Some Properties of Irreducible Ideals

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Throughout this paper all rings will be commutative with identity.

It is well known that in a Noetherian ring every irreducible ideal is primary but a primary ideal need not be irreducible. For instance, in a polynomial ring \( Z[X] \) in one indeterminate over the ring of integers \( Z \), the ideal \( q = (4, 2X, X^2) \) is primary, but we have that \( q \) is the intersection of the two ideals \( (4, X) \) and \( (2, X^2) \).

In this paper we will investigate about the irreducible ideals.

Let \( R \) be a commutative ring with identity. Then we will denote the length of \( R \)-module \( M \) by \( \text{length}(M) \). When we say that "\((R, m)\) is a local ring", we mean that \( R \) is a Noetherian local ring, that \( m \) is the unique maximal ideal of \( R \).

First we give a characterization of irreducible ideals.

**Proposition 1.** Let \((R, m)\) be a local ring and \( q \) be an \( m \)-primary ideal. Then \( q \) is irreducible if and only if \( \text{length} ((q:m)/q) = 1 \) where \( q:m = \{ a \in R \mid am \subset q \} \).

**Proof.** Suppose that \( q \) is irreducible. Put \( b = q:m \). Suppose now that \( \text{length} (b/q) \geq 2 \). From \( mb \subset q \) we deduce that \( b/q \) is a vector space over \( R/m \). There exist two vector subspaces \( V_1 \neq (0) \) and \( V_2 \neq (0) \) of \( b/q \) such that \( V_1 \cap V_2 = (0) \). Let \( \psi : b \rightarrow b/q \) be a natural mapping. Put \( b_1 = \psi^{-1} (V_1) \) and \( b_2 = \psi^{-1} (V_2) \). Then we have \( q = b_1 \cap b_2 \), \( b_1 \not\supset q \) and \( b_2 \not\supset q \). This is contradict to the fact that \( q \) is irreducible. Hence \( \text{length} (b/q) \leq 1 \). Since \( \text{length} (b/q) = 0 \), we have \( \text{length} (b/q) = 1 \).

Conversely, suppose that \( \text{length} ((q:m)/q) = 1 \). We assume that \( q \) is reducible. Then there exist two ideals \( b_1 \) and \( b_2 \) of \( R \) such that \( q = b_1 \cap b_2 \), \( b_1 \not\supset q \) and \( b_2 \not\supset q \). We can
choose the ideals \( b_1 \) and \( b_2 \) such that \( b_1 = q + aR \) and \( b_2 = q + bR \) for some elements \( a \) and \( b \) of \( q: m \). We claim that \( \tilde{a}, \tilde{b} \) are linearly independent over \( R/m \), where \( \tilde{a}, \tilde{b} \) denote the homomorphic image in \((q: m)/q\) of \( a \) and \( b \), respectively. Suppose that \( \tilde{a}, \tilde{b} \) are linearly dependent over \( R/m \). Then there exists an element \( \lambda \neq 0 \) of \( R/m \) such that \( \tilde{a} = \lambda \tilde{b} \). From \( a_1/q \subset (q: m)/q \), it holds that \( a_1/q = \tilde{a} (R/m) \). Similarly \( a_2/q = \tilde{b} (R/m) \). And we can see that \( \tilde{a} (R/m) \cap \tilde{b} (R/m) = (0) \). This is a contradiction. Thus the proof is complete.

In connection with Proposition 1, we give the following.

Let \((R, m)\) be a local ring and \( q = a_1 \cap \ldots \cap a_t \) be a finite intersection of irreducible ideals, \( \psi: q: m \rightarrow (q: m)/q \) be the natural mapping and \( W_1, \ldots, W_n \) be the \( d-1 \) dimensional subspaces of \((q: m)/q\) where \( d = \text{length} ((q: m)/q) \). Put \( a_i = \psi^{-1}(W_i) \). Some \( a_i \) is not an irreducible ideal.

**Corollary 2.** Let \((R, m)\) be a local ring. If \( R \) is neither a field nor a one dimensional regular local ring, then \( m^n \) is reducible for some integer \( n \).

**Proof.** Suppose that \( m^n \) is irreducible for any non-negative integer \( n \). From the facts that \( m^n: m \supset m^{n-1} \) and \( \text{length} (m^{n-1}/m^n) \geq 1 \), we have that \( \text{length} (m^{n-1}/m^n) = 1 \) by Proposition 1. Therefore we see that \( m^{n-1} = aR + m^n \) for some \( a \in m^{n-1} \). Take \( n = 2 \). Then it holds that \( m = aR \), and so \( R \) is a one-dimensional regular local ring. This is a contradiction. Thus the proof is complete.

Next, we give a relation between the irreducible ideals and the generating elements of the maximal ideal.

**Proposition 3.** Let \((R, m)\) be a local ring and \( a \) be an \( m \)-primary ideal. If \( \{a_1, \ldots, a_d\} \) is the minimal generating system of \( m \) and \( a_1, \ldots, a_{d-1} \in a \), then \( a \) is irreducible.

**Proof.** Since \( a \) is \( m \)-primary, there exists some integer \( n \) satisfying \( m^n \subset a \). The subset \( \{a_d, a_d^2, \ldots, a_d^{n-1}\} \) of \( m \) is not contained in \( a \). Therefore we have \( a: m = a + a_d^{n-1}R \) and \( \text{length} ((a: m)/a) = 1 \). Hence \( a \) is irreducible by Proposition 1. Thus the assertion holds.

Concerning the converse of Proposition 3, we give the following Remark.

**Remark.** In general, the converse of Proposition 3 is not true. Let \( k \) be a field, \( k[[X, Y]] \) be the ring of formal power series in two indeterminates \( X \) and \( Y \) over \( k \) and \( R \) be the residue class ring \( k[[X, Y]]/(X^2 - Y^2) = k[[x, y]] \) (where \( x \) and \( y \) are the residue of \( X \) and \( Y \)). Put \( m = (x, y) \) and \( I = (\lambda x + \mu y)R \) for suitable elements \( \lambda, \mu \in k \). We want to prove \( I \supset m^2 \). Since \( (\lambda x + \mu y)(\lambda x - \mu y) \in I \), it follows that \( x^2 \in I \) for \( \lambda \neq \pm \mu \neq 0 \). Thus
y^2 \in I. Since (\lambda x + \mu y)x \in I, we see xy \in I. It then follows that

\[(fx + gy)(px + qy) = fpx^2 + (fq + gp)xy + gqy^2 \in I\]

for any f, g, p and q \in R. Hence we have I \supseteq m^2. By this result, it follows that an irreducible ideal \( a \) such that \( a \supseteq m^2 \) does not contain one element of minimal generating elements of \( m \). Now, the type of ideals in \( R \) are following:

1) \( a = m^n \) for some integer \( n \).

2) \( a = m^n + (\lambda x^d + \mu x^{d-1}y)R \) for some non-zero elements \( \lambda \) and \( \mu \) of \( k \). The ideal of type 2) is irreducible. Since \( a:m = m^{n-1} \) by the following Proposition 4, we can see that \( a \) is irreducible but in general \( a:m = m^{n-1} \) is reducible by Corollary 2.

**Conjecture.** If \((R, m)\) is a regular local ring and \( a \) is an \( m \)-primary and irreducible ideal, then there exist minimal generating elements \((a_1, \ldots, a_d)\) of \( a \) such that \( a_1, \ldots, a_{d-1} \in a \) where \( d = \text{length} ((a:m)/a) \).

**Proposition 4.** Let \((R, m)\) be a local ring and \( a \) be an ideal of \( R \) such that \( m^{n-1} \nsubseteq a \supseteq m^n \) for some positive integer \( n \). If \( a \) is irreducible, then \( a:m = m^{n-1} \).

**Proof.** Since \( a:m \supseteq m^{n-1} \nsubseteq a \) and length \(((a:m)/a)\) = 1 by Proposition 1, we have \( a:m = m^{n-1} \).

We now investigate the irreducibility of quotient ideals.

**Remark.** Let \((R, m)\) be a local ring and \( a \) be an ideal of \( R \). Even if \( a:m \) is irreducible, \( a \) is not necessarily irreducible.

**Example (Zariski-Samuel).** Let \( k \) be a field. Let \( R = k[X, Y] \) be a polynomial ring in two indeterminates \( X, Y \). Let \( a = (X^2, XY, Y^2) \) and \( m = (X, Y) \). From \( a = (X, XY, Y^2) \cap (X^2, XY, Y^2) \), we see that \( a \) is reducible but \( a:m = (X, Y^2) \) is irreducible.

We will give the following main result.

**Theorem 5.** Let \((R, m)\) be a local ring and \( a \) be an \( m \)-primary ideal. If \( a \) is an irredundant representation as finite intersection of irreducible ideals, say, \( a = b_1 \cap \ldots \cap b_t \), then we have length \(((a:m)/a)\) = 1.

**Proof.** Let \( b_1 \supseteq a:m \). We show that length \(((a:m)/(a:m) \cap b_1)\) = 1. We can find two elements \( x \) and \( y \) of \( a:m \) not contained in \( b_1 \). Then we have \( x, y \in b_1:m \). Since length \(((b_1:m)/b_1)\) = 1 by Proposition 1, there exists some element \( \lambda \in R - m \) such that \( x-\lambda y \in b_1 \cap (a:m) \). Therefore length \(((a:m)/(a:m) \cap b_1)\) = 1. Since \( a \subseteq (a:m) \cap b_1 \subseteq a:m \), we
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have length \(((a:m) \cap b_1)/a) = n-1\), where \(n = \text{length } ((a:m)/a)\). Since \(\bigcap_{i=1}^t ((a:m) \cap b_i)/a) = (0)\), we have that \(n \leq t\). We can assume that \(\bigcap_{i=1}^p ((a:m) \cap b_i)/a) = (0)\). Now, putting \(b = b_1 \cap ... \cap b_n\), we will prove that \(b = a\). In fact, suppose that \(b \not\subseteq a\). Then we can take an element \(x \in b\) such that \(x \in a:m\) and \(x \notin a\). On the other hand, \(b \cap (a:m) = \cap ((a:m) \cap b_i) = a\). This is a contradiction. Thus our theorem is completely proved.

For a local ring \((R, m)\) and an ideal \(a\) of \(R\), if the following conditions hold:

(1) \(a \supset m^n\)  (2) \(a \supset m^{n-1}\) for some integer \(n\), then \(n\) is called the index of \(a\). \(a\) is called strongly irreducible if \(a:m^i\) is irreducible for any non-negative integer \(i\).

Finally, in connection with strongly irreducible ideals, we obtain the following. The proof is omitted.

**Proposition 6.** Let \((R, m)\) be a local ring, \(a\) be an \(m\)-primary ideal and \(n\) be the index of \(a\). Then the following assertions are equivalent.

(1) \(a\) is strongly irreducible.
(2) \(\text{length } (m/a) = n\).
(3) \(a \not\subset a:m \subset ... \subset a:m^{n-1} = m\) is the composition series of \(a\).