Flavor Symmetry on Non-Commutative Compact Space and $SU(6) \times SU(2)_R$ Model

Yoshikazu ABE [*], Chuichiro HATTORI[a] [†], Masato ITO[b] [‡],
Masahisa MATSUDA[c] [§], Mamoru MATSUNAGA [¶] and Takeo MATSUOKA[d] [‖]

Department of Physics Engineering, Mie University, Tsu, JAPAN 514-8507
[a]*Science Division, General Education, Aichi Institute of Technology, Toyota, JAPAN 470-0392*
[b]*Department of Physics, Nagoya University, Nagoya, JAPAN 464-8602*
[c]*Department of Physics and Astronomy, Aichi University of Education, Kariya, Aichi, JAPAN 448-8542*
[d]*Kogakkan University, Nabari, JAPAN 518-0498*

Abstract

In the four-dimensional effective theory from string compactification discrete flavor symmetries arise from symmetric structure of the compactified space and generally contain both the R symmetry and non-R symmetry. We point out that a new type of non-Abelian flavor symmetry can also appear if the compact space is non-commutative. Introducing the dihedral group D_4 as such a new type of flavor symmetry together with the R symmetry and non-R symmetry in $SU(6) \times SU(2)_R$ model, we explain not only fermion mass hierarchies but also hierarchical energy scales including the breaking scale of the GUT-type gauge symmetry, intermediate Majorana masses of R-handed neutrinos and the scale of μ-term.

[*]E-mail address: abechan@phen.mie-u.ac.jp

[†]E-mail address: hattori@ge.aitech.ac.jp

[‡]E-mail address: mito@eken.phys.nagoya-u.ac.jp

[§]E-mail address: mmatsuda@auecc.aichi-edu.ac.jp

[¶]E-mail address: matsuna@phen.mie-u.ac.jp

[‖]E-mail address: matsuoka@kogakkan-u.ac.jp

1 Introduction

Quark and lepton masses and mixing angles exhibit apparent characteristic patterns. Many authors have made attempts to explain these characteristic patterns by introducing an appropriate flavor symmetry and by relying on the Froggatt-Nielsen mechanism[1]. As for the flavor symmetry attention has been confined to continuous symmetry such as the gauged $U(1)$. However, in the string theory, which is the only known candidate of the unified theory including gravity, discrete symmetries very likely arise as the flavor symmetry from the symmetric structure of the compactified space. As seen from the geometrical construction of Calabi-Yau space and also from the algebraic construction given by the Gepner model[2], in the string theory it is likely that we have both R and non-R discrete symmetries as the flavor symmetry.

On the other hand, there are several characteristic scales in energy regions ranging from the string scale $M_S(O(10^{18})\text{GeV})$ to the electroweak scale $M_W(O(10^2)\text{GeV})$. First, the breaking scale M_{GUT} of the GUT-type gauge symmetry should be larger than $O(10^{16})\text{GeV}$ to guarantee the longevity of the proton. Second, the seesaw mechanism for neutrinos implies that the Majorana mass scale M_R of R-handed neutrinos is expected to be $O(10^{10\sim 12})\text{GeV}$. Third, in order for the effective theory to be consistent with the standard model, the scale of the μ-term is required to be $O(10^{2\sim 3})\text{GeV}$.

In this paper we concentrate our attension on whether or not these characteristic scale hierarchies are derivable from the flavor symmetry. In our previous work [3], in which we chose a discrete Abelian R symmetry as the flavor symmetry together with $SU(6) \times SU(2)_R$ gauge symmetry, it was shown that the gauge symmetry is spontaneously broken at tree level down to the standard model gauge group $G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$ in two steps. Furthermore, we explained the triplet-doublet splitting as well as fermion mass hierarchies except for neutrinos. However, we could not derive the correct scale of the Majorana mass M_R of R-handed neutrinos. In fact, in the previous paper[3] we obtained the result that M_R is either around colored Higgs mass scale or around the scale of the μ-term. Phenomenologically, the scale M_R is required to be almost equal to the geometrically averaged value between M_S and M_W. In order for us to explain such a scale of M_R, it seemed that we need an additional flavor symmetry.

Recently, extensive studies of string with discrete torsion have been made[4, 5]. The discrete torsion is inherently related to the introduction of nontrivial background

B field. In the presence of background B field coordinates of D-branes become non-commutative operators[6]. The non-commutativity of coordinates is closely linked to quantum fluctuation of compactified space. Due to the quantum fluctuation of the compact space coordinates are described in terms of a projective representation of the discrete symmetry group. In the string theory massless matter fields correspond to the degree of freedom of the deformation of compact space. Various types of the deformation are represented by approriate functions of the coordinates with definite charges under the discrete group. Therefore, massless matter fields are also described in terms of the projective representation. On the other hand, four-dimensional effective Lagrangian of the theory should belong to the center of the non-commutative algebra, which is the subalgebra consisting of commuting elements of the algebra. This means, as we shall see in the following section, that a new type of flavor symmetry also arises in the compact space with non-commutative geometry.

Motivated by phenomenological requirements, in this paper we introduce as the flavor symmetries a discrete non-Abelian symmetry as well as discrete Abelian symmetries. As for Abelian symmetries we choose R-parity and $\mathbf{Z}_3 \times \mathbf{Z}_5$ R symmetry in addition to $\mathbf{Z}_2 \times \mathbf{Z}_7$ non-R symmetry. Further, we introduce the dihedral group D_4 as a non-Abelian symmetry and take its projective representation arising from the non-commutativity of coordinates. Applying this new type of the flavor symmetry to the $SU(6) \times SU(2)_R$ model, we explain not only fermion mass hierarchies but also hierarchical energy scales including the breaking scale of the GUT-type gauge symmetry, intermediate Majorana masses of R-handed neutrinos and the scale of μ-term.

This paper is organized as follows. In section 2 it is pointed out that the string theory naturally provides a discrete symmetry, in which non-Abelian symmetry is likely contained together with Abelian R and non-R symmetries. The discrete symmetry plays a role of the flavor symmetry. When coordinates are described in terms of a projective representation of the discrete symmetry, matter fields also have matrix-valued charges of the discrete symmetry. In section 3 we briefly review the minimal $SU(6) \times SU(2)_R$ model and introduce Abelian R and non-R symmetries as the flavor symmetry. We attempt to explain not only fermion mass hierarchies but also hierarchical energy scales. It is found that an additional selection rule is needed to explain the hierarchical energy scales. In section 4 we introduce a non-Abelian discrete symmetry to yield the additional selection rule and take its projective rep-

resentation. This non-Abelian group turns out to be just the dihedral group D_4. By using the flavor symmetry including D_4 together with Abelian R and non-R symmetries, mass spectra and fermion mixings are studied in section 5. The results come up to phenomenological requirements. Whether we obtain the LMA-MSW solution or the SMA-MSW solution is controlled by the assignment of Abelian flavor charges. Final section is devoted to summary and discussion.

2 Discrete flavor symmetries

In the string theory discrete symmetries stem from the symmetric structure of compact space. As a simple example of Calabi-Yau space let us consider the quintic hypersurface

$$F(z) = \sum_{i=1}^{5} z_i^5 = 0 \tag{1}$$

in CP^4 with the homogeneous coordinates z_i $(i = 1 \sim 5)$. On this Calabi-Yau space we have the discrete symmetry

$$G = S_5 \ltimes (\mathbf{Z}_5)^5 / \mathbf{Z}_5 \,, \tag{2}$$

where five $\mathbf{Z}_5$'s represent the phase transformations $z_i \to \alpha^{n_i} z_i$ with $\alpha^5 = 1$ and with integers n_i. Massless matter fields are classified according to charges of the discrete symmetry. Then the discrete symmetry plays a role of the flavor symmetry. String compactification on this hypersurface corresponds to the 3^5 Gepner model[2]. In Gepner model the compact space is algebraically constructed in terms of a tensor product of discrete series of $N = 2$ superconformal field theory with level k_i. When the trace anomaly condition

$$\sum_{i=1}^{r} \frac{3k_i}{k_i + 2} = 9 \tag{3}$$

is satisfied, we have a complex 3-dimensional Calabi-Yau space with the discrete symmetry

$$G = G_P \ltimes \prod_{i=1}^{r} \mathbf{Z}_{k_i+2} / \mathbf{Z}_l \,, \tag{4}$$

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where l is the l.c.m. of $k_i + 2$ $(i = 1 \sim r)$ and G_P permutes the different variables of the same level k_i. The product $\prod_{i=1}^{r} \mathbf{Z}_{k_i+2}$ is R symmetry, while G_P is non-R symmetry. The Gepner model with $r = 5$ corresponds to string compactification on weighted hypersurfaces in weighted CP^4 [7].

As seen from these examples of compact spaces, in the string theory there can appear various types of discrete R symmetry and discrete non-R symmetry as the flavor symmetry. Since the discrete symmertry group G in the above examples contains semi-direct products of Abelian groups in addition to a permutation group, G is non-Abelian as a whole. Thus it is natural to expect that the symmetry group is non-Abelian in general.

Let us first consider a compact space with an Abelian discrete symmetry. In string with discrete torsion the coordinates become non-commutative operators and are represented by a projective representation of the Abelian discrete symmetry[4]. Hereafter the coordinates are referred to as quantum coordinates. We illustrate quantum coordinates with the above example. In the case of the quintic hypersurface in CP^4 quantum coordinates are described in terms of the projective representation of $(\mathbf{Z}_5)^4$ and given by[8]

$$\hat{z}_1 = z_1\, P, \qquad \hat{z}_2 = z_2\, Q, \qquad \hat{z}_3 = z_3\, P^{p_3}Q^{q_3},$$
$$\hat{z}_4 = z_4\, P^{p_4}Q^{q_4}, \qquad \hat{z}_5 = z_5\, P^{p_5}Q^{q_5}, \tag{5}$$

where p_i and q_i $(i = 3, 4, 5)$ are integers and

$$P = \text{diag}(1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4), \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{6}$$

with $\alpha^5 = 1$. Since P and Q satisfy the relations

$$P^5 = Q^5 = 1, \qquad PQ = \alpha\, QP, \tag{7}$$

we have

$$\hat{z}_1\, \hat{z}_2 = \alpha\, \hat{z}_2\, \hat{z}_1, \quad \cdots. \tag{8}$$

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The quantum fluctuation of coordinates is described in terms of the matrices P and Q, which represent the non-commutativity in the compact space. When the quantum fluctuation is switched off, namely, in the string without discrete torsion, the matrix factors of $\hat{z}_i$ given by the products of P and Q disappear. In this case we are led to a compact space with commutative geometry (classical geometry) **. Massless matter fields in effective theory correspond to the degree of freedom of deformation of the compact space. In the above example the deformation is described in terms of polynomials of homogeneous coordinates z_i with definite charges under the discrete group. This correspondence is expected to hold also in the case of non-commutative geometry. Since massless matter fields correspond to functions of the quantum coordinates $\hat{z}_i$, massless matter fields become matrix-valued. On the other hand, four-dimensional effective Lagrangian of the theory should belong to the center of the non-commutative algebra. This means that a new type of flavor symmetry arises in the compact space with non-commutative geometry.

We next consider an extension of the above argument to the compact space with a non-Abelian discrete symmetry. Based on the analogy to an Abelian discrete symmetry, we take the first ansatz that the quantum fluctuation of the coordinates is represented by a projective representation of the non-Abelian discrete symmetry. This ansatz implies that for coordinates we introduce the nontrivial commutation relation different from the usual canonical commutation relations[9, 10]. The second ansatz is that the degree of freedom of the deformation is expressed by the functions of quantum coordinates in the compact space with a certain non-Abelian discrete symmetry. Therefore, massless matter fields in effective theory are described in terms of quantum coordinates and become matrix-valued. Consequently, we have a new type of the flavor symmetry coming from the quantum fluctuation of coordinates.

In this paper we introduce $\mathbf{Z}_2 \ltimes \mathbf{Z}_4$ symmetry as the discrete non-Abelian flavor symmetry and consider its projective representation, which is motivated by phenomenological requirements as shown in section 4. The $\mathbf{Z}_2$ and $\mathbf{Z}_4$ groups are expressed as

$$\mathbf{Z}_2 = \{1, \ g_1\}, \tag{9}$$

$$\mathbf{Z}_4 = \{1, \ g_2, \ g_2^2, \ g_2^3\}, \tag{10}$$

**This situation is analogous to the appearance of the degree of freedom of spin in quantum mechanics. When we take the limit $\hbar \to 0$, coordinates commute with their conjugate momenta and also the degree of freedom of spin disappears.

respectively and we assume the relation

$$g_1 g_2 g_1^{-1} = g_2^{-1}. \tag{11}$$

The group $\mathbf{Z}_2 \ltimes \mathbf{Z}_4$ is nothing but the dihedral group D_4. The elements g_1 and g_2 mean the reflection and $\pi/2$ rotation of a square, respectively. In view of the fact that $D_4 \subset SO(3) \subset SO(6)$ it is plausible that the dihedral group D_4 is contained in the discrete symmetry from six-dimensional string compactification. A vector representation of D_4 is given by

$$\gamma_{g_1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \qquad \gamma_{g_2} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1. \tag{12}$$

On the other hand, a projective representation of D_4 is given by

$$\gamma_{g_1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \qquad \gamma_{g_2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{13}$$

and

$$\gamma_{g_1} \gamma_{g_2} \gamma_{g_1}^{-1} = i\gamma_{g_2}^{-1}. \tag{14}$$

We will use this projective representation of D_4 in section 4.

3   Minimal $SU(6) \times SU(2)_R$ model

In this paper we take up minimal $SU(6) \times SU(2)_R$ string-inspired model, which has been studied in detail in Refs.[3, 11, 12, 13, 14]. Here we briefly review the main points of the model.

(i). We choose $SU(6) \times SU(2)_R$ as the unification gauge symmetry at the string scale M_S, which can be derived from the perturbative heterotic superstring theory via the flux breaking[15].

(ii). Matter chiral superfields consist of three families and one vector-like multiplet, i.e.,

$$3 \times \mathbf{27}(\Phi_{1,2,3}) + (\mathbf{27}(\Phi_0) + \overline{\mathbf{27}}(\overline{\Phi})) \tag{15}$$

in terms of E_6. Under $G_{gauge} = SU(6) \times SU(2)_R$, the superfields Φ in $\mathbf{27}$ of E_6 are decomposed into two groups as

$$\Phi(\mathbf{27}) = \begin{cases} \phi(\mathbf{15}, \mathbf{1}) & : & Q, L, g, g^c, S, \\ \psi(\mathbf{6}^*, \mathbf{2}) & : & (U^c, D^c), (N^c, E^c), (H_u, H_d), \end{cases} \tag{16}$$

where g, g^c and H_u, H_d represent colored Higgs and doublet Higgs superfields, respectively. N^c is the right-handed neutrino superfield and S is an $SO(10)$-singlet.

(iii). Gauge invariant trilinear couplings in the superpotential W become to be of the forms

$$
\begin{aligned}
(\phi(\mathbf{15}, \mathbf{1}))^3 &= QQg + Qg^cL + g^cgS, \tag{17} \\
\phi(\mathbf{15}, \mathbf{1})(\psi(\mathbf{6}^*, \mathbf{2}))^2 &= QH_dD^c + QH_uU^c + LH_dE^c + LH_uN^c \\
&\quad + SH_uH_d + gN^cD^c + gE^cU^c + g^cU^cD^c. \tag{18}
\end{aligned}
$$

It has been found that this model contains phenomenologically attractive features. In the conventional GUT-type models, unless an adjoint or higher representation matter(Higgs) field develops a non-zero VEV, it is impossible that the large gauge symmetry is spontaneously broken down to the standard model gauge group G_{SM} via Higgs mechanism. On the other hand, as explained above, in the present model matter fields consist only of $\mathbf{27}$ and $\overline{\mathbf{27}}$. The symmetry breaking of $G_{gauge} = SU(6) \times SU(2)_R$ down to G_{SM} can take place via Higgs mechanism without adjoint or higher representation matter fields. In addition, $SU(6) \times SU(2)_R$ is one of the maximal subgroups of E_6. Further, it is noticeable that doublet Higgs and color-triplet Higgs fields belong to different irreducible representations of G_{gauge}. This situation is favorable to solve the triplet-doublet splitting problem.

As the flavor symmetry we introduce Abelian discrete R and non-R symmetries in the first place. Concretely, as for the R symmetry we take R-parity and $\mathbf{Z}_M$. We assign odd R-parity for $\Phi_{1,2,3}$ and even for Φ_0 and $\overline{\Phi}$, respectively. Since ordinary

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Table 1: Assignment of $\mathbf{Z}_{MN}$-charge and R-parity for matter superfields

	$\Phi_i \ (i = 1, 2, 3)$	Φ_0	$\overline{\Phi}$
$\phi(\mathbf{15}, \ \mathbf{1})$	$(a_i, \ -)$	$(a_0, \ +)$	$(\overline{a}, \ +)$
$\psi(\mathbf{6}^*, \ \mathbf{2})$	$(b_i, \ -)$	$(b_0, \ +)$	$(\overline{b}, \ +)$

Higgs doublets have even R-parity, they belong to Φ_0. As for the non-R symmetry we take $\mathbf{Z}_N$. Assuming that M and N are relatively prime, we combine as

$$\mathbf{Z}_M \times \mathbf{Z}_N = \mathbf{Z}_{MN}. \tag{19}$$

In this case Grassmann number θ in superfield formalism has a charge $(-1, \ 0)$ under $\mathbf{Z}_M \times \mathbf{Z}_N$. The charge of θ under $\mathbf{Z}_{MN}$ is denoted as q_θ. $\mathbf{Z}_{MN}$-charges of matter superfields are denoted as a_i and b_i, etc. as shown in Table 1.

In the superpotential there appear various types of non-renormalizable terms which respect both the gauge symmetry and the flavor symmetry. In R-parity even sector the superpotential contains the terms

$$W_1 \sim M_S^3 \sum_{i=0}^{r} \lambda_i \left(\frac{\phi_0 \overline{\phi}}{M_S^2} \right)^{n_i} \left(\frac{\psi_0 \overline{\psi}}{M_S^2} \right)^{m_i}, \tag{20}$$

where $\lambda_i = \mathcal{O}(1)$ and the exponents are non-negative integers which satisfy the $\mathbf{Z}_{MN}$ symmetry condition

$$n_i(a_0 + \overline{a}) + m_i(b_0 + \overline{b}) - 2q_\theta \equiv 0 \qquad \mathrm{mod}\ MN. \tag{21}$$

Through the minimization of the scalar potential with the soft SUSY breaking mass terms characterized by the scale $\widetilde{m}_0 = \mathcal{O}(10^2)\mathrm{GeV}$, matter fields develop non-zero VEV's. In Refs.[16] and [17] we have studied the minimum point of the scalar potential in detail. If and only if the relations

$$n_i = (r - i)n_{r-1}, \qquad m_i = i m_1 \qquad (i = 0 \sim r) \tag{22}$$

are satisfied, the gauge symmetry is spontaneously broken in two steps in a tolerable parameter region of the coefficients λ_i[18]. Further, the scales of the gauge symmetry

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breaking are given by

$$\begin{aligned}
|\langle\phi_0\rangle| = |\langle\overline{\phi}\rangle| &\sim M_S\,\rho^{1/2(n_0-1)}, \\
|\langle\psi_0\rangle| = |\langle\overline{\psi}\rangle| &\sim M_S\,\rho^{n_{r-1}/2(n_0-1)m_1},
\end{aligned}$$
(23)

where $\rho = \widetilde{m}_0/M_S \sim 10^{-16}$. The D-flat conditions require $|\langle\phi_0\rangle| = |\langle\overline{\phi}\rangle|$ and $|\langle\psi_0\rangle| = |\langle\overline{\psi}\rangle|$. Under the assumption $n_{r-1} > m_1$ we have

$$|\langle\phi_0\rangle| > |\langle\psi_0\rangle|.$$
(24)

Then the gauge symmetry is spontaneously broken at the scale $|\langle\phi_0(\mathbf{15},\ \mathbf{1})\rangle|$ and subsequently at the scale $|\langle\psi_0(\mathbf{6}^*,\ \mathbf{2})\rangle|$. This yields the symmetry breakings

$$SU(6) \times SU(2)_R \xrightarrow{\langle\phi_0\rangle} SU(4)_{\mathrm{PS}} \times SU(2)_L \times SU(2)_R \xrightarrow{\langle\psi_0\rangle} G_{SM},$$
(25)

where $SU(4)_{\mathrm{PS}}$ is the Pati-Salam $SU(4)$[19]. Since the fields which develop non-zero VEV's are singlets under the remaining gauge symmetries, they are assigned as $\langle\phi_0(\mathbf{15},\ \mathbf{1})\rangle = \langle S_0\rangle$ and $\langle\psi_0(\mathbf{6}^*,\ \mathbf{2})\rangle = \langle N_0^c\rangle$. Below the scale $|\langle\phi_0\rangle|$ Froggatt-Nielsen mechanism is at work for the non-renormalizable interactions[1].

Majorana masses for R-handed neutrinos are induced from the non-renormalizable terms

$$M_S^{-1}\left(\frac{S_0\overline{S}}{M_S^2}\right)^{\nu_{ij}}(\psi_i\overline{\psi})(\psi_j\overline{\psi}) \qquad (i,j=1,2,3),$$
(26)

where the exponents are given by

$$(a_0+\overline{a})\nu_{ij}+b_i+b_j+2\overline{b}-2q_\theta \equiv 0 \qquad \mathrm{mod}\ MN.$$
(27)

In fact, these terms lead to the Majorana mass terms

$$\mathcal{N}_{ij}N_i^c N_j^c \sim x^{\nu_{ij}}\left(\frac{\langle\overline{N^c}\rangle}{M_S}\right)^2 N_i^c N_j^c$$
(28)

in M_S units, where we use the notation

$$x = \frac{\langle S_0\rangle\langle\overline{S}\rangle}{M_S^2}.$$
(29)

From Eq.(23) we have

$$x^{n_0-1} \sim \rho \sim 10^{-16}. \tag{30}$$

Phenomenologically, it is desirable that the Majorana mass for the third generation is $\mathcal{O}(10^{10\sim12})$GeV. This scale is almost equal to the geometrically averaged value between M_S and M_W, namely,

$$x^{\nu_{33}} \left(\frac{\langle \overline{N^c} \rangle}{M_S} \right)^2 \sim \sqrt{\rho}. \tag{31}$$

This is translated as

$$\nu_{33} + \frac{n_{r-1}}{m_1} \sim \frac{n_0 - 1}{2}. \tag{32}$$

Further, colored Higgs mass coming from

$$\left(\frac{S_0 \overline{S}}{M_S^2} \right)^{\zeta_{00}} S_0 g_0 g_0^c, \tag{33}$$

is given by

$$m_{g_0/g_0^c} = x^{\zeta_{00}} \langle S_0 \rangle. \tag{34}$$

The $\mathbf{Z}_{MN}$ symmetry controls the exponent ζ_{00} as

$$(a_0 + \overline{a})\zeta_{00} + 3a_0 - 2q_\theta \equiv 0 \qquad \mathrm{mod} \ MN. \tag{35}$$

In order to guarantee the longevity of the proton, ζ_{00} should be sufficiently small compared to n_0. On the other hand, the μ-term induced from

$$\left(\frac{S_0 \overline{S}}{M_S^2} \right)^{\eta_{00}} S_0 H_{u0} H_{d0}, \tag{36}$$

is of the form

$$\mu = x^{\eta_{00}} \langle S_0 \rangle. \tag{37}$$

The exponent η_{00} is determined by

$$(a_0 + \overline{a})\eta_{00} + a_0 + 2b_0 - 2q_\theta \equiv 0 \qquad \mathrm{mod} \ MN. \tag{38}$$

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To be $\mu = \mathcal{O}(10^{2\sim3})$GeV, we need to obtain $\eta_{00} \sim n_0$ as a solution.

In order to find out a solution to the condition (32) as well as $0 \le \zeta_{00} \ll n_0$ and $\eta_{00} \sim n_0$, it is assumed that $\mathbf{Z}_{MN}$-charges of all matter superfields but $\overline{\phi}$ are even and that $a_0 + \overline{a} = -4$. Further, if $q_\theta =$ even and $q_\theta \ll MN$ and if $a_0 \equiv b_i + b_j \equiv MN - 2 \equiv 0 \mod 4$, we can expect that Eqs.(27), (35) and (38) are reduced to

$$-4\nu_{ij} + b_i + b_j + 2\overline{b} - 2q_\theta = -MN, \tag{39}$$

$$-4\zeta_{00} + 3a_0 - 2q_\theta = 0, \tag{40}$$

$$-4\eta_{00} + a_0 + 2b_0 - 2q_\theta = -2MN. \tag{41}$$

Here we put $M =$odd and $N \equiv 2 \mod 4$ so as to render q_θ even. To be more specific, we choose a typical example

$$M = 15, \qquad N = 14. \tag{42}$$

In order to get $q_\theta \ll MN$ we take $|M - N| = 1$ and then $q_\theta = N$. Furthermore, for the sake of simplicity, M and N are chosen such that when decomposed into prime factors, all the prime factors are numbers with one figure. Thus we take the R symmetry $\mathbf{Z}_{15} = \mathbf{Z}_3 \times \mathbf{Z}_5$ and the non-R symmetry $\mathbf{Z}_{14} = \mathbf{Z}_2 \times \mathbf{Z}_7$. In this case we obtain $q_\theta = 14$ and $q_\theta \ll MN$. Further we put

$$b_0 + \overline{b} = -49. \tag{43}$$

Under these parametrizations Eq.(21) becomes

$$-4n_i - 49m_i - 28 \equiv 0 \mod 210. \tag{44}$$

Since this equation allows the case

$$-4n_i - 49m_i - 28 = -210, \tag{45}$$

we can not derive the relation (22). Then we need to forbid this case by introducing an additional selection rule. If the additional selection rule requires $m_i \equiv 0 \mod 4$, Eq.(44) is rewritten as

$$-4n_i - 49m_i - 28 = -420. \tag{46}$$

This leads to

$$(n_i,\ m_i) = (98,\ 0), \quad (49,\ 4), \quad (0,\ 8), \tag{47}$$

which are in accord with the relation (22).

There appears similar situation in the μ-term. In addition to the term (36), the non-renormalizable term

$$\left(\frac{S_0\overline{S}}{M_{\tilde{S}}^2}\right)^{\eta'_{00}} \left(\frac{N_0^c\overline{N}^c}{M_{\tilde{S}}^2}\right)^{\xi} S_0 H_{u0} H_{d0} \tag{48}$$

is also allowed and leads to an additional μ-term

$$\mu' = x^{\eta'_{00}+\xi n_{r-1}/m_1} \langle S_0 \rangle. \tag{49}$$

The exponents are determined by

$$-4\eta'_{00} - 49\xi + a_0 + 2b_0 - 28 \equiv 0 \mod 210. \tag{50}$$

This allows the case

$$-4\eta'_{00} - 49\xi + a_0 + 2b_0 - 28 = -210. \tag{51}$$

As a result we obtain an unrealistic solution $\mu' \gg \mu = \mathcal{O}(10^{2\sim3})\text{GeV}$. We also need to forbid this solution. If the additional selection rule requires $\xi \equiv 0 \mod 4$, then Eq.(50) is reduced to

$$-4\eta'_{00} - 49\xi + a_0 + 2b_0 - 28 = -420. \tag{52}$$

In this case we can obtain μ, $\mu' = \mathcal{O}(10^{2\sim3})\text{GeV}$. Thus we need an additional selection rule under which both the exponents m_i and ξ of $(\psi_0\overline{\psi})$ in Eqs.(20) and (48) should be multiples of 4.

4 Discrete non-Abelian symmetry

As an additional flavor symmetry we introduce a discrete non-Abelian symmetry. It is postulated that due to the quantum fluctuation of coordinates this discrete non-Abelian symmetry is described in terms of the projective representation and that massless matter fields are matrix-valued. We denote matrix-valued charges of matter

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Table 2: Assignment of matrix-valued charges

	Φ_i $(i = 1, 2, 3)$	Φ_0	$\overline{\Phi}$
$\phi(\mathbf{15},\ \mathbf{1})$	A_i	A_0	$\overline{A}$
$\psi(\mathbf{6^*},\ \mathbf{2})$	B_i	B_0	$\overline{B}$

fields as A_i and B_i, etc. as shown in Table 2. Since effective Lagrangian of the theory is the center of the non-commutative algebra, all the terms in the superpotential should be proportional to unit matrix provided that θ^2 is neutral under the non-Abelian symmetry. Therefore, we have a new type of selection rule arising from the projective representation of non-Abelian symmetry.

From the superpotential terms (20), (33), (36) and (48) we have the non-Abelian symmetry conditions

$$\left(A_0\overline{A}\right)^{n_i} \left(B_0\overline{B}\right)^{m_i} \propto 1, \tag{53}$$

$$\left(A_0\overline{A}\right)^{\zeta_{00}} A_0^3 \propto 1, \tag{54}$$

$$\left(A_0\overline{A}\right)^{\eta_{00}} A_0 B_0^2 \propto 1, \tag{55}$$

$$\left(A_0\overline{A}\right)^{\eta'_{00}} \left(B_0\overline{B}\right)^{\xi} A_0 B_0^2 \propto 1, \tag{56}$$

respectively. When we put $A_0\overline{A} = 1$, Eq.(54) yields $A_0^3 \propto 1$. Accordingly, we take a simple choice $A_0 = \overline{A} = 1$. In this choice the above conditions are reduced to

$$\left(B_0\overline{B}\right)^{m_i}, \ \left(B_0\overline{B}\right)^{\xi}, \ B_0^2 \propto 1. \tag{57}$$

As discussed in the previous section, we need the selection rule

$$m_i \equiv \xi \equiv 0 \qquad \text{mod } 4, \tag{58}$$

which are expressed as

$$\left(B_0\overline{B}\right)^4 \propto 1, \qquad \left(B_0\overline{B}\right)^2 \not\propto 1. \tag{59}$$

We turn to quark/lepton mass matrices. Mass matrix for up-type quarks comes from the term

$$m_{ij} \left(\frac{S_0\overline{S}}{M_S^2}\right)^{\mu_{ij}} Q_i U_j^c H_{u0} \tag{60}$$

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with $m_{ij} = \mathcal{O}(1)$. Due to the Froggatt-Nielsen mechanism the mass matrix is given by

$$\mathcal{M}_{ij}v_u = m_{ij}\,x^{\mu_{ij}}v_u. \tag{61}$$

The exponent μ_{ij} is determined by

$$-4\mu_{ij} + a_i + b_j + b_0 - 28 \equiv 0 \qquad \text{mod } 210. \tag{62}$$

In addition, the non-Abelian symmetry condition

$$\left(A_0\overline{A}\right)^{\mu_{ij}} A_i B_j B_0 = A_i B_j B_0 \propto 1 \tag{63}$$

is obtained. We here choose a solution under which this non-Abelian condition is satisfied irrespective of i, j $(i, j = 1, 2, 3)$. This choice is in line with the assumption that there is no texture-zero in quark/lepton mass matrices. Thus we put

$$A_1 = A_2 = A_3, \qquad B_1 = B_2 = B_3 \tag{64}$$

and then obtain

$$A_3 B_3 B_0 \propto 1. \tag{65}$$

If we took a different choice for matrix-valued charges A_i and B_i $(i = 1, 2, 3)$, there could appear texture-zeros in quark/lepton mass matrices. Here we do not consider such a case.

In the down-quark sector mass matrix is given by[3, 11, 12, 13]

$$\widehat{\mathcal{M}}_d = \begin{array}{c} \\ g \\ D \end{array} \begin{pmatrix} \overset{g^c}{y_S \mathcal{Z}} & \overset{D^c}{y_N \mathcal{M}} \\ 0 & \rho_d \mathcal{M} \end{pmatrix} \tag{66}$$

in M_S units, where $y_S = \langle S_0 \rangle / M_S$, $y_N = \langle N_0^c \rangle / M_S$ and $\rho_d = v_d / M_S$. Since g^c and D^c have the same quantum number under the standard model gauge group, mixings occur between these fields. Consequently, mass matrix for down-type quarks becomes 6×6 matrix. The above g-g^c submatrix coming from the term

$$z_{ij} \left(\frac{S_0 \overline{S}}{M_S^2}\right)^{\zeta_{ij}} S_0 g_i g_j^c, \tag{67}$$

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is given by

$$\mathcal{Z}_{ij} = z_{ij}\, x^{\zeta_{ij}} \tag{68}$$

with $z_{ij} = \mathcal{O}(1)$. The flavor symmetry requires the conditions

$$-4\zeta_{ij} + a_i + a_j + a_0 - 28 \equiv 0 \qquad \mathrm{mod}\ 210, \tag{69}$$

$$\left(A_0\overline{A}\right)^{\zeta_{ij}} A_i A_j A_0 = A_3^2 \propto 1. \tag{70}$$

In the charged lepton sector mass matrix is of the form[3, 11, 12, 14]

$$\widehat{\mathcal{M}}_l = \begin{array}{cc} & \begin{array}{cc} H_u^+ & E^{c+} \end{array} \\ \begin{array}{c} H_d^- \\ L^- \end{array} & \left(\begin{array}{cc} y_S\mathcal{H} & 0 \\ y_N\mathcal{M} & \rho_d\mathcal{M} \end{array}\right) \end{array} \tag{71}$$

in M_S units. Since H_d and L also have the same quantum number under the standard model gauge group, mixings occur between these fields. The above H_d-H_u submatrix coming from

$$h_{ij}\left(\frac{S_0\overline{S}}{M_S^2}\right)^{\eta_{ij}} S_0 H_{di} H_{uj}, \tag{72}$$

is expressed as

$$\mathcal{H}_{ij} = h_{ij}x^{\eta_{ij}} \tag{73}$$

with $h_{ij} = \mathcal{O}(1)$. From the flavor symmetry we have the conditions

$$-4\eta_{ij} + b_i + b_j + a_0 - 28 \equiv 0 \qquad \mathrm{mod}\ 210, \tag{74}$$

$$\left(A_0\overline{A}\right)^{\eta_{ij}} B_i B_j A_0 = B_3^2 \propto 1. \tag{75}$$

In the neutral sector there exist five types of matter fields H_u^0, H_d^0, L^0, N^c and S. Then we have 15×15 mass matrix[3, 11, 12, 14]

$$\widehat{\mathcal{M}}_{NS} = \begin{array}{cc} & \begin{array}{ccccc} H_u^0 & H_d^0 & L^0 & N^c & S \end{array} \\ \begin{array}{c} H_u^0 \\ H_d^0 \\ L^0 \\ N^c \\ S \end{array} & \left(\begin{array}{ccccc} 0 & y_S\mathcal{H} & y_N\mathcal{M}^T & 0 & \rho_d\mathcal{M}^T \\ y_S\mathcal{H} & 0 & 0 & 0 & \rho_u\mathcal{M}^T \\ y_N\mathcal{M} & 0 & 0 & \rho_u\mathcal{M} & 0 \\ 0 & 0 & \rho_u\mathcal{M}^T & \mathcal{N} & \mathcal{T}^T \\ \rho_d\mathcal{M} & \rho_u\mathcal{M} & 0 & \mathcal{T} & \mathcal{S} \end{array}\right) \end{array} \tag{76}$$

in M_S units, where $\rho_u = v_u/M_S$. In this matrix the 6×6 submatrix

$$\widehat{\mathcal{M}}_M = \begin{pmatrix} \mathcal{N} & \mathcal{T}^T \\ \mathcal{T} & \mathcal{S} \end{pmatrix} \tag{77}$$

play a role of Majorana mass matrix in the seesaw mechanism. The 3×3 submatrix $\mathcal{N}$ has already been given in Eq.(28). The flavor symmetry leads to the conditions

$$- 4\nu_{ij} + b_i + b_j + 2\overline{b} - 28 \equiv 0 \qquad \mathrm{mod}\ 210, \tag{78}$$

$$\left(A_0\overline{A} \right)^{\nu_{ij}} (B_i\overline{B})(B_j\overline{B}) = (B_3\overline{B})^2 \propto 1 \tag{79}$$

The submatrix $\mathcal{S}$ induced from

$$M_S^{-1} \left(\frac{S_0\overline{S}}{M_S^2} \right)^{\sigma_{ij}} (\phi_i\overline{\phi})(\phi_j\overline{\phi}), \tag{80}$$

is expressed as

$$\mathcal{S}_{ij} \sim x^{\sigma_{ij}} \left(\frac{\langle \overline{S} \rangle}{M_S} \right)^2. \tag{81}$$

The exponents are determined by

$$- 4\sigma_{ij} + a_i + a_j + 2\overline{a} - 28 \equiv 0 \qquad \mathrm{mod}\ 210. \tag{82}$$

The condition on matrix-valued charges

$$\left(A_0\overline{A} \right)^{\sigma_{ij}} (A_i\overline{A})(A_j\overline{A}) = A_3^2 \propto 1 \tag{83}$$

is the same as Eq.(70). The submatrix $\mathcal{T}$ induced from

$$M_S^{-1} \left(\frac{S_0\overline{S}}{M_S^2} \right)^{\tau_{ij}} (\phi_i\overline{\phi})(\psi_j\overline{\psi}), \tag{84}$$

is given by

$$\mathcal{T}_{ij} \sim x^{\tau_{ij}} \frac{\langle \overline{S} \rangle \langle \overline{N^c} \rangle}{M_S^2}. \tag{85}$$

The flavor symmetry yields the conditions

$$-4\tau_{ij} + a_i + b_j + \overline{a} + \overline{b} - 28 \equiv 0 \qquad \text{mod } 210, \tag{86}$$

$$\left(A_0\overline{A}\right)^{\tau_{ij}} (A_i\overline{A})(B_j\overline{B}) = (A_3\overline{A})(B_3\overline{B}) \propto 1. \tag{87}$$

However, since only $\overline{b}$ is taken as an odd integer, we have no solution to satisfy Eq.(86). This means the 3×3 matrix $\mathcal{T} = 0$.

Here we summarize the constraints on matrix-valued charges. First, we choose

$$A_0 = \overline{A} = 1, \qquad A_1 = A_2 = A_3, \qquad B_1 = B_2 = B_3. \tag{88}$$

From Eqs.(57), (59), (65), (70), (75) and (79) the conditions are put in order as

$$A_3^2, \ B_3^2, \ B_0^2, \ A_3B_3B_0, \ (B_3\overline{B})^2, \ (B_0\overline{B})^4 \propto 1, \quad (B_0\overline{B})^2 \not\propto 1. \tag{89}$$

If $[\,B_3, \ \overline{B}\,] = [\,B_0, \ \overline{B}\,] = 0$, these conditions are inconsistent. Consequently, it is necessary for us to introduce a non-Abelian symmetry as the flavor symmetry. The above conditions are realized provided that B_3, $\overline{B}$, B_0 and A_3 correspond to the elements g_1, g_2, g_2^2 and $g_1 g_2^2$ in the dihedral group D_4 discussed in section 2. By taking the projective representation of D_4

$$A_3 = \sigma_1, \quad B_3 = \sigma_2, \quad B_0 = \sigma_3, \quad \overline{B} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \tag{90}$$

where σ_i's represent Pauli matrices, we obtain the relations

$$A_3^2 = B_3^2 = B_0^2 = 1, \tag{91}$$

$$A_3B_3B_0 = (B_3\overline{B})^2 = i, \tag{92}$$

$$(B_0\overline{B})^2 = \sigma_3, \quad (B_0\overline{B})^4 = 1. \tag{93}$$

On the other hand, the conditions (89) are not satisfied in the case of the vector representation of D_4 given in Eq.(12) [††].

[††]If we take 4×4 matrices, we can obtain the vector representation of D_4 which satisfies the conditions (89). However, we are interested in the projective representation associated to the non-commutativity of coordinates.

5   Mass spectra and fermion mixings

In the previous section the dihedral group D_4 was introduced as an additional flavor symmetry. By using the dihedral flavor symmetry as well as Abelian R and non-R symmetries, we now proceed to study mass spectra and fermion mixings.

In section 3 we take the parametrization $M = 15$, $N = 14$ and

$$a_0 + \overline{a} = -4, \qquad b_0 + \overline{b} = -49. \tag{94}$$

Further, it was assumed that $\mathbf{Z}_{210}$-charges of matter fields other than $\overline{b}$ are even and that $a_0 \equiv b_i + b_j \equiv 0 \bmod 4$. By virtue of the dihedral flavor symmetry offered here the superpotential (20) becomes

$$W_1 \sim M_S^3 \left[\lambda_0 \left(\frac{\phi_0 \overline{\phi}}{M_S^2} \right)^{98} + \lambda_1 \left(\frac{\phi_0 \overline{\phi}}{M_S^2} \right)^{49} \left(\frac{\psi_0 \overline{\psi}}{M_S^2} \right)^{4} + \lambda_2 \left(\frac{\psi_0 \overline{\psi}}{M_S^2} \right)^{8} \right] \tag{95}$$

and leads to

$$\begin{aligned}
|\langle \phi_0 \rangle| = |\langle \overline{\phi} \rangle| \quad &\sim \quad M_S \, \rho^{1/194}, \\
|\langle \psi_0 \rangle| = |\langle \overline{\psi} \rangle| \quad &\sim \quad M_S \, \rho^{49/776}.
\end{aligned} \tag{96}$$

From the relation $x^{97} \sim \rho \sim 10^{-16}$ we obtain

$$x^4 \simeq 0.23 \simeq \lambda. \tag{97}$$

As mentioned above, the gauge symmetry breaking takes place as

$$SU(6) \times SU(2)_R \xrightarrow{\langle \phi_0 \rangle} SU(4)_{\mathrm{PS}} \times SU(2)_L \times SU(2)_R \xrightarrow{\langle \psi_0 \rangle} G_{SM}. \tag{98}$$

At the first step of the symmetry breaking fields Q_0, L_0, $\overline{Q}$, $\overline{L}$ and $(S_0 - \overline{S})/\sqrt{2}$ are absorbed by gauge fields. Through the subsequent symmetry breaking fields U_0^c, E_0^c, $\overline{U}^c$, $\overline{E}^c$ and $(N_0^c - \overline{N}^c)/\sqrt{2}$ are absorbed.

The scale of colored Higgs mass is controlled by ζ_{00}, which is determined by

$$-4\zeta_{00} + 3a_0 - 28 \equiv 0 \qquad \bmod 210. \tag{99}$$

The μ is also controlled by η_{00}, which is determined by

$$-4\eta_{00} + a_0 + 2b_0 - 28 \equiv 0 \qquad \bmod 210. \tag{100}$$

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Therefore, if we put $420 \gg 3a_0 - 28 > 0$ and $-420 \ll a_0 + 2b_0 - 28 < 0$, then we have

$$-4\zeta_{00} + 3a_0 - 28 = 0, \tag{101}$$

$$-4\eta_{00} + a_0 + 2b_0 - 28 = -420, \tag{102}$$

where we used $a_0 \equiv 2b_0 \equiv 0 \mod 4$. In the case $a_0 = 12 \sim 20$, we have $\zeta_{00} = 2 \sim 8$ and then $m_{g_0/g_0^c} = \mathcal{O}(10^{17 \sim 18}) \text{GeV}$, which is consistent with the longevity of the proton. On the other hand, the parametrization $a_0 + 2b_0 = 0 \sim -20$ leads to a phenomenologically viable value of μ, namely, $\mu = \mathcal{O}(10^{2 \sim 3}) \text{GeV}$.

We now return to quark/lepton mass matrices. Mass matrix for up-type quarks is given by $\mathcal{M}_{ij} \sim x^{\mu_{ij}}$ with

$$-4\mu_{ij} + a_i + b_j + b_0 - 28 \equiv 0 \qquad \mod 210. \tag{103}$$

In order to explain the experimental fact that top-quark mass is of $\mathcal{O}(v_u)$, we put

$$a_3 + b_3 + b_0 - 28 = 0. \tag{104}$$

Further, we take the parametrization

$$
\begin{aligned}
a_1 - a_3 &= 48, & a_2 - a_3 &= 32, \\
b_1 - b_3 &= 64, & b_2 - b_3 &= 32.
\end{aligned}
\tag{105}
$$

As a result, the mass matrix $\mathcal{M}$ becomes

$$\mathcal{M} \sim \begin{pmatrix} \lambda^7 & \lambda^5 & \lambda^3 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{pmatrix}, \tag{106}$$

where we used Eq.(97). Eigenvalues of $\mathcal{M}$ yield up-type quark masses as

$$(m_u, \ m_c, \ m_t) \sim (\lambda^7 v_u, \ \lambda^4 v_u, \ v_u). \tag{107}$$

Down-type quark mass is derived from Eq.(66) with $\mathcal{Z}_{ij} \sim x^{\zeta_{ij}}$, where

$$-4\zeta_{ij} + a_i + a_j + a_0 - 28 \equiv 0 \qquad \mod 210. \tag{108}$$

Then, under the parametrization (105) we have

$$\mathcal{Z} \sim \lambda^\zeta \begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \tag{109}$$

where

$$\zeta = \frac{1}{16}(a_3 - b_3 + a_0 - b_0) \tag{110}$$

and we put $\zeta > 0$. Eigenstates of mass matrix (66) contain three heavy modes with their masses $\mathcal{O}(10^{17})$GeV and three light modes. When we choose

$$\zeta \sim 2.5, \tag{111}$$

light mode spectra turn out to be[13]

$$(m_d,\ m_s,\ m_b) \sim (\lambda^7 v_d,\ \lambda^6 v_d,\ \lambda^3 v_d). \tag{112}$$

In addition, the above choice of ζ leads to the CKM matrix[13]

$$V_{CKM} \sim \begin{pmatrix} 1 & \lambda & \lambda^5 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \tag{113}$$

at the string scale[13]. It should be noticed that in the present model the element (1, 3) of V_{CKM}, i.e., V_{ub} is suppressed compared with the element (3, 1), i.e., V_{td}.

In the charged-lepton sector the mass matrix is of the form (71) with $\mathcal{H}_{ij} \sim x^{\eta_{ij}}$, where

$$- 4\eta_{ij} + b_i + b_j + a_0 - 28 \equiv 0 \quad \text{mod } 210. \tag{114}$$

The matrix $\mathcal{H}$ becomes

$$\mathcal{H} \sim \lambda^\eta \begin{pmatrix} \lambda^8 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{pmatrix}, \tag{115}$$

where

$$\eta = \frac{1}{16}(b_3 - a_3 + a_0 - b_0) \tag{116}$$

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Table 3: Assignment of $\mathbf{Z}_{210}$-charges for matter superfields

	$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$	$\begin{pmatrix} a_0 & \overline{a} \\ b_0 & \overline{b} \end{pmatrix}$
LMA − MSW solution	$\begin{pmatrix} 78 & 62 & 30 \\ 74 & 42 & 10 \end{pmatrix}$	$\begin{pmatrix} 12 & -16 \\ -12 & -37 \end{pmatrix}$
SMA − MSW solution	$\begin{pmatrix} 72 & 56 & 24 \\ 84 & 52 & 20 \end{pmatrix}$	$\begin{pmatrix} 20 & -24 \\ -16 & -33 \end{pmatrix}$

and we put $\eta \geq 0$. Eigenstates of mass matrix (71) also contain three heavy modes with their masses $\mathcal{O}(10^{17})$GeV and three light modes. When we choose $\eta \sim 0$, light mode spectra are[14]

$$(m_e, \ m_\mu, \ m_\tau) \sim (\lambda^7 v_d, \ \lambda^{4.5} v_d, \ \lambda^2 v_d). \tag{117}$$

As seen soon later, this case corresponds to the LMA-MSW solution[20, 21]. In the case $\eta \sim 2$, light modes become[14]

$$(m_e, \ m_\mu, \ m_\tau) \sim (\lambda^{8.5} v_d, \ \lambda^5 v_d, \ \lambda^{2.5} v_d), \tag{118}$$

which corresponds to the SMA-MSW solution[20, 21]. In Table 3 we show interesting examples of assignment of $\mathbf{Z}_{210}$-charges for matter superfields. From this Table we find

$$\zeta_{00} = 2, \qquad \eta_{00} = 95, \qquad \zeta = 2.75, \qquad \eta = 0.25 \tag{119}$$

in the LMA-MSW solution and

$$\zeta_{00} = 8, \qquad \eta_{00} = 95, \qquad \zeta = 2.5, \qquad \eta = 2 \tag{120}$$

in the SMA-MSW solution.

In the neutral sector the mass matrix is given by Eq.(76) with $\mathcal{T} = 0$. The mass matrix $\mathcal{S}$ is given by

$$\mathcal{S}_{ij} \sim x^{\sigma_{ij}} \left(\frac{\langle \overline{S} \rangle}{M_S} \right)^2 = x^{\sigma_{ij}+1} \tag{121}$$

with

$$-4\sigma_{ij} + a_i + a_j + 2\overline{a} - 28 \equiv 0 \quad \mod 210. \tag{122}$$

In the LMA-, SMA-MSW solutions we obtain $\sigma_{33} = 0,\ 98$, respectively, which lead to

$$\mathcal{S} \sim x \begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \quad x^2 \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda \\ \lambda^3 & \lambda^2 & 1 \\ \lambda & 1 & \rho \end{pmatrix}, \tag{123}$$

respectively. From the above mass matrix eigenvalues of $\mathcal{S}$ are $\mathcal{O}(10^{14\sim18})$GeV and then sufficiently heavy compared to those of $\mathcal{N}$. Therefore, in seesaw mechanism an important role is played by the submatrix $\mathcal{N}$. The Majorana mass matrix $\mathcal{N}$ is given by

$$\mathcal{N}_{ij} \sim x^{\nu_{ij}} \left(\frac{\langle \overline{N^c} \rangle}{M_S} \right)^2 \sim x^{\nu_{ij}+49/4} \tag{124}$$

with

$$-4\nu_{ij} + b_i + b_j + 2\overline{b} - 28 \equiv 0 \quad \mod 210. \tag{125}$$

From Table 3 this yields $\nu_{33} = 32,\ 39$ in the LMA-, SMA-MSW solutions, respectively. Then the Majorana mass for the third generation is obtained as

$$m_{N_3^c} = \mathcal{O}(10^{11})\text{GeV (LMA)}, \qquad \mathcal{O}(10^{10})\text{GeV (SMA)}. \tag{126}$$

Thus we find the hierarchical Majorana mass matrix

$$\mathcal{N} \sim (x^{44},\ x^{51}) \times \begin{pmatrix} \lambda^8 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{pmatrix} \tag{127}$$

in the LMA-, SMA-MSW solutions, respectively. It is worth noting that Dirac mass hierarchies in the neutrino sector cancel out with the Majorana sector in large part

due to seesaw mechanism. Eigenstates of the 15×15 mass matrix (76) contain twelve heavy modes and three light modes. Light mode spectra turn out to be[14]

$$(m_{\nu 1}, \ m_{\nu 2}, \ m_{\nu 3}) \quad \sim \quad \frac{v_u^2}{m_{N_3^c}} \times (\lambda^6, \ \lambda^5, \ \lambda^4), \qquad (128)$$

$$(m_{\nu 1}, \ m_{\nu 2}, \ m_{\nu 3}) \quad \sim \quad \frac{v_u^2}{m_{N_3^c}} \times (\lambda^9, \ \lambda^6, \ \lambda^5) \qquad (129)$$

in the LMA-, SMA-MSW solutions, respectively. Furthermore, the mixing angles in the MNS matrix become[14]

$$\tan \theta_{12} \sim \sqrt{\lambda}, \quad \tan \theta_{23} \sim \sqrt{\lambda}, \quad \tan \theta_{13} \sim \lambda, \qquad (130)$$

in the LMA-MSW solution and

$$\tan \theta_{12} \sim \lambda^{1.5}, \quad \tan \theta_{23} \sim \sqrt{\lambda}, \quad \tan \theta_{13} \sim \lambda^2, \qquad (131)$$

in the SMA-MSW solution. It should be emphasized that the mixing angle $\tan \theta_{13}$ in the lepton sector becomes $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^2)$ for the LMA-, SMA-MSW solutions, respectively. Whether we obtain the LMA-MSW solution or the SMA-MSW solution depends on the Abelian flavor charge assignment. Concretely, the solutions are governed by the parameter $\eta = (b_3 - a_3 + a_0 - b_0)/16$.

In the present model the massless sector at the string scale contains extra particles beyond the minimal supersymmetric standard model. In the course of the gauge symmetry breakings many particles become massive or are absorbed by gauge fields via Higgs mechanism at the intermediate energy scales. Therefore, integrating out these heavy modes we derive the low-energy effective theory in which large extra-particle mixings cause an apparant change of the Yukawa hierarchies for leptons and down-type quarks.

6   Summary and discussion

String theory naturally provides discrete symmetries in which non-Abelian symmetry as well as Abelian R and non-R symmetries are contained as the flavor symmetry

in the four-dimensional effective theory. Dihedral group D_4 introduced here is a possible example of non-Abelian symmetry expected in the string theory. In non-commutative compact space the coordinates become non-commutative operators and then are described in terms of the projective representation of the discrete symmetries. Explicitly, we take the projective representation of D_4. Since the deformation of the compact space is expressed by functions of the coordinates, massless matter fields in the effective theory are also described in terms of the projective representation and become matrix-valued. Thus we have a new type of selection rule coming from the projective representation of the discrete symmetry. This selection rule plays an important role in explaining the phenomenological fact that the scale of Majorana mass of R-handed neutrinos is almost equal to the geometrically averaged value between M_S and M_W.

By using the dihedral flavor symmetry as well as Abelian R and non-R symmetries, we studied mass spectra and fermion mixings. Under an appropriate parametrization of the flavor charges our results come up to our expectations and are phenomenologically viable. The breaking scale of GUT-type gauge symmetry becomes $\mathcal{O}(10^{17\sim18})$GeV and the longevity of the proton is guaranteed. The scale of μ becomes $\mathcal{O}(10^{2\sim3})$GeV. Majorana mass of R-handed neutrinos for the third generation turns out to be $\mathcal{O}(10^{10\sim11})$GeV. An attractive account is also given of characteristic patterns of quark/lepton masses and mixing angles. Whether we obtain the LMA-MSW solution or the SMA-MSW solution depends on the Abelian charge parameter $\eta = (b_3 - a_3 + a_0 - b_0)/16$.

Finally, we touch upon the anomaly of the discrete symmetry $\mathbf{Z}_{MN}$. If the $\mathbf{Z}_{MN}$ symmetry arises from certain gauge symmetries and if the anomaly cancellation does not occur via the Green-Schwartz mechanism[22], the $\mathbf{Z}_{MN}$ symmetry should be nonanomalous[23, 24]. Since the gauge symmetry at the string scale is assumed to be $SU(6) \times SU(2)_R$, the mixed anomaly conditions $\mathbf{Z}_{MN} \cdot (SU(6))^2$ and $\mathbf{Z}_{MN} \cdot (SU(2)_R)^2$ are imposed on $\mathbf{Z}_{MN}$-charges of massless matter fields. In the present model the matter fields are $(\mathbf{15},\ \mathbf{1})$, $(\mathbf{6^*},\ \mathbf{2})$ and their conjugates under $SU(6) \times SU(2)_R$. Then we obtain the mixed anomaly conditions

$$4a_T + 2b_T \equiv 18q_\theta, \qquad 6b_T \equiv 26q_\theta \qquad \mathrm{mod}\ MN \tag{132}$$

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for $SU(6)$ and $SU(2)_R$, respectively, where

$$a_T = \sum_{i=0}^{3} a_i + \overline{a}, \qquad b_T = \sum_{i=0}^{3} b_i + \overline{b}. \tag{133}$$

The present parametrization with $M = 15$ and $N = 14$ turns out to be inconsistent with the above anomaly conditions. The anomaly conditions yield stringent constraints on M, N and $\mathbf{Z}_{MN}$-charge assignments for matter fields. Exploration into nonanomalous solutions with the discrete group $\mathbf{Z}_M(R) \times \mathbf{Z}_N(\mathrm{non}-R)$ together with the dihedral group D_4 will be discussed elsewhere.

Acknowledgements

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