1), \ldots, D_G(a_i), D_G(b_1), \ldots, D_G(b_j)\right),
\]

(4.10) \[ (D_{Fak}^{-1}(a_1), \ldots, D_{Fak}^{-1}(a_i), D_{Fak}^{-1}(b_1), \ldots, D_{Fak}^{-1}(b_j)) = (D_F(a_1), \ldots, D_F(a_i), D_F(b_1), \ldots, D_F(b_j))\]

and

\[
(D_{Fak}(c_0), \ldots, D_{Fak}(c_n)) = (D_F(c_0), \ldots, D_F(c_n)).
\]

Here, $k_i = \tilde{d}_i \circ \tilde{a}_{i+1} \circ \tilde{a}_i^{-1}$ (1 \leq i \leq g - 1) and $h$ is any orientation preserving diffeomorphism in $\text{Diff}^+_\partial(V)$. Because of (4.8), (4.9) and (4.11), Theorem 4.1 (1) and (2) are obvious.
Suppose that \( g > 1 \). By (4.8) and (4.9), we have that
\[
\sum_{j=1}^{g} \left( D_{F \circ a_{j}}(a_{j}) + 1 \right) \left( D_{F \circ a_{j}}(b_{j}) + 1 \right) - \sum_{j=1}^{g} \left( D_{F}(a_{j}) + 1 \right) \left( D_{F}(b_{j}) + 1 \right)
= (D_{F}(a_{j}) + 1) (D_{F}(b_{j}) + 1) - (D_{F}(a_{j}) + 1) (D_{F}(b_{j}) + 1)
= \pm D_{F}(a_{j}) (D_{F}(a_{j}) + 1) \equiv 0 \pmod{2}
\]
and
\[
\sum_{j=1}^{g} \left( D_{F \circ a_{j}}(a_{j}) + 1 \right) \left( D_{F \circ a_{j}}(b_{j}) + 1 \right) - \sum_{j=1}^{g} \left( D_{F}(a_{j}) + 1 \right) \left( D_{F}(b_{j}) + 1 \right)
= (D_{F}(a_{j}) \pm D_{F}(b_{j}) + 1) (D_{F}(b_{j}) + 1) - (D_{F}(a_{j}) + 1) (D_{F}(b_{j}) + 1)
= \pm D_{F}(b_{j}) (D_{F}(b_{j}) + 1) \equiv 0 \pmod{2}.
\]
By (4.10), we have that
\[
\sum_{j=1}^{g} \left( D_{F \circ a_{j}}(a_{j}) + 1 \right) \left( D_{F \circ a_{j}}(b_{j}) + 1 \right) - \sum_{j=1}^{g} \left( D_{F}(a_{j}) + 1 \right) \left( D_{F}(b_{j}) + 1 \right)
= (D_{F}(a_{j}) + 1) (D_{F}(b_{j}) + 1) + (D_{F}(a_{j}+1) + 1) (D_{F}(b_{j}+1) - D_{F}(a_{j}))
- (D_{F}(a_{j}) + 1) (D_{F}(b_{j}) + 1) - (D_{F}(a_{j}+1) + 1) (D_{F}(b_{j}+1) + 1)
= (D_{F}(a_{j}) + 1) (D_{F}(a_{j}+1) - 1) - (D_{F}(a_{j}+1) + 1) (D_{F}(a_{j}) + 1) \equiv 0 \pmod{2}
\]
and
\[
\sum_{j=1}^{g} \left( D_{F \circ a_{j}}^{-1}(a_{j}) + 1 \right) \left( D_{F \circ a_{j}}^{-1}(b_{j}) + 1 \right) - \sum_{j=1}^{g} \left( D_{F}(a_{j}) + 1 \right) \left( D_{F}(b_{j}) + 1 \right)
= (D_{F}(a_{j}) + 1) (D_{F}(b_{j}) + 1) - (D_{F}(a_{j}) + 1) + (D_{F}(a_{j}+1) + 1) (D_{F}(a_{j}) + 1) + 2
- (D_{F}(a_{j}) + 1) (D_{F}(b_{j}+1) + 1) - (D_{F}(a_{j}+1) + 1) (D_{F}(b_{j}+1) + 1)
= (D_{F}(a_{j}) + 1) (-D_{F}(a_{j}+1) + 1) + (D_{F}(a_{j}+1) + 1) (D_{F}(a_{j}) + 1) \equiv 0 \pmod{2}.
\]
These four relations and (4.11) lead Theorem 4.1 (3). This completes the proof. \(\square\)

From Theorem 4.1, we have two examples.

**Example 4.2.** Let \( f : \partial D^{2} \times [0, 1] \to \mathbb{R} \) be a Morse function and \( F_{1}, F_{2} : D^{2} \to \mathbb{R}^{2} \) immersion extension-lifts over \( f \) which are defined in Example 2.5. It is easy to see that there exists a Morse homotopy \( f_{s} (s \in [0, 1]) \) such that \( f_{0} = f_{1} = f \) and \( F_{1} \) and \( F_{2} \) are image homotopic over this Morse homotopy \( f_{s} \). See Figure 7.

**Example 4.3.** Let \( f : \partial V \times [0, 1] \to \mathbb{R} \) be a Morse function such that the genus of \( V \) is one and the number of connected components of \( \partial V \) is two. Let \( F \) and \( G : V \to \mathbb{R}^{2} \) be immersion extension-lifts over \( f \) which are depicted in Figure 8. From Figure 8, we can take oriented simple closed curves \( a_{1} \) and \( b_{1} \) such that \((D_{F}(a_{1}), D_{F}(b_{1})) = (0, 1)\) and \((D_{G}(a_{1}), D_{G}(b_{1})) = (0, 0)\). Since
\[
\gcd(D_{F}(a_{1}), D_{F}(b_{1})) = 1 \neq 0 = \gcd(D_{G}(a_{1}), D_{G}(b_{1})),
\]
\( F \) and \( G \) are not image homotopic over any Morse homotopy \( f_{s} (s \in [0, 1]) \) such that \( f_{0} = f_{1} = f \).
Figure 7. Image homotopy between $F_1$ and $F_2$ over a Morse homotopy.

Figure 8. Two immersion extension-lifts $F$ and $G : V \to \mathbb{R}^2$ over the same Morse function which are not image homotopic over any Morse homotopy.

References


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