Remarks on some homological results of certain rings

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All rings are commutative with identity and all modules are unitary. Let $R$ be a ring and $R[X]$ be the polynomial ring. If $M$ is a $R$-module then we let $hd_R(M)$ denote the homological dimension of $M$.

The purpose of this note is to answer the following question: Which rings $R$ have the property that $R[X]/P$ is $R$-projective ($R$-flat or $hd(R[X]/P) \leq 1$) for each prime (or maximal) ideal of $R$?

We start with the following results.

**Theorem 1.** Let $R$ be a commutative ring. Then the following statements are equivalent.

1. $R[X]/P$ is a projective $R$-module for each prime ideal $P$ of $R[X]$.
2. $R[[X]]/P$ is a projective $R$-module for each prime ideal $P$ of $R[[X]]$.
3. $R[X]/M$ is a projective $R$-module for each maximal ideal $M$ of $R[X]$.
4. $R[[X]]/M$ is a projective $R$-module for each maximal ideal $M$ of $R[[X]]$.
5. $R/P$ is a projective $R$-module for each prime ideal $P$ of $R$.
6. $R/M$ is a projective $R$-module for each maximal ideal $M$ of $R$.
7. $R$ is a direct sum of a finite number of fields, that is, $gl. \dim(R) = 0$.
8. $R$ is an Artinian reduced ring.
9. $R$ is a von Neumann regular ring with Noetherian spectrum.

[Proof].

$(1) \rightarrow (5)$.

Let $P$ be any prime ideal of $R$ and set $P* = P + XR[X]$. Then $R[X]/P* \cong R/P$ which is an integral domain. So $P*$ is a prime ideal of $R[X]$. Therefore, by the assumption, $R/P$ is $R$-projective.

$(2) \rightarrow (5)$. The same proof as proof that $(1)$ implies $(5)$.

$(7) \rightarrow (1), (2)$, $(2) \rightarrow (4) \rightarrow (6)$, $(1) \rightarrow (3)$, $(5) \rightarrow (6)$, $(7) \rightarrow (8) \rightarrow (9)$ and $(3) \rightarrow (6)$. Trivial.

$(6) \rightarrow (7)$.

Let $M$ be any maximal ideal of $R$ and set $A = R_M$, $N = MR_M$. Then $A/N \cong R/M$ which is $R$-projective. Therefore
is a split exact sequence of $R$-modules. Since $A$ is a connected ring, $A$ is a field. Consequently $\dim R = 0$.

Since $R/M$ is finitely generated $R$-projective and finitely presented, $M$ is a finitely generated ideal. As $\dim R = 0$, $R$ is Noetherian by Cohen's theorem. Since $R_M/\mathfrak{M}_M \cong R/\mathfrak{M}$ is $R_M$-projective, $R_M$ is a zero-dimensional regular local ring, that is, a field. Consequently, $R$ is a von Neumann (Noetherian) regular ring. Hence $R$ is a semi-simple ring.

(9) $\rightarrow$ (8).

Since $\text{Spec}(R)$ is a Noetherian $T_1$-space, it is a finite set. Let $\text{Spec}(R) = \{M_1, M_2, \ldots, M_n\}.$

Thus

$$0 = \text{rad}(R) = M_1 \cap M_2 \cap \cdots \cap M_n = M_1M_2 \cdots M_n.$$ 

By structure theorem for Artinian rings, $R$ is a direct sum of a finite number of Artinian local rings. Since $R$ is reduced, $R$ is a direct sum of a finite number of fields.

We now give a characterization of von Neumann regular rings.

**Theorem 2.** Let $R$ be a commutative ring. Then the following statements are equivalent.

(1). $R[X]/P$ is a flat $R$-module for each prime ideal $P$ of $R[X].$

(2). $R[[X]]/P$ is a flat $R$-module for each prime ideal $P$ of $R[[X]].$

(3). $R[X]/M$ is a flat $R$-module for each maximal ideal $M$ of $R[X].$

(4). $R[[X]]/M$ is a flat $R$-module for each maximal ideal $M$ of $R[[X]].$

(5). $R/P$ is a flat $R$-module for each prime ideal $P$ of $R.$

(6). $R/M$ is a flat $R$-module for each maximal ideal $M$ of $R.$

(7). $R$ is a von Neumann regular ring.

(8). $R[X]$ is a semi-hereditary ring.

(9). $R$ is a reduced ring and $\text{Spec}(R)$ is a $T_1$-space.

[Proof]. (7) $\rightarrow$ (1), (2), (1) $\rightarrow$ (3) $\rightarrow$ (6), (2) $\rightarrow$ (4) $\rightarrow$ (6), (1) $\rightarrow$ (5) $\rightarrow$ (6). Trivial.

(6) $\rightarrow$ (7).

Let $M$ be any maximal ideal of $R.$ Then $R/M \cong R_M/\mathfrak{M}_M$ is $R_M$-flat. So $R/M$ is $R_M$-free. And so $R_M$ is a field. Thus $R$ is a von Neumann regular ring.

(7) $\leftrightarrow$ (8) follows from Theorem of [1].

That (7) is equivalent to (9) is due to Kaplansky.

Next we consider locally regular rings.

We need the following remark and definition.
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Remark. If $\text{hd}_R(R[X]/P^*) \leq 1$ for each prime ideal $P^*$ of $R[X]$, then $\text{hd}_R(R/P) \leq 1$ for each prime ideal $P$ of $R$.

Definition. A locally regular ring is a (not necessarily Noetherian) ring $R$, having the property that $R_P$ is a regular local ring for each prime ideal $P$ of $R$.

Theorem 3. If $\text{hd}_R(R/P) \leq 1$ for each prime ideal $P$ of $R$, then $R$ is a locally regular ring of Krull dimension $\leq 1$.

[Proof]. Let $M$ be a maximal ideal of $R$. Then we have an exact sequence

$$0 \rightarrow PR_M \rightarrow R_M \rightarrow R_M/PR_M \rightarrow 0$$

for each prime ideal $P$ contained in $M$.

By the assumption, $PR_M$ is $R_M$-free and so $PR_M = 0$ or $R_M$. Since $PR_M$ is a finitely generated ideal, $R_M$ is a Noetherian ring by Cohen's theorem. Since $\text{hd}_R(R_M/(MR_M)) \leq 1$, $R_M$ is a regular local ring of Krull dimension $\leq 1$.

Corollary. For any Noetherian ring $R$, the following statements are equivalent.

1. $\text{hd}_R(R[X]/P^*) \leq 1$ for each prime ideal $P$ of $R[X]$.
2. $\text{hd}_R(R[[X]]/P^*) \leq 1$ for each prime ideal $P$ of $R[[X]]$.
3. $\text{hd}_R(R/P) \leq 1$ for each prime ideal $P$ of $R$.
4. $R$ is a regular ring of Krull dimension $\leq 1$, that is, $R$ is a direct sum of a finite number of fields and Dedekind domains.

If we replace "fields" by "special primary rings" in this result, it is not true in general.

By Theorem 2.3 of [3] and Theorem 9.10 of [2], we have the following results.

For any ring $R$, the following statements are equivalent.

(a). $R$ is a direct sum of a finite number of Dedekind domains and special primary rings.

(b). Each ideal of $R$ can be represented as a finite product of prime ideals.

(c). Each finitely generated ideal of $R$ can be represented as a finite product of prime ideals.

Now, let $R$ be a special primary ring with the unique maximal ideal $M$. Then $\text{gl.dim } R = 0$ or $\infty$. Since $\text{gl.dim } R = \text{hd}_R(R/M)$, $\text{hd}_R(R/M) = 0$ or $\infty$.

Finally, we give the following.

Remark. Is the converse of Theorem 3 true? If $R$ has the property "$M_P$ is $R_P$-projective for each prime ideal $P$ of $R$ implies $M$ is $R$-projective", then the converse of Theorem 3 is true. In particular, if $R$ is a Noetherian or local ring, then it is true.

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References.
