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On Generalized Trigonometric Functions
<修士論文要旨>

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1　Introduction

We give a definition of generalized trigonometric functions. The notations are due to Lang and Edmunds [1]. For real numbers $p, q > 1$, we define a function F_{pq} by

$$F_{pq}(y) = \int_0^y (1 - t^q)^{-1/p} dt, \tag{1.1}$$

where $y \in [0, 1]$. We introduce generalized pi's as follows:

$$\pi_{pq} = 2F_{pq}(1) = \frac{2}{q} B\Big(\frac{1}{p^*}, \frac{1}{q}\Big) = \frac{p^*}{q} \pi_{q^*p^*}, \tag{1.2}$$

where $1/p^* = 1 - 1/p$ and $B(.,.)$ denotes the beta function. We define a function $y = \sin_{pq} t$ on $[0, \pi_{pq}/2]$ by the inverse function of $t = F_{pq}(y)$, called generalized sine function. We define a function $x = \cos_{pq} t$ on $[0, \pi_{pq}/2]$ by the following equality:

$$\cos_{pq} t = (1 - (\sin_{pq} t)^q)^{1/p}, \tag{1.3}$$

called generalized cosine function. We can define $\sin_{pq} t$, $\cos_{pq} t$ on the whole real numbers by the following equalities:

$$\begin{aligned} &\sin_{pq} t = \sin_{pq}(\pi_{pq} - t),\ \sin_{pq} t = -\sin_{pq}(-t), \\ &\cos_{pq} t = -\cos_{pq}(\pi_{pq} - t),\ \cos_{pq} t = \cos_{pq}(-t). \end{aligned} \tag{1.4}$$

When $p = q$, we abbreviate $\sin_{pp} t$ to $\sin_p t$, $\cos_{pp} t$ to $\cos_p t$ and π_{pp} to π_p. When $p = q = 2$, the functions $\sin_{pq} t$, $\cos_{pq} t$ and π_{pq} are obviously reduced to the usual sin, cos and pi.

2　Properties of Generalized Trigonometric Functions

The functions $\sin_{pq} t, \cos_{pq} t$ satisfy the following proposition.

Proposition 2.1. For every $t \in [0, \pi_{pq}/2]$, the following equalities hold:

(i) $(\sin_{pq} t)' = \cos_{pq} t$, (ii) $(\cos_{pq} t)' = -\dfrac{q}{p}(\sin_{pq} t)^{q-1}(\cos_{pq} t)^{-p+2}$,

(iii) $(\cos_{pq} t)^p + (\sin_{pq} t)^q = 1$, (iv) $\cos_{pq} t = \left(\sin_{q^*p^*}\left(\dfrac{\pi_{q^*p^*}}{\pi_{pq}}\left(\dfrac{\pi_{pq}}{2} - t\right)\right)\right)^{p^*-1}$.

3 Properties of Generalized Pi's

We know some equalities containing two generalized pi's as follows:

$$\text{(a)}\ \frac{\pi_{q^*,p^*}}{\pi_{p,q}} = \frac{q}{p^*}, \quad \text{(b)}\ \frac{\pi_{p^*,p}}{\pi_{2,p}} = 2^{-2/p+1}. \tag{3.1}$$

The first equality is already mentioned in Eq.(1.2), and the second can be proved directly from Legendre Duplication Formula. (It is already pointed out by Takeuchi [5].) In this paper, we give other relations containing two generalized pi's.

Theorem 3.1 ([2]). For every real number $p \in (1, \infty)$, the following equalities hold:

$$\text{(i)}\ \frac{\pi_{2p^*,2p}}{\pi_{p^*,p}} = 2^{1/p-1}, \ \text{(ii)}\ \frac{\pi_{p^*,2p^*}}{\pi_{p,2p}} = (p-1)2^{2/p-1}, \ \text{(iii)}\ \frac{\pi_{2p^*,p^*}}{\pi_{2p,p}} = 2^{-2/p+1}.$$

4 Generalized Elliptic Integrals of Four Parameters.

4.1 Generalized Legendre Relation.

Takeuchi [6] defined generalized elliptic integrals of three parameters. In this paper, we define generalized elliptic integrals of four parameters as follows:

$$E_{pqrs}(k) = \int_0^1 \frac{(1-k^s t^q)^{1/r}}{(1-t^q)^{1/p}} dt, \ K_{pqrs}(k) = \int_0^1 \frac{(1-k^s t^q)^{1/r-1}}{(1-t^q)^{1/p}} dt.$$

Obviously, the following equation holds:

$$E_{pqrs}(0) = K_{pqrs}(0) = \int_0^1 \frac{1}{(1-t^q)^{1/p}} dt = \frac{\pi_{pq}}{2}. \tag{4.1}$$

The following formula is Generalized Legendre Relations of four parameters.

Theorem 4.1. Let $p \in (-\infty, 0) \cup (1, \infty)$, $q, r \in (1, \infty)$. For every $k \in [0, 1]$, we denote $k' = (1-k^s)^{1/s}$. Then the following equality holds:

$$E_{pqrs}(k)K_{prqs}(k') + K_{pqrs}(k)E_{prqs}(k') - K_{pqrs}(k)K_{prqs}(k') = \frac{\pi_{pq}\pi_{\sigma r}}{4}, \tag{4.2}$$

where $1/\sigma = 1/p - 1/q$.

5 Similar Results to Salamin-Brent Formula

5.1 Similar Results to Gauss AGM Formula

Gauss found an important formula concerning elliptic integrals and arithmetic-geometric mean.

Theorem 5.1 (Gauss). For $a_0 \geqq b_0 > 0$, we define two sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$a_{n+1} = \frac{a_n + b_n}{2}, \ b_{n+1} = \sqrt{a_n b_n}. \tag{5.1}$$

Then two sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit. We denote it by $M_2(a_0, b_0)$ and called the arithmetic-geometric mean (AGM) of a_0 and b_0. Then the following formula holds:

$$\frac{1}{a_0} K_{2222}\Big(\frac{c_0}{a_0}\Big) = \frac{\pi/2}{M_2(a_0, b_0)}. \tag{5.2}$$

J. M. Borwein and P. B. Borwein [4] found two formulas which can give analogous results to Gauss's AGM Formula. Recently, Takeuchi [6], [7] made their theorems those of generalized pi's.

Theorem 5.2 (Borwein-Takeuchi). For $a_0 \geqq b_0 > 0$, we define three sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ as follows:

$$a_{n+1} = \frac{a_n + 2b_n}{3}, \ c_{n+1} = \frac{a_n - b_n}{3}, \ b_n^3 = a_n^3 - c_n^3. \tag{5.3}$$

Then two sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit. We denote it by $M_3(a_0, b_0)$. Then the following formula holds:

$$\frac{1}{a_0} K_{3333}\Big(\frac{c_0}{a_0}\Big) = \frac{\pi_3/2}{M_3(a_0, b_0)}. \tag{5.4}$$

Theorem 5.3 (Borwein-Takeuchi). For $a_0 \geqq b_0 > 0$, we define three sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ as follows:

$$a_{n+1} = \frac{a_n + 3b_n}{4}, \ c_{n+1} = \frac{a_n - b_n}{4}, \ b_n^2 = a_n^2 - c_n^2. \tag{5.5}$$

Then two sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit. We denote it by $A_4(a_0, b_0)$. Then the following formula holds:'

$$\frac{1}{\sqrt{a_0}} K_{4442}\Big(\frac{c_0}{a_0}\Big) = \frac{\pi_4/2}{\sqrt{A_4(a_0, b_0)}}. \tag{5.6}$$

5.2 Salamin-Brent-Like Formula

In 1985-86, Salamin and Brent independently found a fast convergence formula for computing the value of π. The following is the Salamin-Brent Formula.

Theorem 5.6 (Salamin-Brent Formula). Let $a_0 = 1$ and $b_0 = 1/\sqrt{2}$, then

$$\pi = \frac{2\big(M_2\big(1, 1/\sqrt{2}\big)\big)^2}{\dfrac{1}{2} - \displaystyle\sum_{j=1}^{\infty} 2^j (a_j^2 - b_j^2)},$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences defined by Eq.(5.1).

Recently, Takeuchi [6], [7] found two Salamin-Brent-like formulas for π_3 and π_4.

Theorem 5.7 (Takeuchi [6]). Let $a_0 = 1$ and $b_0 = 1/2^{1/3}$, then

$$\pi_3 = \frac{2\big(M_3\big(1, 1/2^{1/3}\big)\big)^2}{1 - 2\displaystyle\sum_{j=1}^{\infty} 3^j (a_j + c_j)c_j},$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are the sequences defined by Eq.(5.3).

Theorem 5.8 (Takeuchi [7]). Let $a_0 = 1$ and $b_0 = 1/\sqrt{2}$, then

$$\pi_4 = \frac{2A_4\big(1, 1/\sqrt{2}\big)}{1 - \displaystyle\sum_{j=0}^{\infty} 2^j (a_j - b_j)},$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences defined by Eq.(5.5).

The statement of the above result is different from that of Takeuchi [7]. However, it is the same. In the past few months, the author has tried to find Salamin-Brent-like formula for another π_{pq}. Unfortunately, I could not find it. However, I found a simpler proof of Theorem 5.8.

5.3 Proof of Theorem 5.8

To prove Theorem 5.8, we consider the case when $p = q = r = 4$, $s = 2$. That is,

$$\begin{aligned} K_{4442}(k) &= \int_0^1 (1 - k^2 t^4)^{-3/4} (1 - t^4)^{-1/4} dt, \\ E_{4442}(k) &= \int_0^1 (1 - k^2 t^4)^{1/4} (1 - t^4)^{-1/4} dt. \end{aligned} \tag{5.13}$$

From now, we abbreviate the suffices of K_{4442} and E_{4442}. By Theorem 4.1, we obtain the following corollary .

Corollary 5.9. For every $k \in (0,1)$, we denote $k' = \sqrt{1-k^2}$. Then the following equality holds:

$$E(k)K(k') + K(k)E(k') - K(k)K(k') = \frac{\pi_4}{2}.$$

Lemma 5.10. For every real number $k \in (0,1)$, the following equalities hold:

(i) $K(k) = \dfrac{1}{\sqrt{1+3k}} K(m'),$ where $m = \dfrac{1-k}{1+3k}.$

(ii) $E(k) = \dfrac{\sqrt{1+3k}}{2} E(m') + \dfrac{1-k}{2\sqrt{1+3k}} K(m').$

Outline of Proof. (i) We can show that the left- and right-hand sides of (i) satisfy the same differential equation. By using it, we can prove (i).

(ii) We can prove it by differentiating both sides of (i). □

By putting $k = \dfrac{c_{n+1}}{a_{n+1}}$ into Lemma 5.10, we obtain the following lemma.

Lemma 5.11. If $\{a_n\}$ and $\{c_n\}$ are the sequences defined by Eq.(5.5), then the following equalities hold:

(i) $\dfrac{1}{\sqrt{a_{n+1}}} K\Big(\dfrac{c_{n+1}}{a_{n+1}}\Big) = \dfrac{1}{\sqrt{a_n}} K\Big(\dfrac{c_n}{a_n}\Big),$

(ii) $2\sqrt{a_{n+1}} E\Big(\dfrac{c_{n+1}}{a_{n+1}}\Big) = \sqrt{a_n} E\Big(\dfrac{c_n}{a_n}\Big) + \dfrac{b_n}{\sqrt{a_n}} K\Big(\dfrac{c_n}{a_n}\Big).$

Theorem 5.12. If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are the sequences defined by Eq.(5.5) and $a_0 = a$, $b_0 = b$, $c_0 = c$. Then the following formula holds:

(i) $\dfrac{1}{\sqrt{a}} K\Big(\dfrac{c}{a}\Big) = \dfrac{\pi_4/2}{\sqrt{A_4(a,b)}},$ (ii) $E\Big(\dfrac{c}{a}\Big) = \Big\{1 - \dfrac{1}{a}\displaystyle\sum_{j=1}^{\infty} 2^j c_j\Big\} K\Big(\dfrac{c}{a}\Big).$

Outline of Proof. (i) By repeating Lemma 5.11 (i), and taking limit $n \to \infty$, we obtain that

$$\frac{1}{\sqrt{a}} K\Big(\frac{c}{a}\Big) = \lim_{n\to\infty} \frac{1}{\sqrt{a_n}} K\Big(\frac{c_n}{a_n}\Big) = \frac{1}{\sqrt{A_4(a,b)}} K(0) = \frac{\pi_4/2}{\sqrt{A_4(a,b)}}.$$

(ii) We can show that, by repeating Lemma 5.11 (ii), we obtain that

$$\sqrt{a}\Big\{E\Big(\frac{c}{a}\Big) - K\Big(\frac{c}{a}\Big)\Big\} = 2^n\sqrt{a_n}\Big\{E\Big(\frac{c_n}{a_n}\Big) - K\Big(\frac{c_n}{a_n}\Big)\Big\} - \frac{1}{\sqrt{a}}\sum_{j=1}^{n} 2^j c_j K\Big(\frac{c}{a}\Big).$$

Then we can prove that $2^n\sqrt{a_n}\Big\{K\Big(\dfrac{c_n}{a_n}\Big) - E\Big(\dfrac{c_n}{a_n}\Big)\Big\}$ vanishes when $n \to \infty$. By using it, we can prove (ii). □

Proof of Theorem 5.8. By putting $k = c/a = 1/\sqrt{2}$, we have that $k' = 1/\sqrt{2}$. By putting them into Corollary 5.9 we obtain that

$$2K\Big(\frac{1}{\sqrt{2}}\Big)E\Big(\frac{1}{\sqrt{2}}\Big) - \Big(K\Big(\frac{1}{\sqrt{2}}\Big)\Big)^2 = \frac{\pi_4}{2}. \tag{5.23}$$

By Theorem 5.12, we obtain that

$$K\Big(\frac{1}{\sqrt{2}}\Big) = \frac{\pi_4/2}{\sqrt{A_4\Big(1, \frac{1}{\sqrt{2}}\Big)}},\ E\Big(\frac{1}{\sqrt{2}}\Big) = \Big\{1 - \sum_{j=1}^{\infty} 2^j c_j\Big\}K\Big(\frac{1}{\sqrt{2}}\Big). \tag{5.24}$$

By substituting Eq.(5.24) into (5.23), we obtain that

$$\Big\{1 - 2\sum_{j=1}^{\infty} 2^j c_j\Big\}K\Big(\frac{1}{\sqrt{2}}\Big)^2 = \frac{\pi_4}{2}.$$

So Theorem 5.8 is proved. □

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