On a Relation between Sine Formula and Radii of Circumcircles for Spherical Triangles

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Introduction

For a triangle \( \triangle ABC \) on a plane, we know well the cosine and the sine formulae:

\[
\begin{align*}
da^2 &= b^2 + c^2 - 2bc \cos A, \\
\frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. 
\end{align*}
\] (1)

On the other hand, for a spherical triangle \( \triangle ABC \) on a unit sphere, there are also the cosine and the sine formulae:

\[
\begin{align*}
\cos a &= \cos b \cos c + \sin b \sin c \cos A, \\
\frac{\sin a}{\sin A} &= \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.
\end{align*}
\] (2)

In the above formulae, we denote by \( a, b, c \) the length of the edges \( BC, CA, AB \), respectively, by \( A, B, C \) the inner angles at the vertices \( A, B, C \), respectively, and by \( R \) the radius of the circumcircle. Since every spherical triangle can be considered to lie in a half sphere, these values must satisfy \( 0 < a, b, c, A, B, C < \pi \) and \( 0 < R < \pi/2 \).

A question arises to the author. For plane triangles, there is a relation that \( a / \sin A = 2R \). Then, for spherical triangles, is \( f = \sin a / \sin A \) a function of \( R \)? — However, the answer is negative. Figure 1 and 2 show the trace of \( (R, f) \) for spherical triangles which are generated at random. Spherical triangles of Figure 1 are generated to contain its circumcenter, and those of Figure 2 not to contain it. We find from the figures that each trace is not a curve!

A further question arises to the author. What curves bound the traced areas? — On the question, Figure 3 shows the graphs of the following functions:

\[
\begin{align*}
f_1(R) &= 2\tan R \left( 1 - \frac{3}{4} \sin^2 R \right)^{\frac{3}{2}}, \\
f_2(R) &= 2\sin R \left( 1 - \frac{1}{2} \sin^2 R \right), \\
f_3(R) &= \sin 2R, \\
f_4(R) &= 2\tan R.
\end{align*}
\] (3)

It seems that the traced area of Figure 1 is bounded by \( f_1, f_2 \) and \( f_3 \), and that of Figure 2 by \( f_3 \) and \( f_4 \). The following are the answer to the question.
Theorem 1. If the circumcenter $O \in \triangle ABC$, then the following hold:

1. $f_1(R) \leq \frac{\sin a}{\sin A} \leq f_2(R) \quad \text{when } 0 < \sin R \leq \frac{22 - 2\sqrt{13}}{27}$
2. $f_3(R) < \frac{\sin a}{\sin A} \leq f_2(R) \quad \text{when } \frac{22 - 2\sqrt{13}}{27} < \sin R \leq \frac{14 - 2\sqrt{3}}{11}$
3. $f_3(R) < \frac{\sin a}{\sin A} \leq f_1(R) \quad \text{when } \frac{14 - 2\sqrt{3}}{11} < \sin R \leq 1.$

Theorem 2. If the circumcenter $O \in \triangle ABC$, then $f_1(R) < \frac{\sin a}{\sin A} < f_2(R)$.

In the paper, we will prove the above theorems by applying the method of the maximum and minimum problem to a two-variable function.

Proof of theorems

The following formula is the key to prove Theorems 1 and 2:

$$\left( \frac{\sin a}{\sin A} \right)^2 = 4 \tan^2 R \cos^2(\alpha/2) \cos^2(\beta/2) \cos^2(\gamma/2)$$

(Part 1)

Proof of (4). We denote by $L, M, N$ the midpoints of the edges $BC, CA, AB$, respectively, and by $\alpha, \beta, \gamma$ the angles $\angle BOL, \angle COM, \angle AON$, respectively. Since we have $\angle LOC = \alpha, \angle MOA = \beta$ and $\angle NOB = \gamma$, we obtain that $\alpha + \beta + \gamma = \pi$. By applying the sine formula to three triangles $\triangle OBL, \triangle OCM$ and $\triangle OAN$, we obtain that

$$\frac{\sin (\alpha/2)}{\sin \alpha} = \frac{\sin (\beta/2)}{\sin \beta} = \frac{\sin (\gamma/2)}{\sin \gamma} = \frac{\sin R}{\sin A}.$$  

(Part 2)

By using $\beta = \pi - \gamma - \alpha$, we obtain that

$$\sin^2 \gamma - \sin^2 \alpha = \sin^2 \gamma (1 - \sin^2 \alpha) - (1 - \sin^2 \gamma) \sin^2 \alpha$$

$$= \sin^2 \gamma \cos^2 \alpha - (\cos^2 \gamma - \sin^2 \alpha)^2$$

$$= \sin^2 \gamma \cos^2 \alpha - (\sin^2 \beta - \sin^2 \gamma \cos^2 \alpha)^2$$

$$= 2 \sin \beta \sin \gamma \cos \alpha - \sin^2 \beta.$$

(Part 3)

By using (6) and (5), we obtain that

$$\cos^2 \alpha = \frac{\sin^2 \beta + \sin^2 \gamma + \sin^2 \alpha}{2 \sin \beta \sin \gamma} = \frac{q^2 + r^2 - p^2}{2qr},$$

(Part 4)

where $p = \sin (\alpha/2), q = \sin (\beta/2), r = \sin (\gamma/2)$. By using (5) and (7), we obtain that
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\[
\sin^2 R = \frac{\sin^2 \frac{a}{2}}{1 - \cos \alpha} = \frac{p^2 (2 qr)^2}{(2 qr)^2 - (q^2 + r^2 - p^2)^2} = \frac{4p^2 q^2 r^2}{2p^2 q^2 + 2q^2 r^2 + 2r^2 p^2 - p^2 - q^2 - r^2}.
\]

(8)

By using it, we obtain that

\[
\tan^2 R = \frac{4p^2 q^2 r^2}{2p^2 q^2 + 2q^2 r^2 + 2r^2 p^2 - p^2 - q^2 - r^2 - 4p^2 q^2 r^2}.
\]

(9)

On the other hand, by using the cosine formula, we obtain that

\[
\left( \frac{\sin \beta}{\sin A} \right)^2 = \frac{\sin^2 \alpha}{1 - \cos A} = \frac{\sin^2 \alpha \cdot (\sin b \cdot \sin c)^2}{(\cos a - \cos b \cdot \cos c)^2} = \frac{4q^2 (1 - p^2) \cdot 4q^2 (1 - q^2) \cdot 4r^2 (1 - r^2)}{4q^2 (1 - q^2) \cdot 4r^2 (1 - r^2) - (1 - 2p^2) (1 - 2q^2) (1 - 2r^2)} = \frac{16p^2 q^2 r^2 (1 - p^2) (1 - q^2) (1 - r^2)}{2p^2 q^2 + 2q^2 r^2 + 2r^2 p^2 - p^2 - q^2 - r^2 - 4p^2 q^2 r^2}.
\]

(10)

By putting (9) to (10), we have proved (4). □

**Proof of Theorem 1.** We fix the radius \( R \) of the circumscribed circle, and put \( k = \sin R \). Then the pair \((\alpha, \beta)\) of the angles determines the shape of the triangle. Since \( O \in \triangle ABC \), all of \( \alpha, \beta, \gamma \) are acute or right. So we assume that the variables \((\alpha, \beta)\) vary in the following domain:

\[
D_1 = \{ (\alpha, \beta) \in [0, \pi/2]^2 | \alpha + \beta \geq \pi/2 \}.
\]

(11)

By putting (5) to (4), and by using \( \gamma = \pi - \alpha - \beta \), we obtain that

\[
\left( \frac{\sin \alpha}{\sin A} \right)^2 = \frac{4k}{1 - k} \left( 1 - k \sin \alpha \right) \left( 1 - k \sin \beta \right) \left( 1 - k \sin (\alpha + \beta) \right).
\]

(12)

We can regard \( f = \sin \alpha / \sin A \) as a function of \((\alpha, \beta)\), and so we denote it by \( f(\alpha, \beta) \). So we can reduce the problem to that of finding the maximum and the minimum of \( f(\alpha, \beta) \) on \( D_1 \).

To do it, we partially differentiate (12) and obtain that

\[
\frac{\partial}{\partial \alpha} \left( f(\alpha, \beta)^2 \right) = \frac{4k}{1 - k} \left( 1 - k \sin \beta \right) \left( 1 - k \sin (\alpha + \beta) \right) \left( 1 - k \frac{1 - \cos 2\alpha}{2} \right) - \frac{8k^2}{1 - k} \left( 1 - k \sin \beta \right) \left( 1 - k \sin (\alpha + \beta) \right) \left( 1 - k \sin \alpha \right) \left( 1 - k \cos (\alpha + \beta) \right).
\]

(13)

Similarly, we obtain that

\[
\frac{\partial}{\partial \beta} \left( f(\alpha, \beta)^2 \right) = -\frac{8k^2}{1 - k} \left( 1 - k \sin \alpha \right) \sin (\alpha + 2\beta) \left( \cos (\alpha - k \sin (\alpha + \beta) \sin \alpha) \right).
\]

(14)

To get the maximum or the minimum of \( f(\alpha, \beta) \), we must find the points \((\alpha, \beta)\) at which both (13) and (14) vanish. We obtain the following four cases:
Proof of Theorem 1. From So the function \( f(\alpha, \beta) \) possibly takes the maximum or the minimum on the boundary of \( D_r \), which consists of the following three line segments:

\[
l_r = \{(\alpha, \pi/2) \mid 0 \leq \alpha \leq \pi/2\}, \quad l = \{(\pi/2, \beta) \mid 0 \leq \beta \leq \pi/2 \},
\]

\[
l = \{(\alpha, \pi/2 - \alpha) \mid 0 \leq \alpha \leq \pi/2\}.
\]

For \( (\alpha, \pi/2) \in l \), we can calculate as follows:

\[
f(\alpha, \pi/2)^2 = 4k(1 - k \sin \alpha)(1 - k \cos \alpha) = 4k(1 - k + k^2 \sin^2 \alpha).
\]

So the function \( f(\alpha, \beta) \) on \( l \) takes the maximum \( f_2(R) \) at \( (\pi/4, \pi/2) \) and the minimum \( f_3(R) \) at \( (0, \pi/2) \) and \( (\pi/2, \pi/2) \). Similar conclusions hold on \( l_1 \) and \( l_2 \).

Therefore we have three values \( f_1(R), f_2(R), f_3(R) \) and 1 (only when \( R \geq \pi/4 \)) as the candidates for the maximum and the minimum. By comparing these values, we have proved the inequalities. Finally, we remark that, since each of \( (0, \pi/2) \), \( (\pi/2, \pi/2) \) and \( (\pi/2, 0) \) cannot make any triangles, the left inequalities of (2) and (3) must be strict. □

**Proof of Theorem 2.** Since \( O \in \triangle ABC \), one of \( \alpha, \beta, \gamma \) is obtuse, say, \( \gamma = \pi - \alpha - \beta \geq \pi/2 \). So we assume that the variables \( \langle \alpha, \beta \rangle \) vary in the following domain:

\[
D_\gamma = \{(\alpha, \beta) \in [0, \pi/2] \mid \alpha + \beta \leq \pi/2\}.
\]

There are no points in \( D_\gamma \) at which both (13) and (14) vanish. So the function \( f(\alpha, \beta) \) must take the maximum and the minimum on the boundary of \( D_\gamma \), which consists of the following three line segments:

\[
l_r, l = \{(\alpha, 0) \mid 0 \leq \alpha \leq \pi/2\}, \quad l_\gamma = \{(0, \beta) \mid 0 \leq \beta \leq \pi/2\}.
\]

For \( (\alpha, 0) \in l_r \), we can calculate as follows:

\[
f(\alpha, 0)^2 = \frac{4k}{1 - k}(1 - k \sin \alpha)^2.
\]
So the function \( f(\alpha, \beta) \) on \( l_i \) takes the maximum \( f_i(R) \) at \((0, 0)\) and the minimum \( f_i(R) \) at \((\pi/2, 0)\). A similar conclusion holds on \( l_i \). Moreover, the function \( f(\alpha, \beta) \) on \( l_i \) takes the maximum \( f_i(R) \) at \((\pi/4, \pi/4)\) and the minimum \( f_i(R) \) at \((0, \pi/2)\) and \((\pi/2, 0)\).

Therefore we have three values \( f_1(R), f_2(R) \) and \( f_3(R) \) as the candidates for the maximum and the minimum. By comparing these values, we have proved the inequalities. Finally, we remark that, since each of \((0, 0), (\pi/2, 0)\) and \((0, \pi/2)\) can not make any triangles, the right and the left inequalities must be strict. □

References


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