Measurements of acoustic particle velocity in a coaxial duct and its application to a traveling-wave thermoacoustic heat engine

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Citation: Review of Scientific Instruments 85, 094902 (2014); doi: 10.1063/1.4893639
View online: http://dx.doi.org/10.1063/1.4893639
View Table of Contents: http://scitation.aip.org/content/aip/journal/rsi/85/9?ver=pdfcov
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I. INTRODUCTION

Sound propagation in gas-filled ducts is a fundamental problem commonly encountered in many areas of acoustics, such as organ pipes, absorbing materials, and regenerators employed in acoustic heat engines. Central to the basic analysis are the viscous and thermal interactions between the gas and the duct walls, because these interactions govern the radial-dependent acoustic variables such as velocity and temperature. For a plane pressure wave in a duct, theoretical solutions for those oscillatory values have been given for various cross-sectional shapes: circle, rectangle, equilateral triangle, parallel plates, and pin arrays. Experiments have proven the validity of the solution for circular ducts by direct measurements of the axial acoustic particle velocity and by measurements of the propagation constants.

In this study, we examine the axial velocity oscillations in an annular duct that is formed between two concentric cylinders with different diameters. Such a coaxial alignment has been adopted in recent thermoacoustic heat engines, aiming at a simplified design while maintaining a traveling wave field that has so far been achieved by using a looped tube. In a thermoacoustic heat engines with a looped wave field that has so far been achieved by using a looped tube, experimental confirmation of the acoustic power flow in a coaxial duct, which provides experimental evidence for acoustic power feedback in the coaxial duct.

II. THEORETICAL ANALYSIS

A. Laminar flow theory

We consider small amplitude acoustic gas oscillations in rigid wall ducts with angular frequency \( \omega \). The characteristic transverse length of the duct is assumed to be much shorter than the acoustic wavelength. The axial coordinate of the duct is expressed by \( x \), and the axial acoustic particle velocity is denoted by \( u \). The linearized Navier-Stokes (NS) equation for the axial component is given by

\[
\frac{\partial u}{\partial t} - v \Delta_{\perp} u = -\frac{1}{\rho_m} \frac{\partial p}{\partial x},
\]

where \( v \) and \( \rho_m \) are the kinematic viscosity and the temporal mean density of the gas, respectively, and \( p \) denotes acoustic pressure; \( \Delta_{\perp} \) represents the cross-sectional component of the Laplace operator, which, in cylindrical coordinates with radial axis \( r \), is given by

\[
\Delta_{\perp} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}.
\]

We express \( u \) as \( u = \text{Re}[U e^{i\omega t}] \) by using the complex amplitude \( U \), and write the solution of the NS equation in the following form:

\[
U = \frac{1 - f_v}{1 - \chi_v} V,
\]

where \( V \) is the cross-sectional mean of \( U \). Here the function \( f_v \) satisfies the following equation:

\[
i \omega f_v - v \Delta_{\perp} f_v = 0
\]

and \( \chi_v \) is the cross-sectional mean of \( f_v \). The nonslip condition for \( u \) requires \( f_v = 1 \) at the duct wall surface.

B. Solution for an annular duct

We consider acoustic gas oscillations in the annular duct created between two circular cylinders with radii \( r_1 \) and \( r_2 \).
The unknown constants and the parallel plates.

Changing the variable from the boundary conditions to the plane of the page. The flow channel spacing is $2h$ for the coaxial duct and the parallel plates.

$r_1 \leq r \leq r_2$, as shown in Fig. 1. By using a characteristic transverse length $\delta_v = \sqrt{2\nu/\omega}$, we introduce a new variable

$$\eta \equiv i (i + 1) \frac{r}{\delta_v}. \tag{5}$$

Changing the variable from $r$ to $\eta$ simplifies Eq. (4) as follows:

$$\frac{\partial^2 f_v}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial f_v}{\partial \eta} + f_v = 0. \tag{6}$$

Equation (6) is known as the Bessel differential equation and its solutions are zero-order Bessel functions of the first and second kinds, $J_0$, and $Y_0$. Therefore,

$$f_v = C_1 J_0(\eta) + C_2 Y_0(\eta). \tag{7}$$

The unknown constants $C_1$ and $C_2$ are determined by the boundary conditions $f_v = 1$ at $r = r_1$ and $r = r_2$, which yield

$$C_1 J_0(\eta_1) + C_2 Y_0(\eta_1) = 1 \tag{8}$$

for $\eta_j = i (i + 1) r_j/\delta_v$ with $j = 1, 2$. Solving Eq. (8) with respect to $C_1$ and $C_2$ yields

$$f_v = \frac{J_0(\eta) \Delta Y_0 - Y_0(\eta) \Delta J_0}{J_0(\eta_1) Y_0(\eta_2) - J_0(\eta_2) Y_0(\eta_1)}, \tag{9}$$

with

$$\Delta J_0 \equiv J_0(\eta_2) - J_0(\eta_1) \tag{10}$$

and

$$\Delta Y_0 \equiv Y_0(\eta_2) - Y_0(\eta_1). \tag{11}$$

The dimensionless form of $U$ can be expressed by using the central velocity $U_C = U(\eta_C)$ at $\eta_C = (\eta_1 + \eta_2)/2$ as

$$\frac{U}{U_C} = \frac{1 - f_v}{1 - f_v}, \tag{12}$$

where $f_{v,C} = f_v(\eta_C)$. This equation will be used for comparison with experiments in Sec. II C.

The cross sectional mean of $f_v$ is expressed as

$$\chi_v = \frac{1}{\pi (r_2^2 - r_1^2)} \int_{r_1}^{r_2} 2\pi r f_v dr. \tag{13}$$

By change of variable from $r$ to $\eta$, and by use of mathematical formulas

$$\int z J_0(z) dz = z J_1(z) \tag{14}$$

and

$$\int z Y_0(z) dz = z Y_1(z), \tag{15}$$

($J_1$ and $Y_1$ are the first-order Bessel functions of the first and second kinds, respectively) we obtain

$$\chi_v = \frac{2}{\eta_2^2 - \eta_1^2} \cdot \frac{\Delta(\eta J_1) \Delta Y_0 - \Delta(\eta Y_1) \Delta J_0}{J_0(\eta_1) Y_0(\eta_2) - J_0(\eta_2) Y_0(\eta_1)}, \tag{16}$$

where

$$\Delta(\eta J_1) \equiv \eta_2 J_1(\eta_2) - \eta_1 J_1(\eta_1), \tag{17}$$

and

$$\Delta(\eta Y_1) \equiv \eta_2 Y_1(\eta_2) - \eta_1 Y_1(\eta_1). \tag{18}$$

Hereafter we use two non-dimensional parameters $h/\delta_v$ and $R$ to specify the coaxial duct; $h$ denotes the half of the spacing ($2h = r_2 - r_1$) and $R$ is the radius ratio ($R = r_1/r_2$). Because $r_1$ and $r_2$ are respectively given as $r_1 = 2hR/(1 - R)$ and $r_2 = 2h/(1 - R)$, $\eta_1$ and $\eta_2$ in $f_v$ and $\chi_v$ are rewritten as

$$\eta_1 = i (i + 1) \frac{2R}{1 - R} \cdot \frac{h}{\delta_v} \tag{19}$$

and

$$\eta_2 = i (i + 1) \frac{2}{1 - R} \cdot \frac{h}{\delta_v} \tag{20}$$

respectively.

C. Comparison with experiments

In order to verify the theoretical solution in Sec. II B, we measured the axial acoustic velocity $u = \text{Re}[U e^{i\omega t}]$ in annular ducts by using a laser Doppler velocimeter (LDV). The experimental set up is shown in Fig. 2. The duct is between an outer cylindrical tube with inner radius $r_2 = 20.5$ mm and an inner cylindrical tube with outer radius $r_1$. The outer tube is made of transparent acrylic pipe in order to allow measurement of the velocity oscillations by the LDV. The air in the set up was at ambient temperature and pressure. The acoustic driver, located at one end of the coaxial duct, generated acoustic oscillations of the internal gas column with a frequency of $2.0$ Hz. This slow frequency was chosen to increase $\delta_v = \sqrt{2\nu/\omega}$. The processor of the LDV accumulated the velocity and, at the same time, the built-in AD convertor recorded the voltage signals used to drive the acoustic driver.
D. Derivation of the cross sectional mean velocity

The acoustic power delivered with acoustic pressure $p$ and axial acoustic particle velocity $u$ is given by the surface integral of the acoustic intensity over a cross section:

$$W = 2\pi \int_{r_1}^{r_2} \langle pu \rangle r dr,$$

where the angled brackets represent the time averaged value. Since the acoustic pressure $p = \text{Re}[Pe^{i\omega t}]$ can be seen to be uniform at a given cross section, $W$ can be expressed in terms of the cross sectional mean velocity $u = \text{Re}[Ve^{i\omega t}]$ and the cross sectional area $A$ by $W = A \langle pv \rangle$. That is,

$$W = \frac{A}{2} |P| |V| \cos \Phi,$$

where $\Phi = \text{arg}(V/P)$ denotes the phase lead of $V$ over $P$. Therefore, one can obtain $W$ experimentally through measurements of the amplitudes $|P|$ and $|V|$ and the phase lead $\Phi$.

We have seen that the theoretical solution predicts the actual acoustic velocity distribution in the annular duct. On the basis of this fact, the cross sectional mean velocity $V$ can be obtained by simultaneous measurements of the central velocity $U_c$ and acoustic pressure $P$. The theoretical factor $\Gamma$ that links $V$ and $U_c$ through

$$V = \frac{1}{\Gamma} U_c,$$

is given by

$$\Gamma = \frac{1 - f_{v,c}}{1 - \chi_v}.$$  

Here, $f_{v,c}$ represents $f_v$ of Eq. (9) when $r = (r_1 + r_2)/2$.

Figure 4 shows the magnitude $|\Gamma|$ and the phase $\Theta$ of $\Gamma$ as a function of $h \delta_v$. The ratio $R = r_1/r_2$ of the inner radius to the outer one in the annulus does not have critical influence on the values of $|\Gamma|$ and $\Theta$, except for the regions with $h \delta_v < 2$ for $|\Gamma|$ and with $0.3 < h \delta_v < 3$ for $\Theta$. When $h \delta_v \gg 1$, $\Gamma$ goes to unity and when $h \delta_v \ll 1$, $\Gamma \approx 1.5$. At these extreme cases, $\Theta$ becomes zero, whereas $\Theta$ presents a minimum of $-10.7^\circ$ with $h \delta_v = 3.2$. By tabulating the $h \delta_v$ dependence of $\Gamma$ in advance, the central velocity $U_c$ only needs to be measured to determine the cross sectional mean velocity $V$.

For comparison, we have plotted the corresponding values for parallel plates with distance $2h$ shown in Fig. 1(b). For parallel plates, $f_{v,c}$ and $\chi_v$ are given, respectively, by

$$f_{v,c} = \frac{1}{\cosh[(i + 1)h/\delta_v]},$$

and

$$\chi_v = \frac{\tanh[(i + 1)h/\delta_v]}{(i + 1)h/\delta_v}.$$  

It is found that the values of $|\Gamma|$ and $\Theta$ for the annular duct rapidly converge on those for the parallel plates, as shown in Fig. 4. When $h \delta_v = 4.5$ and $R = r_1/r_2 > 0.5$, for example, the relative error of $|\Gamma|$ is less than 0.1\%, and the difference in $\Theta$ between the annulus and the parallel plates is less than

FIG. 3. Experimental and theoretical axial acoustic particle velocities. Symbols represent the measured amplitude (*) and phase (o). Solid and dotted curves are the theoretical amplitude and phase, respectively.

The driver signal was used as a reference to determine the phase of $U$ measured at different $r$.

Figures 3(a)–3(c) show the amplitude and phase of the axial velocity oscillations determined by the theoretical solution and by the optical measurements, when $r_1$, the outer diameter of the inner duct, is (a) 1.0 mm, (b) 2.5 mm, and (c) 12.5 mm, respectively. The velocity amplitude is normalized with respect to that of the central velocity $U_c$ at $r = (r_1 + r_2)/2$, and the phase $\theta$ represents the phase lead relative to $U_c$. The characteristic length $\delta_v = \sqrt{2v/\omega}$ in the experiment is 1.5 mm. When $\delta_v$ is much smaller than $h$, the half of the spacing between the two cylinders (see Fig. 1), a core region is visible as shown in Figs. 3(a) and 3(b), but the velocity profile becomes parabolic in Fig. 3(c) in which $h \delta_v$ has decreased to 2.7. Such dependence of the velocity profile in the annular duct is qualitatively the same as that between the parallel plates, but it should be noted that while the symmetrical velocity profile with respect to the central axis is expected in the parallel plates, it is not guaranteed for the annular duct; a slight distortion of the velocity profile is visible in Figs. 3(b) and 3(c). The agreement between the experimental results shown in Figs. 3(a)–3(c) and the theoretical result in Eq. (12) is sufficiently good.
FIG. 4. Theoretical factor $\Gamma$ for converting the central velocity $U$ to the cross-sectional mean velocity $V$ for annular region with various $R$ values ($R = 0.1, 0.3$ and $0.5$), and for parallel plates (PP).

0.7°. Thus, instead of using the mathematically complicated equations in Eqs. (9) and (16), we can determine $V$ from $U_C$ by using the simpler equations in Eqs. (25) and (26). This holds in most of the realistic cases with $h/\delta_v > 3$, because of the smallness of $\delta_v$. Furthermore, at high $h/\delta_v$ values, a wide core region is created in the annulus, where $U$ becomes spatially uniform. This means that the measured $U_C$ values are not influenced by a possible misalignment of the measuring position of the LDV from the center $[r = (r_1 + r_2)/2]$. In Sec. III, we apply the method for determining the acoustic power in the coaxial duct in order to analyze a prototypical thermoacoustic heat engine.

III. APPLICATION TO A THERMOACOUSTIC HEAT ENGINE

One of the reasons for using a looped tube in the thermoacoustic heat engine is the ability to feedback the acoustic power, which enables the recovery of the acoustic power\textsuperscript{15} that is lost in the acoustic coolers with in-line configurations such as an orifice pulse tube refrigerator.\textsuperscript{19} The other reason is the excitation of a traveling wave field where the acoustic pressure and velocity can oscillate in phase. The gas parcels oscillating with a traveling wave phasing can execute the thermodynamic Stirling cycle through reversible thermodynamic processes, and thereby realize the acoustic power amplification of the traveling wave passing through the regenerator from cold to hot.\textsuperscript{14, 16} The thermal efficiency of a traveling wave engine is essentially superior to that of standing wave thermoacoustic engines using a resonance tube, because the standing wave engines rely on irreversible thermodynamic processes. The objective of this section is to investigate whether the coaxial duct can replace the looped tube.

A. Experimental setup

Figure 5 shows the experimental setup, where the axial coordinate $x$ is directed from the bottom flange of the acoustic driver to the solid plate at the top. The setup consists of a transparent acrylic cylindrical duct with inner radius $r_2 = 20.5$ mm. An open-ended 210-mm long glass tube with inner radius of 10.5 mm and the outside radius $r_1 = 12.5$ mm is installed coaxially within the acrylic duct. The bottom of the glass tube is at $x = 0$. Thus, the coaxial duct is formed in the region with $0 < x < 210$ mm. A 20-mm long regenerator made of tightly stacked stainless-steel mesh screen (the wire diameter is 0.14 mm, and mesh number per inch is 60) is installed at $x = 170$ mm, as shown in Fig. 5(b). We used air at atmospheric pressure ($1.0 \times 10^5$ Pa) and ambient temperature as the working fluid.

The acoustic field in the setup was externally excited by using an acoustic driver, which was controlled by a function generator and a power amplifier. The pressure amplitude at the solid plate ($x = 220$ mm) was maintained at 0.5 kPa, independently of the driving frequency $f$ ($50$ Hz $\leq f \leq 200$ Hz). When a steady oscillation state was achieved, the acoustic pressure $P$ was measured via small short ducts mounted on the tube walls with small pressure transducers, and the axial central velocity $U_C$ in the regions with ($0 < x < 210$ mm) and ($-450 < x < 0$) without the inner duct was measured at the same axial positions as the pressure. Here $U_C$ represents the velocity at $r = (r_1 + r_2)/2$ in the coaxial duct region, and that on the
central axis in the regions of a single cylindrical tube. The central velocity \( U_C \) was measured simultaneously with \( P \) at the same axial position and the signal from the function generator, and was converted to the cross sectional mean velocity \( V \) by using \(|\Gamma|\) and \( \Theta \) shown in Figs. 4(a) and 4(b), in order to obtain the acoustic power. Here, because of the driving frequency \( f \geq 50 \text{ Hz} \), \( h\phi_1 \) values always exceeded 13 in the annular region, which assures the use of the solution for the parallel plates.

B. Experimental results

Figure 6 shows the axial distribution of the acoustic power \( W \) for different frequencies from \( f = 50 \text{ Hz} \) to \( 200 \text{ Hz} \). The direction of flow of \( W \) is represented by its sign; a positive (negative) sign represents the flow in the positive (negative) direction of the \( x \) axis. The acoustic power \( W \) in the annular area is shown by triangles, whereas those in the inner duct are represented by circles. The acoustic power in the single cylindrical duct with \( x < 0 \) is shown by squares. The validity of the acoustic power measurements in the annulus is confirmed by the continuity of the acoustic powers at \( x = 0 \). Namely, we see that the relation \( W_1 \approx W_2 + W_3 \) holds where \( W_1 \) at \( x = -20 \text{ mm} \) represents the power supplied from the driver through the single duct, and \( W_2 \) and \( W_3 \) are the powers at the driver end (\( x = 19 \text{ mm} \)) of the inner duct and in the annular region, respectively.

The acoustic power \( W_1 \) supplied from the driver produces a circulating flow of the acoustic power, flowing upward into the inner duct (\( W_2 > 0 \)) and going back through the regenerator in the annular duct (\( W_3 < 0 \)). Therefore, there is a possibility of realizing a traveling wave engine by using the coaxial ducts as well as a looped tube, if a temperature gradient from cold to hot can be created along the regenerator in the direction of flow of \( W \). Thus, as shown in Fig. 5(b), we placed the cold and hot heat exchangers with temperatures \( T_C \) and \( T_H \) on the sides of the regenerator such that \( W \) flows in the direction of the positive temperature gradient. The cold heat exchanger temperature \( T_C \) was maintained at 293 K by a circulating water, and the hot heat exchanger temperature \( T_H \) was increased by using an electrical heater.

We operated the acoustic driver at \( f = 50 \text{ Hz} \), because the ratio \( W_2/W_1 \) was greatest at this frequency in the frequency range studied. Figure 7 shows the distribution of the acoustic power \( W \) when the temperature difference \( \Delta T = T_H - T_C \) was changed while keeping the acoustic pressure amplitude at the top plate to 0.5 kPa. Heat power supply of 50, 100, and 150 W through the electrical heater around the hot heat exchanger resulted in \( \Delta T \) of 164, 274, and 355 K, respectively. The flow direction of the acoustic power \( W \) in the annular region was maintained upon increasing \( \Delta T \), but the magnitude of \( W \) increased significantly, resulting in \(|W_2/W_3| = 0.58\) for \( \Delta T = 0 \) while \(|W_3/W_2| = 0.95\) for \( \Delta T = 164 \text{ K} \). Further increase of \( \Delta T \) above 274 K realized \(|W_1| > |W_2|\), signifying amplification of the acoustic power, and caused the sign reversal of \( W_1 \). In other words, the acoustic power produced by a thermodynamic cycle equivalent to the Stirling thermodynamic cycle executed in a regenerator\(^{14,17,20}\) becomes sufficiently large so that the acoustic power can flows out of the annular duct towards the driver. Therefore, we are able to construct a coaxial thermoacoustic engine by replacing the cylindrical duct at \( x < 0 \) and the acoustic driver with an acoustic element having an acoustic impedance identical to that of the upper part (\( x > 0 \)) of the present setup shown in Fig. 5.\(^{21}\)

We attempted to build a thermoacoustic engine by connecting an open-ended cylindrical duct with an inner radius of 20.5 mm at \( x = 0 \). The length of the tube, 1.12 m, was chosen so that the fundamental frequency of the combined system was close to 50 Hz. We moved the regenerator unit location from \( x = 170 \text{ mm} \) to \( x = 70 \text{ mm} \) and successfully observed the spontaneous oscillations of 55.5 Hz when \( \Delta T = 264 \text{ K} \). Because a thermoacoustic engine with coaxial ducts can operate as a traveling wave thermoacoustic heat engine, by optimizing the engine design, it would be possible to efficiently drive the acoustic cooler to achieve low temperatures or use an electroacoustic transducer for generating electrical power.
IV. SUMMARY

Theoretical solutions for the axial acoustic particle velocity in the annular region of a coaxial duct were presented using two non-dimensional parameters $h/\delta_v$ and $R$. The validity of the solutions was verified by optical measurements. It was also shown that the velocity profile in the annular region can be approximated by that between the parallel plates when the values of $h/\delta_v$ and $R$ are sufficiently large. Based on these results, measurements of the acoustic power were conducted for a prototypical thermoacoustic engine with a coaxial duct. The results presented experimental evidence for the feedback of acoustic power in the coaxial configuration.

ACKNOWLEDGMENTS

This work was supported by JSPS KAKENHI Grant No. 23656141.
