Remark on classical logic and intuitionistic logic

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In a former paper [2], it was proved that if a formula $A$ is provable in the classical predicate logic $LK$ then $\neg\neg A^*$ (strong double negation of $A$) is provable in the intuitionistic predicate logic $LJ$ where $A^*$ is the formula obtained from $A$ substituing $\forall x \neg\neg B$ and $\exists y \neg\neg B$ for $\forall x B$ and $\exists y B$ respectively in a formula $A$ at all places where $\forall x$ or $\exists y$ appears.

In this paper, we shall prove precisely that for any formula $A$, $A$ is provable in the classical logic if and only if $A^\circ$ is provable in the intuitionistic logic.

$A^\circ$ is the formula obtained from $A$ replacing $\neg\neg B$, $7 (7B \land 7C)$ and $\forall x \neg\neg 7B$ for $\forall x B$, $B \lor C$ and $\exists y B$ respectively in a formula $A$. Thus $A^\circ$ does not contain logical symbol $\lor$ or $\exists$.

As to most of the notions and notations we refer to [1] throughout this paper.

Definition For every formula (in LK) $A$, $A^\circ$ is defined recursively as follows:

1. If $A$ is prime, then $A^\circ$ is $\neg\neg A$.
2. If $A$ is of the forms $\neg B$, $B \land C$, $B \lor C$, or $B \supset C$, then $A^\circ$ is $\neg B^\circ$, $B^\circ \land C^\circ$, $\neg(7B^\circ \land 7C^\circ)$ or $B^\circ \supset C^\circ$ respectively.
3. If $A$ is of the form $\forall x B(x)$ or $\exists y B(y)$, then $A^\circ$ is $\forall x B^\circ(x)$ or $\forall y B^\circ(y)$ respectively.

Lemma 1 For any formula $A$, the sequent $\neg\neg A^\circ \rightarrow A^\circ$ is provable in the intuitionistic logic $LJ$.

Proof. We prove this lemma by the induction on the number of logical symbols in $A$. As other cases are easy, we shall treat only the following cases (1) $\sim$ (5).

1. $A$ is prime: $A^\circ$ is $\neg\neg A$ and $\neg\neg\neg\neg A \rightarrow \neg\neg A$ is clearly provable in the intuitionistic logic.
2. $A$ is $\neg B$: $A^\circ$ is $\neg B^\circ$ and $\neg\neg\neg\neg B^\circ \rightarrow \neg B^\circ$ is clearly provable in the intuitionistic logic.
3. $A$ is $B \land C$: $A^\circ$ is $B^\circ \land C^\circ$. By the induction hypothesis, the following two sequents are provable in the intuitionistic logic:

\[
\begin{align*}
\neg\neg B^\circ & \rightarrow B^\circ \\
\neg\neg C^\circ & \rightarrow C^\circ
\end{align*}
\]
We can then prove the lemma as follows:

<table>
<thead>
<tr>
<th>P₁</th>
<th>P₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>77(B° ⊃ C°) → B° ⊃ C°</td>
<td></td>
</tr>
</tbody>
</table>

Here P₁ and P₂ are

\[
\begin{align*}
B° & \rightarrow B° \\
B° \land C° & \rightarrow B° \\
\text{some negations} & \quad \\
77(B° \land C°) & \rightarrow 77B° \\
77B° & \rightarrow B° \\
\end{align*}
\]

and

\[
\begin{align*}
C° & \rightarrow C° \\
B° \land C° & \rightarrow C° \\
\text{some negations} & \quad \\
77(B° \land C°) & \rightarrow 77C° \\
77C° & \rightarrow C° \\
\end{align*}
\]

respectively.

(4) \( A \) is \( B \supset C \):

By the induction hypothesis, \( 77C° \rightarrow C° \) is provable in the intuitionistic logic. We can see the sequent

\[
77(B° \supset C°) \rightarrow B° \supset C°
\]

is provable in the intuitionistic logic as follows:

\[
\begin{align*}
C° & \rightarrow C° \\
\quad & \quad \\
B° & \rightarrow B° \\
\quad & \quad \\
B° \supset C°, 7C°, B° & \rightarrow \\
7C°, 7C°, B° & \rightarrow 7(B° \supset C°) \\
7C°, B°, 77(B° \supset C°) & \rightarrow \\
77(B° \supset C°) & \rightarrow 77C° \\
77C° & \rightarrow C° \\
\end{align*}
\]

(5) \( A \) is \( \forall x B(x) \):

By induction hypothesis, \( 77B°(a) \rightarrow B°(a) \) is provable in the intuitionistic logic. We can see the sequent \( 77\forall x B°(x) \rightarrow \forall x B°(x) \) is provable in the intuitionistic logic as follows:
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\[
\begin{array}{c}
\frac{B^o(a) \rightarrow B^o(a)}{
\forall x B^o(x) \rightarrow B^o(a)}
\end{array}
\]

\[
\frac{\forall x B^o(x) \rightarrow B^o(a)}{\forall x B^o(x) \rightarrow B^o(a)}
\]

Lemma 2 If the sequent

\[A_1, A_2, ..., A_m \rightarrow B_1, B_2, ..., B_n\]

is provable in LK, then the sequent

\[A_1^o, A_2^o, ..., A_m^o, \neg B_1^o, \neg B_2^o, ..., \neg B_n^o \rightarrow\]

is provable in LJ.

Proof. We prove the lemma by the induction on the number of inference-figures to S, where S is \(A_1, ..., A_m \rightarrow B_1, ..., B_n\). As other cases are easy or similar, we shall treat only the following some cases:

1) S is an initial sequent:
   In this case, the lemma is trivial.

2) S is the lower sequent of an inference I:
   In this case, we can assume I is not 'cut' by Gentzen's Hauptsatz [1].

We shall use often the following notations: If \(\Gamma\) is the series of formulae \(C_1, C_2, ..., C_n\), then

\[\Gamma^o\] means \(C_1^o, C_2^o, ..., C_n^o\]

and

\[\neg \Gamma^o\] means \(\neg C_1^o, \neg C_2^o, ..., \neg C_n^o\).

(i) I is \(7\)-right:
   Let the last inference-figure to \(A_1, ..., A_m \rightarrow B_1, ..., B_n\) be of the form

\[
\begin{array}{c}
D, \Gamma \rightarrow \Delta
\end{array}
\]

\[
\frac{\Gamma \rightarrow \Delta, 7D}{\Gamma \rightarrow 7D}
\]

Here \(\Gamma\) is \(A_1, ..., A_m\) and \(7D, \Delta\) is \(B_1, ..., B_n\).

By the induction hypothesis,

\[D^o, \Gamma^o, 7\Delta^o \rightarrow\]

is provable in LJ. By the following proof-figure, we can see the sequent

\[A_1^o, ..., A_m^o, \neg B_1^o, ..., \neg B_n^o \rightarrow\]

is provable in LJ:
(ii) \( I \) is \( \land \)-right:
Let the last inference-figure be of the form

\[
\frac{\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \land B}
\]

By the induction hypothesis, the following two sequents are provable in the minimal logic:

\[
\Gamma^\circ, 7\Delta^\circ, 7A^\circ \rightarrow \\
\Gamma^\circ, 7\Delta^\circ, 7B^\circ \rightarrow
\]

Moreover, by Lemma 1, the following two sequents are provable in LJ:

\[
77A^\circ \rightarrow A^\circ \\
77B^\circ \rightarrow B^\circ
\]

By the following proof-figure, we can see the sequent

\[
\Gamma^\circ, 7\Delta^\circ, 7(A^\circ \land B^\circ) \rightarrow
\]

is provable in LJ:

\[
\frac{P_1 \quad P_2}{\Gamma^\circ, 7\Delta^\circ \rightarrow A^\circ \land B^\circ} \\
\frac{7(A^\circ \land B^\circ), \Gamma^\circ, 7\Delta^\circ \rightarrow}{\Gamma^\circ, 7\Delta^\circ, 7(A^\circ \land B^\circ) \rightarrow}
\]

Here \( P_1 \) and \( P_2 \) are

\[
\frac{\Gamma^\circ, 7\Delta^\circ, 7A^\circ \rightarrow}{\Gamma^\circ, 7\Delta^\circ, 7A^\circ \rightarrow 77A^\circ} \\
\frac{77A^\circ \rightarrow A^\circ}{\Gamma^\circ, 7\Delta^\circ \rightarrow A^\circ}
\]

and
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\[
\begin{array}{c}
\Gamma^\circ, 7\Delta^\circ, 7B^\circ \\
\text{some interchanges and negation}
\end{array}
\]

\[
\begin{array}{c}
\Gamma^\circ, 7\Delta^\circ \rightarrow 7\overline{7}B^\circ \\
\Gamma^\circ, 7\Delta^\circ \rightarrow B^\circ
\end{array}
\]

respectively.

(iii) \( \vdash \)-right:

Let the last inference-figure be of the form

\[
\frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B}
\]

By induction hypothesis, the sequent

\[
A^\circ, \Gamma^\circ, 7\Delta^\circ, 7B^\circ \rightarrow
\]

is provable in LJ. Moreover, by Lemma 1, so is the sequent \( 77A^\circ \rightarrow A^\circ \). Hence we can see the sequent

\[
\Gamma^\circ, 7\Delta^\circ, 7(A^\circ \supset B^\circ) \rightarrow
\]

is provable in LJ as follows:

\[
\begin{array}{c}
B^\circ \rightarrow B^\circ \\
A^\circ, B^\circ \rightarrow B^\circ \\
B^\circ \rightarrow A^\circ \supset B^\circ \\
7(A^\circ \supset B^\circ), B^\circ \rightarrow \\
B^\circ, 7(A^\circ \supset B^\circ) \rightarrow \\
7(A^\circ \supset B^\circ), 7B^\circ \rightarrow
\end{array}
\]

\[
\begin{array}{c}
\text{P} \\
A^\circ, \Gamma^\circ, 7\Delta^\circ, 7B^\circ \rightarrow \\
7(A^\circ \supset B^\circ), \Gamma^\circ, 7\Delta^\circ \rightarrow \\
7(A^\circ \supset B^\circ), 7(A^\circ \supset B^\circ), \Gamma^\circ, \Delta^\circ \rightarrow \\
\Gamma^\circ, 7\Delta^\circ, 7(A^\circ \supset B^\circ) \rightarrow
\end{array}
\]

Here \( \text{P} \) is the following subproof:

\[
\begin{array}{c}
A^\circ \rightarrow A^\circ \\
7A^\circ, A^\circ \rightarrow (\text{weakening-right}) \\
A^\circ, 7A^\circ \rightarrow B^\circ \\
7A^\circ \rightarrow A^\circ \supset B^\circ \\
7(A^\circ \supset B^\circ), 7A^\circ \rightarrow \\
7A^\circ, 7(A^\circ \supset B^\circ) \rightarrow \\
7(A^\circ \supset B^\circ) \rightarrow 77A^\circ \\
77A^\circ \rightarrow A^\circ
\end{array}
\]

(iv) \( \top \)-right:

Let the last inference be of the form

\[
\frac{\Gamma \rightarrow \Delta, F(a)}{\Gamma \rightarrow \Delta, \forall x F(x)}
\]
By the induction hypothesis, the sequent

$$\Gamma^\circ, 7\Delta^\circ, 7F^\circ(a) \rightarrow$$

is provable in LJ. Then we can see the sequent

$$\Gamma^\circ, 7\Delta^\circ, 7\forall xF^\circ(x) \rightarrow$$
is provable in LJ as follows:

$$\frac{\Gamma^\circ, 7\Delta^\circ, 7F^\circ(a)}{7F^\circ(a), \Gamma^\circ, 7\Delta^\circ \rightarrow 7\forall xF^\circ(x)}$$

$$\frac{\Gamma^\circ, 7\Delta^\circ \rightarrow F^\circ(a)}{\forall xF^\circ(x), \Gamma^\circ, 7\Delta^\circ \rightarrow}$$

(v) \( I \) is \( \exists \)-right:
Let the last inference-figure be of the form

$$\frac{\Gamma \rightarrow \Delta, F(t)}{\Gamma \rightarrow \Delta, \exists xF(x)}$$

By the induction hypothesis, the sequent

$$\Gamma^\circ, 7\Delta^\circ, 7F^\circ(t) \rightarrow$$
is provable in LJ. Then we can construct a proof as follows:

$$\frac{F^\circ(t) \rightarrow F^\circ(t)}{7F^\circ(t), F^\circ(t) \rightarrow}$$

$$\frac{\forall x7F^\circ(x), F^\circ(t) \rightarrow}{F^\circ(t) \rightarrow 7\forall x7F^\circ(x)}$$

$$\frac{7\forall x7F^\circ(x), F^\circ(t) \rightarrow}{7\forall x7F^\circ(x) \rightarrow 7F^\circ(t)}$$

$$\frac{7\forall x7F^\circ(x) \rightarrow 7F^\circ(t)}{\Gamma^\circ, 7\Delta^\circ, 7F^\circ(t) \rightarrow}$$

\textbf{Theorem}  If a formula \( A \) is provable in LK, then \( A^\circ \) is provable in LJ.

\textbf{Proof:}  By the assumption of the Theorem and Lemma 2, the sequent

$$7A^\circ \rightarrow$$
is provable in LJ. Moreover the sequent \( 77A^\circ \rightarrow A^\circ \) is provable in LJ. Then we can see \( A^\circ \) is provable in LJ as follows:
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\[
\begin{align*}
7A^\circ & \rightarrow \\
77A^\circ & \rightarrow A^\circ \\
777A^\circ & \rightarrow A^\circ
\end{align*}
\]

References

(2) H. Kodera: On Glivenko's Theorem in the first order predicate calculus of Gentzen style, Bulletin
of Aichi University of Education Vol 27. 43 ~ 47, 1978.