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On Ulam's Floating Body Problem of Two Dimension

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1 Introduction

S. M. Ulam posed a problem: *If a body of uniform density floats in water in equilibrium in every direction, must it be a sphere?* See [3] for detail. The problem is still open. However, in two dimensional case of the problem, Auerbach [1] gives a counter-example.

Theorem 1. ([1]) *There is a non-circular figure $D \subset \mathbb{R}^2$ of density $\rho = 1/2$ which floats in equilibrium in every direction.*

Before we state our result, we define some terminology of two-dimensional floating bodies. Consider a figure $D \subset \mathbb{R}^2$ whose perimeter ∂D is a simple closed curve, and take a number $0 < \rho < 1$. For a given angle $0 \leqq \theta \leqq 2\pi$, there is a directed line L_θ of slope angle θ which divides the area of D in the ratio $\rho : 1-\rho$. In this paper, we assume the following three conditions:

(C1) ∂D is of class C^1.
(C2) L_θ meets ∂D at exactly two points, say, P and Q.
(C3) Neither the tangent at P nor at Q is not parallel to the line PQ.

We call ρ the *density* of D, and the segment PQ the *water line* of slope angle θ. We denote by D_u and D_a the divided figures of area ratio $\rho : 1-\rho$. We call D_u and D_a the *underwater and abovewater* parts of D, respectively. We denote by G_u and G_a the centroids of D_u and D_a, respectively. We say that D *floats in equilibrium* in direction $\mathbf{e}_2(\theta) = (-\sin\theta, \cos\theta)$ if the line G_uG_a is parallel to $\mathbf{e}_2(\theta)$.

If the figure D of density ρ floats in equilibrium in every direction, we call $D \subset \mathbb{R}^2$ an *Auerbach figure* of an *Auerbach density* ρ. It is known that, if $D \subset \mathbb{R}^2$ is an Auerbach figure, then the water surface divides ∂D in constant ratio, say, $\sigma : 1-\sigma$. See (ii) of Corollary 7. We call σ the *perimetral density* of the Auerbach figure D.

If D is an Auerbach figure of density $\rho = 1/2$, then the water lines L_θ and $L_{\theta+\pi}$ are the same but opposite directed lines. Thus it is of perimetral density $\sigma = 1/2$. In the proof of Theorem 1, the condition $\rho = 1/2$ is essential. It is dificult to make an Auerbach figures of density $\rho \neq 1/2$. So a question arises: *Is there a non-circular Auerbach figure of density $\rho \neq 1/2$?*

Recently, Wegner [7] gave a positive answer to this question. Wegner's examples exhibit more interesting fact. That is, for given integer $p \geqq 3$, one of his examples has $(p-2)$ different Auerbach densities. So one Auerbach figure can have many perimetral densities.

On the other hand, Bracho, Montejano and Oliberos [2] gave a following result.

Theorem 2. ([2]) *If there is an Auerbach figure $D \subset \mathbb{R}_2$ of perimetral density $\sigma = 1/3$ or $1/4$, then it is a circle.*

The purpose of this paper is to prove the following theorem.

Theorem 3. (1) *If an Auerbach figure $D \subset \mathbb{R}^2$ has three perimetral densities σ_1, σ_2 and σ_3, and if $\sigma_1 + \sigma_2 + \sigma_3 = 1$, then it is a circle. (These σ_i's are not necessarily different.)*

(2) *If an Auerbach figure $D \subset \mathbb{R}^2$ has four perimetral densities σ_1, σ_2, σ_3 and σ_4, and if $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 1$, then it is a circle. (These σ_i's are not necessarily different.)*

The above theorem is a generalization of Theorem 2. Certainly, putting $\sigma_1 = \sigma_2 = \sigma_3 = 1/3$ gives the $1/3$ case of Theorem 2, and putting $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1/4$ gives the $1/4$ case of Theorem 2.

2 Auerbach Figures

In this section, we give a short survey of Auerbach figures.

Theorem 4. ([1], [7]) *If a figure $D \subset \mathbb{R}^2$ is Auerbach, then the water line is of constant length.*

Theorem 5. *If a figure $D \subset \mathbb{R}^2$ is Auerbach, and if PQ is the water line of slope angle θ, then there is a 2π-periodic function f of class C^2 such that the position vectors of P and Q are given by*

$$\mathbf{p}(\theta) = f(\theta)\,\mathbf{e}_2(\theta) + (f'(\theta) - 1)\,\mathbf{e}_1(\theta); \qquad \mathbf{q}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + (f'(\theta) + l)\,\mathbf{e}_1(\theta) \tag{1}$$

where $\mathbf{e}_1(\theta) = (\cos\theta, \cos\theta)$, $\mathbf{e}_2(\theta)\,(-\sin\theta, \cos\theta)$, *and l is half the length of PQ.*

Proof. Assume that D is an Auerbatch figure. Since $\{\mathbf{e}_1(\theta), \mathbf{e}_2(\theta)\}$ is a basis of $\mathbb{R}^2$, we can represent the position vectors of the points P and Q as follows:

$$\mathbf{p}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + g(\theta)\mathbf{e}_1(\theta), \qquad \mathbf{q}(\theta) = -f(\theta)\mathbf{e}_2(\theta) + (g(\theta) + 2l)\,\mathbf{e}_1(\theta). \tag{2}$$

Suppose that the chord P^*Q^* of C is the water line of slope angle $\theta + h$. Then the position vector of the intersection H of the chords PQ and P^*Q^* are given by

$$\overrightarrow{OH} = -f(\theta)\mathbf{e}_2(\theta) + \lambda\mathbf{e}_1(\theta) = -f(\theta+h)\mathbf{e}_2(\theta+h) + \mu\mathbf{e}_1(\theta+h). \tag{3}$$

By taking the inner product of (3) and $\mathbf{e}_2\,(\theta+h)$, we have that $f(\theta+h) = \lambda\sin h + f(\theta)\cos h$. Thus we obtain that

$$f'(\theta) = \frac{f(\theta+h) - f(\theta)}{h} + o(1) = \lambda\frac{\sin h}{h} - f(\theta)\frac{1-\cos h}{h} + o(1) = \lambda + o(1). \tag{4}$$

We can evaluate the areas of the sectors HPP^* and HQQ^* by

$$\frac{1}{2}HP^2h + o(h) = \frac{1}{2}\,|\,g(\theta) - f'(\theta)|^2h + o(h), \qquad \frac{1}{2}HQ^2h + o(h) = \frac{1}{2}\,|\,g(\theta) - f'(\theta) + 2l|^2h + o(h), \tag{5}$$

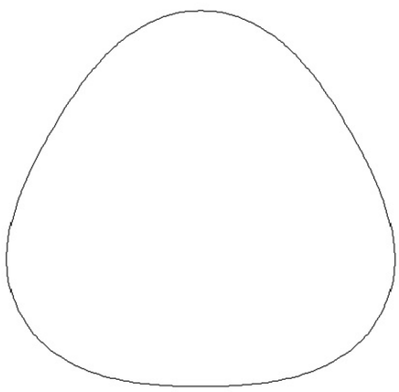
respectively. Since these two areas are equal, we obtain that $g(\theta) = f'(\theta) - l$. Hence we have proved (1). By taking the inner product of (1) and $\mathbf{e}_1(\theta)$, we have that $f'(\theta) = \mathbf{p}(\theta)\cdot\mathbf{e}_1(\theta) + l$. Thus the function $f(\theta)$ is of class C^2. □

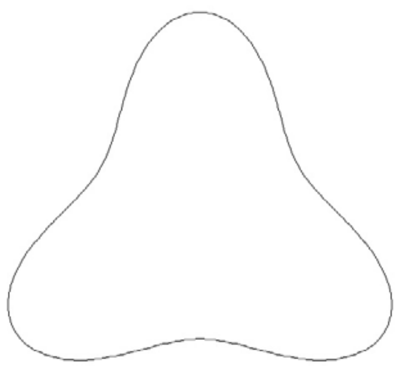
The following result is a "proof" of Theorem 1.

Corollary 6. *If a function f satisfies $f(\theta+\pi) = -f(\theta)$ for every θ, and if the closed curve given by $\mathbf{p}(\theta)$ of Equation (1) is simple, then it surrounds an Auerbach figure of density* $\mathrm{P} = 1/2$.

Proof. Since $\mathbf{p}(\theta+\pi) = \mathbf{q}(\theta)$, two position vectors $\mathbf{p}(\theta)$ and $\mathbf{q}(\theta)$ draw a same closed curve. Thus it surrounds an Auerbach figure. Moreover, if the water line rotates by π, the underwater and abovewater parts change these roles. Thus these areas are equal. Hence we obtain that $\rho = 1/2$. □

Example. Put $f(\theta) = -k\cos 3\theta$ in Equation (1). Then, by Corollary 6, the curve surrounds an Auerbach figures of density $1/2$. The figures of $k/l = 0.03$ and $k/l = 0.1$ are drawn as follows:





The following result gives geometric properties of Auerbach figures.

Corollary 7. *If a figure $D \subset \mathbb{R}^2$ is Auerbach, and if PQ is the water line of slope angle θ, then:*

(i) *The vectors* $\mathbf{p}'(\theta)$ *and* $\mathbf{q}'(\theta)$ *are symmetric with respect to the line* PQ.
(ii) *The arc* PQ of ∂D *is of constant length.*

Proof. By differentiating (1), we have that

$$\mathbf{p}'(\theta) = s(\theta)\, \mathbf{e}_1(\theta) - l\,\mathbf{e}_2(\theta), \qquad \mathbf{q}'(\theta) = s(\theta)\, \mathbf{e}_1(\theta) + l\,\mathbf{e}_2(\theta), \tag{6}$$

where $s(\theta) = f(\theta) + f''(\theta)$. Since the line PQ is parallel to the vector $\mathbf{e}_1(\theta)$, we have proved (i).

(ii) By (6), we have that $|\mathbf{p}'(\theta)| = |\mathbf{q}'(\theta)| = \sqrt{s(\theta)^2 + l^2}$. This implies that the points P and Q move at the same speed along ∂D. Thus we have proved (ii). □

Remark. By integrating (6), we have that

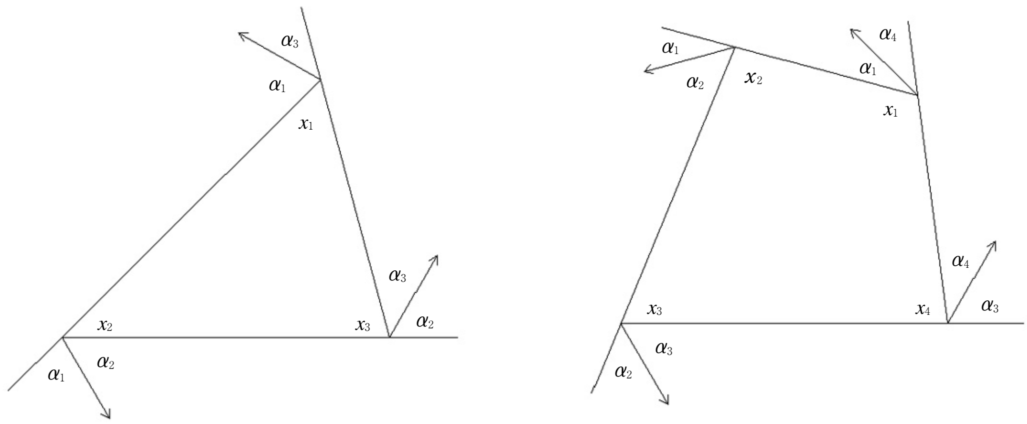
$$\mathbf{p}(\theta) = \mathbf{c} + \int_0^\theta s(\phi)\, \mathbf{e}_1(\phi)\, d\phi - l\,\mathbf{e}_1(\theta), \qquad \mathbf{q}(\theta) = \mathbf{c} + \int_0^\theta s(\phi)\mathbf{e}_1(\phi)\, d\phi + l\,\mathbf{e}_1(\theta) \tag{7}$$

where $\mathbf{c}$ is a constant vector. These formulas are same as those given in Section 2 of [7].

3 Proof of Theorem 3

Proof of Theorem 3. (i) Let P_1, P_2 and P_3 be three points of ∂D such that for each $i = 1, 2, 3$, the line P_iP_{i+1} can be a water surface of perimetral density σ_i. (The indices are taken cyclic in modulo 3.) For each $i = 1, 2, 3$, we denote by $\mathbf{p}_i(\theta)$ the position vector of P_i, by x_i the angle $\angle P_{i-1}\,P_i\,P_{i+1}$ and by α_i the angle between $\mathbf{p}'_i(\theta)$ and P_iP_{i+1}. By (i) of Corollary 7, the angle between $P_{i-1}P_i$ and $\mathbf{p}'_i(\theta)$ is equal to α_i. So we obtain that $x_1 + \alpha_3 + \alpha_1 = \pi$, $x_2 + \alpha_1 + \alpha_2 = \pi$ and $x_3 + \alpha_2 + \alpha_3 = \pi$. Since $x_1 + x_2 + x_3 = \pi$, we have that $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. See Figure 2. So we obtain that $\alpha_1 = x_3$. By the converse of Alternate Segment Theorem, $\mathbf{p}'_1(\theta)$ tangents to the circumcircle of the triangle $P_1\,P_2\,P_3$. Thus P_1 varies on the circumcircle. Hence D is a circle.

(ii) Let P_1, P_2, P_3 and P_4 be four points of ∂D such that for each $i = 1, 2, 3, 4$, the line P_iP_{i+1} can be a water surface of perimetral density σ_i. (The indices are taken cyclic in modulo 4.) By the same notation and argument used in (i) of this theorem, we otain that $x_1 + \alpha_4 + \alpha_1 = \pi$, $x_2 + \alpha_1 + \alpha_2 = \pi$, $x_3 + \alpha_2 + \alpha_3 = \pi$ and $x_4 + \alpha_3 + \alpha_4 = \pi$. See Figure 2. Since $x_1 + x_2 + x_3 + x_4 = 2\pi$, we have that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi$. So we obtain that $x_1 + x_3 = \pi$. By the converse of Inscribed Quadrangle Theorem, the quadrangle $P_1P_2P_3P_4$ inscribes to a circle. Thus P_3P_1 is of constant length, and therefore, it can be a water line of perimetral density $\sigma_3 + \sigma_4$. Hence, by (i) of this theorem, D is a circle. □



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