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# MAXIMA AND MINIMA IN PLANE GEOMETRY

<修士論文要旨>

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## Contents



## 0 Introduction

The idea of maxmima and minima is important in mathematics education. The student must understand the maximun and minimum from elementary to high. We have to think max and mini but also in a daily life not only school.

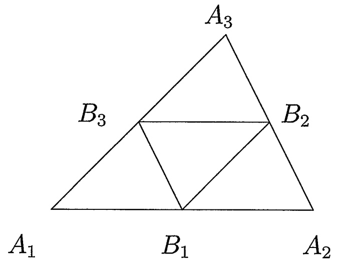
The purpose of the thesis is to investigate the importance of maxmima and minima in mathematics education, and as an example, we investigate the ratio of areas of pentagon and its midpoint pentagon.

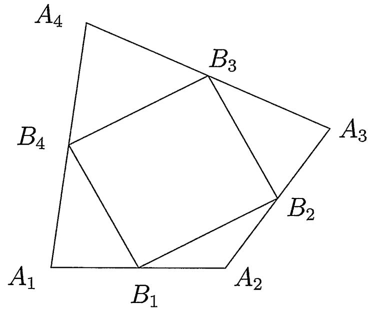
## 1 Mid point polygon

For a polygon $A_1A_2A_3\cdots A_n$, let $B_i$ be the midpoint of side $A_iA_{i+1}$ $(i = 1, 2, \cdots, n)$. Then we obtain new polygon $B_1B_2B_3\cdots B_n$, which we call the *mid-point polygon* of $A_1A_2A_3\cdots A_n$.

There is famous theorem in Elementary Geometry.

(1) The midpoint polygon divides the triangular region up into four congruent triangles so the area ratio is $1/4$.
(2) The midpoint polygon of quadrilateral is a parallelogram and the area ratio is always $1/2$.

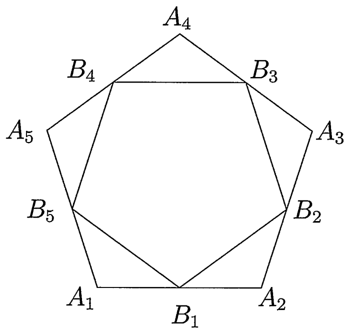



Although for triangles and quadrilaterals we can find easily the ratio of areas of mid point polygon and the original polygon, for pentagons, it is not easy to solve it.

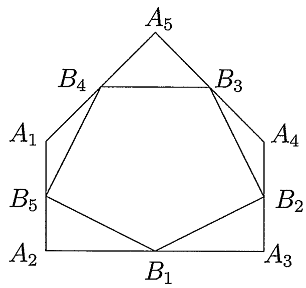
For pentagons, the ratio of the area $T$ of mid point polygon and the area $S$ of original pentagon, is not constant.

Regular pentagons




$$\frac{T}{S} = \frac{(B_1B_2)^2}{(A_1A_2)^2} \sim \frac{0.6545}{1}$$

Case of another pentagon




$$\frac{T}{S} = 12 \times \frac{4}{3} = 0.6666\cdots$$

## 2 Problem

**Problem.**
Find the maximun and the minimum of ratio $T/S$ of the areas of mid point polygon and original pentagon.

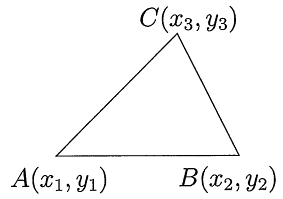
It is difficult to find the ratio of area of pentagon and its mid-point pentagon for general pentagon.

## 3 Degree of Freedom

The *degrees of freedom* in a problem is the number of parameters which may be independently varied.

**Examples.** The degree of freedom of the triangles in plane is 6. Because we need to decide point $A$, pont $B$ and point $C$ with $x_1, y_1, x_2, y_2, x_3, y_3$.

Similarly, the degree of freedom of the quadrilaterals in the plane is 8, and the degree of freedom of the pentagons in the plane is 10.




The degree of freedom of the regular triangles in plane is 4. Because when we choose two points $A, B$ with $x_1, y_1, x_2, y_2$, we can decide the last point $x_3, y_3$ by using angle 60° and length $AB$.

The same way, the degree freedom of the squares, we need 4, and of rectrangles, we need 4 plus 1 (the height). And the degree of freedom of the regular pentagons in the plane is 4.

## 4 Plan for solving the problem

The degree of freedom of the pentagon in the plane is 10. It is difficult to find the ratio of area of pentagon and its mid-point pentagon for general pentagon. Therefore we chooes 3 vertices to be fixed. Then the degree of freedom of pentagons we must consider reduces to 4, and the problem will be easy to solve.

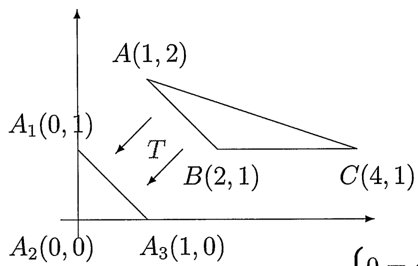
(1) **Reduce of degree of freedom using Linear Transformation**

(2) **Estimate the value of ratio using Computer**

## 5 Reducing of degree of freedom using Linear Transformation

Any triangle $ABC$ can be mapped to the triangle$A_1A_2A_3$ by using linear transformation.

**Example**




To find linear transformation

$$T : \begin{cases} x' = ax + by + p \\ y' = cx + dy + q \end{cases}$$

such that

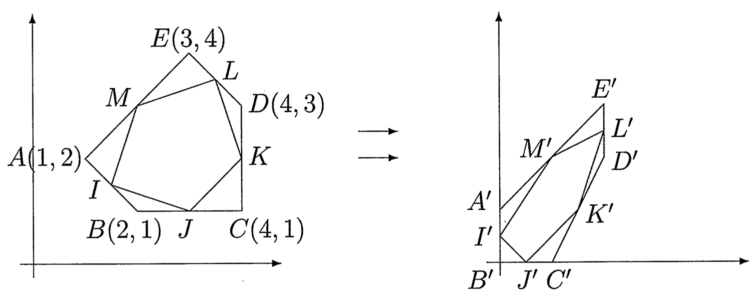
1. $T(A) = A_1$
2. $T(B) = A_2$
3. $T(C) = A_3$

is equivalent to find $a, b, c, d, p, q$

such that $\begin{cases} 0 = a + 2b + p \\ 1 = c + 2d + q \end{cases}$ $\begin{cases} 0 = 2a + b + p \\ 0 = 2c + d + q \end{cases}$ $\begin{cases} 1 = 4a + b + p \\ 0 = 4c + d + q \end{cases}$

Similarly, pentagon $ABCDE$ can be mapped to the pentagon $A'B'C'D'E'$ by using linear transformation.

**Example**



**Theorem**

Linear transformation does not change the ratio of areas.

$$T'/S' = T/S$$

This theorem follows from following lemma.

**Lemma**

By linear transformation $\begin{cases} x' &= ax + by + p \\ y' &= cx + dy + q \end{cases}$, square becomes parallelogram whose area is $|ad - bc|$ times the area of the square.

Since area of any figure is measured by square, the area of the figure after the above linear transformation is also $|ad-bc|$ times the area of original figure. Therefore the ratio of areas of any two figures is not changed by the linear transformation.

## 6 Level line and Gradient vector

The *level line* of a function is a line (curve) on which the value of the function remains unchanged.

The *gradient vector* of a function $f$ at a point $P$ is the vector with components

$$\Big(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)\Big),$$

and it is perpendicular to the level line of $f$ passing through $P$.

We apply these ideas to investigate the ratio of area of mid point polygon and the ratio of area of origin polygon.

## 7 On the Ratio $T/S$

Let a pentagon $ABCDE$ be given, and $IJKLM$ be its mid-point pentagon. We want to investigate the ratio of areas of $IJKLM$ and $ABCDE$. Since the degree of freedom of pentagons is 10, that is, the ratio is a function of 10 variables, it is not easy to find the maximum and minimum by calculus in usual way. We reduce the problem to a problem of function of two variables.

From the study above, we know that a linear transformation preserves the ratio of areas of two pentagons. So we can assmue that

$$A = (0,1), \qquad B = (0,0), \qquad C = (1,0)$$

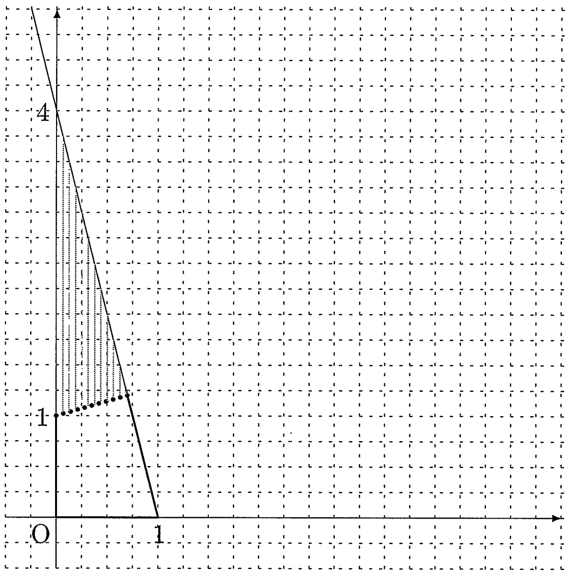
The problem is reduced to a problem of function of four variables. If we fix two variables, then we get function of two variables. Let $D = (a,b)$ and $E = (x,y)$, and we fix $a$ and $b$. Then the ratio $T/S$ becomes a function of two variable $x$ and $y$. So we can see level lines and grandient vectors on the screen of computer.

## 8 Domain of the function $T/S$

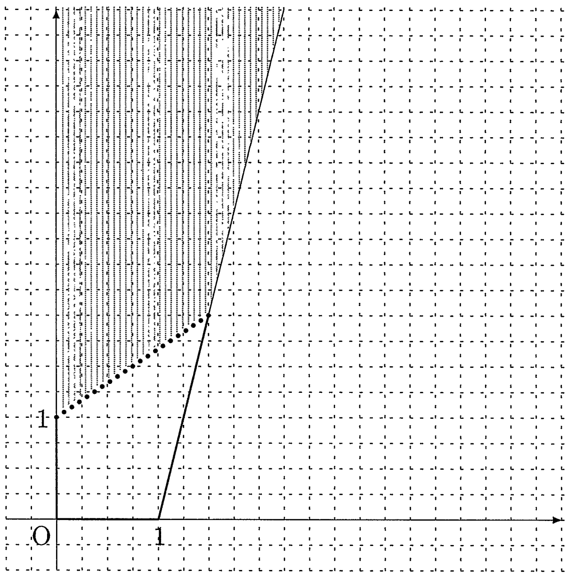
We consider only convex polygons. So the domain of the function $T/S$ is

(1) a triangle when $a < 1$ ;
(2) infinite region bounded by
segment $AD$,
ray(=half-line) from $A$ with direction $(0, 1)$,
and ray from $D$ with direction $(a - 1, b)$, when $a \geq 1$.

**Example** (1) at $a = 0.7, b = 1.2$




**Example** (2) at $a = 1.5, b = 2$



## 9 Expriments

We investigate the value of ratio $T/S$ using BASIC program.

### 10 Results and Conjecture

**Table of the values $T/S$**

| $a, b$ | $\frac{2a+1}{4a}$ | $\frac{2a+3b-2)}{4(a+b-1)}$ | $\frac{3a+2b}{4(a+b)}$ | $\frac{3a+3b-1}{4(a+b)}$ |
|---|---|---|---|---|
| $1, 0.5$ | 0.75000 | 0.75000 | 0.66667 | 0.58333 |
| $2, 0.5$ | 0.62500 | 0.58333 | 0.70000 | 0.65000 |
| $3, 0.5$ | 0.58333 | 0.55000 | 0.71429 | 0.67857 |
| $4, 0.5$ | 0.56250 | 0.53571 | 0.72222 | 0.69444 |
| $5, 0.5$ | 0.55000 | 0.52778 | 0.72727 | 0.70455 |
| $1, 1$ | 0.75000 | 0.75000 | 0.62500 | 0.62500 |
| $2, 1$ | 0.62500 | 0.62500 | 0.66667 | 0.66667 |
| $3, 1$ | 0.58333 | 0.58333 | 0.68750 | 0.68750 |
| $4, 1$ | 0.56250 | 0.56250 | 0.70000 | 0.70000 |
| $5, 1$ | 0.55000 | 0.55000 | 0.70833 | 0.70833 |
| $1, 2$ | 0.75000 | 0.75000 | 0.58333 | 0.66667 |
| $2, 2$ | 0.62500 | 0.66667 | 0.62500 | 0.68750 |
| $3, 2$ | 0.58333 | 0.62500 | 0.65000 | 0.70000 |
| $4, 2$ | 0.56250 | 0.60000 | 0.66667 | 0.70833 |
| $5, 2$ | 0.55000 | 0.58333 | 0.67857 | 0.71429 |

| $a, b$ | $\frac{2a+1}{4a}$ | $\frac{2a+3b-2)}{4(a+b-1)}$ | $\frac{3a+2b}{4(a+b)}$ | $\frac{3a+3b-1}{4(a+b)}$ |
|---|---|---|---|---|
| $1, 3$ | 0.75000 | 0.75000 | 0.56250 | 0.68750 |
| $2, 3$ | 0.62500 | 0.68750 | 0.60000 | 0.70000 |
| $3, 3$ | 0.58333 | 0.65000 | 0.62500 | 0.70833 |
| $4, 3$ | 0.56250 | 0.62500 | 0.64286 | 0.71429 |
| $5, 3$ | 0.55000 | 0.60714 | 0.65625 | 0.71875 |
| $1, 4$ | 0.75000 | 0.75000 | 0.55000 | 0.70000 |
| $2, 4$ | 0.62500 | 0.70000 | 0.58333 | 0.70833 |
| $3, 4$ | 0.58333 | 0.66667 | 0.60714 | 0.71429 |
| $4, 4$ | 0.56250 | 0.64286 | 0.62500 | 0.71875 |
| $5, 4$ | 0.55000 | 0.62500 | 0.63889 | 0.72222 |
| $1, 5$ | 0.75000 | 0.75000 | 0.54167 | 0.70833 |
| $2, 5$ | 0.62500 | 0.70833 | 0.57143 | 0.71429 |
| $3, 5$ | 0.58333 | 0.67857 | 0.59375 | 0.71875 |
| $4, 5$ | 0.56250 | 0.65625 | 0.61111 | 0.72222 |
| $5, 5$ | 0.55000 | 0.63889 | 0.62500 | 0.72500 |

**Level Lines and Gradient Vectors of $T/S$**

**Conjecture 1**
(1) When $b < 1$ and $a = 1$, Gradient vectors look like circles with common center, and level lines pass through the center. The coordinates of the center is $(c, 0)$ with $c < 0$.
(2) When $b = 1$, Gradient vectors are vertical and level lines become horizontal. Direction of gradient vectors is
upward if $a < 1.618$, downward if $a > 1.619$
(3) When $b > 1$ and $a = 1$, Gradient vectors look like circles with common center, and level lines pass through the center. The coordinates of the center is $(c, 0)$ with $c > 1$.

**Conjecture 2**
The value of boundary between "upward" and "downward" will be the golden ratio $\frac{1+\sqrt{5}}{2} = 1.6180339887...$

The conjectures can be proved by calculation.

## References

[1] F. Klein: "Elementary mathematics from an advanced standpoint", Dover, 2004
[2] G. Polya: "Induction and analogy in mathematics", Princeton University Press, 1954
[3] G. Polya: "How to solve it", Princeton University Press, 1957