Hermite-Fejér Interpolation by Trigonometric Polynomials

Rokuzi SHIBATA

(Department of Mathematics)

And

Ryozi SAKAI

(Department of Mathematics of Attached High School)

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§ 0. Introduction.

Let us denote by $\theta_i = \theta_n = (2i-1)\pi/n, \ i = 1,2,\ldots,n$, the $n$ distinct zeros of the function $C_n(\theta) = \cos(n\theta/2), \ \theta \in K = [0,2\pi)$, and let

$$K_{ni}(\theta) = \{\cos(n\theta/2)\}/ \sin[(\theta-\theta_i)/2].$$

Then, for each $i=1,2,\ldots,n$ we obtain a unique interpolation polynomial of degree $mn-1$;

(1) $H_i(\theta) = \sum_{k=0}^{n-1} A(n,k)\sin^{2k}[(\theta-\theta_i)/2]^m K_{ni}(\theta)^{2m}$

such that

(2) $H_i(\theta_j) = \delta_{ij}, \quad H_i^s(\theta_j) = 0, \ \ i,j = 1,2,\ldots,n, \ s = 1,2,\ldots,2m-1.$

Let $\lambda = \{\lambda_i^{[n]}\}, \ i = 1,2,\ldots,n, \ j = 0,1,\ldots,m-1$ be a $(n,m)$-matrix with bounded real components uniformly with respect to $n = 1,2,3,\ldots$. If we define a new matrix $a = \{a_i^{[n]}\}$ by

$$\{a_i^{[n]} + a_i^{[n]}\sin(\theta-\theta_i) + \cdots + a_i^{[n]}\sin^{i}(\theta-\theta_i)\}_{i,j}^{n,j} = \lambda_i^{[n]},$$

$i = 1,2,\ldots,n, \ j = 0,1,\ldots,m-1$

then for $\lambda$ we obtain a trigonometric interpolation polynomial of degree $m(n+1)-2$;

(3) $T_{m(n+1)-2}[\lambda;\theta] = \sum_{i=1}^{n} \sum_{j=0}^{m-1} a_i^{[n]}\sin^{i}(\theta-\theta_i) \cdot H_{ni}(\theta).$
The polynomial of this kind is called a Hermite-Fejér interpolation polynomial. For the algebraic case many authors study this problem ([1], [2], etc.). Our purpose in this paper is to approximate a continuous function by this interpolation polynomial (3).

§ 1. Main theorem.
We denote the modulus of continuity of \( f \in C[K] \) by \( w(f; t) \), and we select an arbitrarily modulus of continuity \( \mathcal{Q} \). Let define a class of functions;

\[
C(\mathcal{Q}) = \{ f \in C[K] : w(f; t) \leq \mathcal{Q}(t), \ t \geq 0 \}.
\]

When we define \( \lambda \) like \( \lambda^{[\frac{m}{n}]}_i = f(\theta_i), \ i = 1, 2, \ldots, n, \), where \( f \in C[K] \), we denote \( T_{m(n+1)-2}[\lambda] \) in (3) by \( T_{m(n+1)-2}[f, \lambda] \).

Our main theorem is the following:

**Theorem.** Let \( f \in C(\mathcal{Q}) \) and let \( \lambda \) be bounded. Then we have

\[
T_{m(n+1)-2}^{(s)}[f, \lambda; \theta] = \lambda^{[\frac{n}{s}]}_i, \ i = 1, 2, \ldots, n, \ s = 0, 1, \ldots, m-1
\]

and

\[
| f(\theta) - T_{m(n+1)-2}[f, \lambda; \theta] |
\]

\[
= O(1)((C_n(\theta)/n) \sum_{i=1}^{n} \mathcal{Q}(1/i) + \mathcal{Q}(|C_n(\theta)|/n) + \{|C_n(\theta)| \log n|/n). \]

§ 2. Lemmas.
In this section we state the several fundamental lemmas.

**Lemma 1.** We have

(i) \( (\sin \theta)^{[s]}_{t=0} = \begin{cases} (-1)^{t+1} & \text{if } s = 2t - 1, \\ 0 & \text{if } s = 2t, \ t = 1, 2, \ldots, \end{cases} \)

(ii) \( (\sin^2 \theta)^{[s]}_{t=0} = \begin{cases} (-1)^{t+1}2^{2t-1} & \text{if } s = 2t, \\ 0 & \text{if } s = 0, 2t - 1, \ t = 1, 2, \ldots. \end{cases} \)

For general,
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\[ \text{Proof. (i) and (ii) are trivial. By induction we show \( \text{iii}. \)} \]

\[ (\sin^{2p+1}\theta)_{\theta=0}^{(S)} = \sum_{\gamma=0}^{S} (\sin^{2p-1}\theta)^{(S)}_{\theta=0} (\sin^2 \theta)^{(\gamma)}_{\theta=0} \]

\[ = \sum_{\gamma=0}^{S} 2q \left( \sin^{2p-1}\theta \right)^{(\gamma-2q)}_{\theta=0} (-1)^{S+1} 2^{q-1}. \]

Let \( s = 2t - 1, \ t \geq p + q, \) then

\[ \text{sign}(\sin^{2p-1}\theta)^{(2t-2q-1)}_{\theta=0} (-1)^{q+1} = (-1)^{p+r+1}. \]

Thus,

\[ \text{sign}(\sin^{2p+1}\theta)_{\theta=0}^{(S)} = \begin{cases} (-1)^{p+r+1} & \text{if } s = 2t - 1, \ t \geq p + 1, \\ 0 & \text{if } s = 0,1,\ldots,2p, \ 2t, \ t \geq p + 1, \ t = 1,2,\ldots. \end{cases} \]

Likewise we can show (iv). (q.e.d.)

Lemma 2. We have

(i) \[ K_{n}(\theta) = \begin{cases} (-1)^{i} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \]

(ii) \[ \text{sign}K_{n}^{(\theta)}(\theta) = \begin{cases} (-1)^{i+p} & \text{if } s = 2t \\ 0 & \text{if } s = 2t - 1, \ t = 1,2,\ldots, \end{cases} \]

and

(iii) \[ |K_{n}^{(\theta)}(\theta)| = \begin{cases} 0(1)n^{s+1} & \text{if } s = 2t, \\ 0 & \text{if } s = 2t - 1, \ t = 1,2,\ldots. \end{cases} \]

Proof. We know
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\[(4) \quad T_{n-1}(\theta) = \begin{cases} \frac{\sum_{p=1}^{[(n+1)/2]} (-1)^{p+1} n(n^2-1)\cdots[n^2-(2p-3)^2]}{(2p-1)!} \sin^{2p-2}\theta & \text{if } n \text{ is odd}, \\ \frac{\sum_{p=1}^{[n/2]} (-1)^{p+1} n(n^2-2^2)\cdots[n^2-(2p-2)^2]}{(2p-1)!} \sin^{2p-2}\theta \cos\theta & \text{if } n \text{ is even}, \end{cases} \]

where the coefficient for \( p = 1 \) is \( n \). Then we see \( K_{ni}(\theta) = (-1)^{t} T_{n-1}((\theta-\theta_i)/2) \).

We use Lemma 1. (i) is trivial. (ii) Let \( n \) be odd, and let \( s = 2t, \ t = 1, 2, \ldots \). Then,

\[ \text{sign} K_{ni}^{(s)}(\theta_i) = \text{sign} (-1)^{t} T_{n-1}^{(s)}(0) = (-1)^{t+s} (p-1 \leq t). \]

If \( n \) is even and \( s = 2t - 1 \) (ii) is trivial. Let \( n \) be even, and let \( s = 2t \).

\[ (\sin^{2p-2}\theta \cos\theta)^{2t}_{k=0} = \sum_{t-r}^{2t} (\sin^{2p-2}\theta)^{2t-2q}_{s=0} (-1)^{q}. \]

Here, we see

\[ \text{sign}(\sin^{2p-2}\theta \cos\theta)^{2t-2q}_{k=0}(-1)^{q} = (-1)^{t+p-1} (p-1 \leq t), \]

thus for \( p-1 \leq t \)

\[ \text{sign}(\sin^{2p-2}\theta \cos\theta)^{2t}_{k=0} = (-1)^{t+p-1}. \]

Thus, we have

\[ \text{sign} K_{ni}^{(s)}(\theta_i) = \text{sign} (-1)^{t} T_{n-1}^{(s)}(0) = (-1)^{t+s} (p-1 \leq t). \]

\[ \hat{\theta} \quad \text{We may show this for } s = 2t. \text{ Let } n \text{ be odd.} \]

\[ | K_{ni}^{(s)}(\theta_i) | = 0(1)n(n^2-1)\cdots[n^2-(2t-1)^2] \]

\[ = 0(1)n^{2t+1} \]

\[ = 0(1)n^{s+1}. \] (q.e.d.)

**Lemma 3.** We have

(i) \( \langle K_{ni}(\theta)^{2m} \rangle_{\theta} = 0 \) if \( j \neq i, \ s \leq 2m - 1, \)
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(ii) \( \text{sign} \{(K_n(\theta))^2\}_{[s,t]}^{(s)} = \begin{cases} (-1)^{t} & \text{if } s = 2t, \\ 0 & \text{if } s = 2t - 1, \ t = 1,2,\ldots, \end{cases} \)

and

\[ \text{sign} \{(K_n(\theta))^2\}_{[s,t]}^{(s)} = \begin{cases} 0\{1\}^{2m+8} & \text{if } s = 2t, \\ 0 & \text{if } s = 2t - 1, \ t = 1,2,\ldots. \end{cases} \]

Proof. We use Lemma 2. (i) is trivial.

(ii) \( \text{sign} \{(K_n(\theta))^2\}_{[s,t]}^{(s)} \)

\[ = \text{sign} \sum_{r=0}^{S} \sum_{i=0}^{(s-i)} \{K_n(\theta)\}_{i-\delta}^{(s-i)} \{K_n(\theta)\}_{\delta-\delta}^{(s)} \]

\[ = \text{sign} \sum_{r=2q}^{S} \sum_{2q}^{(s-2q)} \{K_n(\theta)\}_{i-\delta}^{(s-2q)} \{K_n(\theta)\}_{\delta-\delta}^{(s)} \]

\[ = \begin{cases} (-1)^{t} & \text{if } s = 2t, \\ 0 & \text{if } s = 2t - 1, \ t = 1,2,\ldots, \end{cases} \]

because

\[ \text{sign} \{(K_n(\theta))^2\}_{[s,t]}^{(s-2q)} \{K_n(\theta)\}_{[s,t]}^{(s-2q)} \]

\[ = (-1)^{s-2q}(-1)^{s} \]

\[ = (-1)^{t}. \]

By induction,

\[ \text{sign} \{(K_n(\theta))^2\}_{[s,t]}^{(s+1)} \]

\[ = \text{sign} \sum_{r=0}^{S} \sum_{i=0}^{(s-i)} \{K_n(\theta)\}_{i-\delta}^{(s-i)} \{K_n(\theta)\}_{i-\delta}^{(s)} \]

\[ = \text{sign} \sum_{r=2q}^{S} \sum_{2q}^{(s-2q)} \{K_n(\theta)\}_{i-\delta}^{(s-2q)} \{K_n(\theta)\}_{i-\delta}^{(s)} \]

\[ = \begin{cases} (-1)^{t} & \text{if } s = 2t, \\ 0 & \text{if } s = 2t - 1, \ t = 1,2,\ldots, \end{cases} \]

because

\[ \text{sign} \{(K_n(\theta))^2\}_{[s,t]}^{(s-2q)} \{K_n(\theta)\}_{[s,t]}^{(s-2q)} \]

\[ = (-1)^{s-2q}(-1)^{s} \]

\[ = (-1)^{t}. \]
The next lemma is important for the proof of our theorem.

**Lemma 4.** When each \( m, n \) and \( i \) is fixed, \( nH_{ni}(\theta) \) is decided uniquely. If we fix \( m \) and \( i \) we have

(i) \( A(n,0) = n^{-2m} \)

generally,

(ii) \( |A(n,k)| = 0(1)n^{-2m+2k}, \ k = 1, 2, ..., m-1. \)

Proof. We use Lemma 3, and show Lemma 4 by induction with respect to \( k \). By \( H_i(\theta_j) = 1 \) we have \( A(n,0) = n^{-2m} \). Then we see

\[
H_i(\theta_j) = \delta_{ij}, H_{i}^{[s]}(\theta_j) = 0, \ i,j = 1, 2, ..., n.
\]

Now, we assume \( |A(n,k)| = 0(1)n^{-2m+2k} \) for each \( k \leq s-1 \), then we have

\[
0 = H_{i}^{[2s]}(\theta_i)
\]

\[
= \sum_{r=0}^{2s} \binom{2s}{r}^{s} \sum_{k=0}^{m-1} A(n,k) \sin^{2k}(\theta - \theta_i)/2 \right)^{(s-r)}_{\theta - \theta_i} |(K_{ni}(\theta))^{2m}_{\theta = \theta_i}
\]

\[
= A(n,s) \sin^{2s}(\theta - \theta_i)/2 \right)^{(2s)}_{\theta = \theta_i} |(K_{ni}(\theta))^{2m}_{\theta = \theta_i}
\]
Here, 

$$|\Sigma| = 0(1) \sum_{k=0}^{s-1} n^{-2m+2k} \sum_{q=\frac{k}{2}}^{2q} = 0(1) \sum_{k=0}^{s-1} n^{2(k-q)}$$

Thus we have 

$$|A(n,s)| = 0(1)n^{-2m+2s},$$

and furthermore we see 

$$H^{(2s+r)}(\theta_j) = 0, \quad r=0,1, \quad j=1,2,...,n.$$

Consequently, we obtain a $H_t$ satisfying (i) and (ii), and we see that $H_t$ satisfies the condition (2), Hermite condition so called, thus $H_t$ decided uniquely. \hspace{1cm} (q.e.d.)

**Lemma 5.** We have $\sum_{i=1}^{n} mH_{n}(\theta)=1.$

Proof. $T(\theta) = \sum_{i=1}^{n} H_{i}(\theta)$ is a trigonometric polynomial of degree $mn-1$, and it satisfies the Hermite condition;

$$T(\theta_i) = 1, \quad T^{(s)}(\theta_i) = 0, \quad i=1,2,...,n, \quad s=1,2,...,2m-1.$$

Thus $T(\theta)$ is decided uniquely like $T(\theta) = 1.$ \hspace{1cm} (q.e.d.)

For simplicity we use the following definitions: For $\theta, \theta_0 \in [0,2\pi)$ we define $\theta(i)$ such that $\theta(i) \in \{\theta, \theta \pm 2\pi\}$ and

$$|\theta(i) - \theta_0| = \min\{|\theta - \theta_0|, |\theta \pm 2\pi - \theta_0|\}.$$

Furthermore we define the number $\nu = \nu(\theta)$ by

$$|\theta - \theta_0| = \min\{|\theta - \theta_i| : i=1,2,...,n\},$$
and for $\theta \in [0, 2\pi]$, $i \in \{1, 2, \ldots, n\}$ we define

\[
\theta(i) = \begin{cases} 
\theta_i & \text{if } \theta(i) = \theta, \\
\theta_i + 2\pi & \text{if } \theta(i) = \theta + 2\pi, \\
\theta_i - 2\pi & \text{if } \theta(i) = \theta - 2\pi.
\end{cases}
\]

The following lemma is trivial.

**Lemma 6.** For each $\theta \in [0, 2\pi)$, we have $|\theta(i) - \theta| \leq \pi$, and

\[
\left(\frac{1}{\pi}\right)^{1/2} |\theta(i) - \theta| \leq \sin((\theta - \theta_i)/2) \leq \frac{1}{2} |\theta(i) - \theta|.
\]

The next is our main lemma.

**Lemma 7.** Let $\theta \in [0, 2\pi)$.

(i) If $(2j)\pi/n = |\theta(i) - \theta| \neq 0$ we have

\[
|H_i(\theta)| = O(1)(C_n(\theta)/j^2),
\]

and

\[
|(\theta(i) - \theta_i)H_i(\theta)| = O(1)(C_n(\theta)/nj).
\]

(ii) If $(2j)\pi/n = |\theta(i) - \theta| = |\theta_i - \theta| = 0$ we have

\[
|f(\theta) - f(\theta_i)| = O(1)\mathcal{O}(\|C_n(\theta)\|/n)
\]

and

\[
|(\theta - \theta_i)H_i(\theta)| = O(1)(\|C_n(\theta)\|/n).
\]

**Proof.** We use the lemmas mentioned above.

(i) $|A(n,k)\sin^{2k}((\theta - \theta_i)/2)(C_n(\theta)/\sin((\theta - \theta_i)/2))^{2m}|$

\[
= O(1)n^{2m - 2k}C_n(\theta)(n/j)^{2m - 2k}
\]

\[
= O(1)(C_n(\theta)/j^2).
\]

\[
|(\theta(i) - \theta_i)|A(n,k)\sin^{2k}((\theta - \theta_i)/2)(C_n(\theta)/\sin((\theta - \theta_i)/2))^{2m}|
\]

\[
= O(1)n^{2m - 2k}C_n(\theta)(n/j)^{2m - 2k - 1}
\]
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\[ = 0(1)(C^2_n(\theta)/nf). \]

(ii) First, we see by [3, Lemma 3]

\[ |\theta - \theta_v| \leq (\pi/n) |\cos((n\theta)/2)|. \]

Thus we have

\[ |f(\theta) - f(\theta_v)| = 0(1)Q(|\theta - \theta_v|) \]

\[ = 0(1)Q(\{C_n(\theta)/n\}). \]

Furthermore we see

\[ |(\theta - \theta_v)A(n,k)\sin^{2k}((\theta - \theta_v)/2)/C_n(\theta)/\sin((\theta - \theta_v)/2))^{2m}| \]

\[ = 0(1)n^{-2m-2k}C_n(\theta)^{2m} - 2k - 1 \]

\[ = 0(1)|C_n(\theta)|/n. \] (q.e.d.)

The following is known by R.B. Saxena [2].

**Lemma 8 ([2])**. We have

\[ \sum_{j=1}^{n}(1/j^2)Q(j/n) \leq (16/n)\sum_{i=1}^{n}Q(1/i). \]

§ 3. Proof of the theorem

\[ |f(\theta) - T_{m(n+1)-2}(f,\lambda; \theta)| \]

\[ = 0(1)\sum_{i=1}^{n} |f(\theta) - f(\theta(i))H_i(\theta)| + |f(\theta) - f(\theta_v))H_v(\theta)| \]

\[ + \sum_{i=1}^{n} |\theta(i) - \theta_v)H_i(\theta)| \]

\[ \equiv \Sigma_1 + \Sigma_2 + \Sigma_3. \]

Now we use Lemma 7.

\[ \Sigma_1 = 0(1)\sum_{i=1}^{n} Q(|\theta(i) - \theta_v|)H_i(\theta) | \]

\[ = 0(1)\sum_{j=1}^{n} Q(j/n)(C^2_n(\theta)/j^2) \]

\[ = 0(1)C^2_n(\theta)\sum_{j=1}^{n}(1/j^2)Q(j/n) \]

\[ = 0(1)/n. \]
\[ =0(1)C_n^2(\theta)/n \sum_{j=1}^{n} \mathcal{O}(1/j), \]
because of Lemma 8.

\[ \Sigma_2 = 0(1)\mathcal{O}(|C_n(\theta)|/n) \]
and

\[ \Sigma_3 = 0(1)\{|C_n(\theta)|/n \{|C_n(\theta)| \log n + 1 \}. \]

Thus we have the theorem. \hfill (q.e.d.)

References