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Generalized Gergonne's Trick and its Continuous Approximation

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1. A motivation from a card trick

Suppose that you have eight cards in hand and choose a card from them in your mind. Hold the deck face-up in one hand, and deal it into two piles with face up, like left, right, left, right.... After that pick the pile containing the chosen card with face up, and stack the remaining pile face-up on it. Repeat this process three times. Then hold the deck face-down and turn off the top card. Is it the card you choose? - NO. Turn off the next. Is it? - NO. Do it again. Is it? - YES! Figure 1 illustrates the process, where the number 1 is the target.

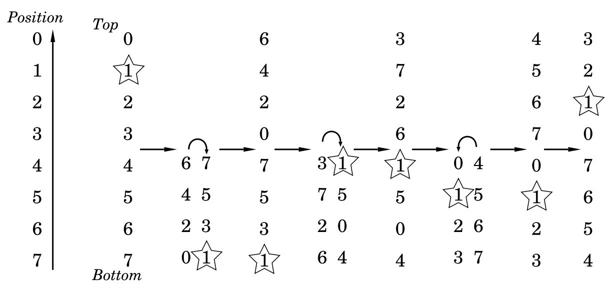

FIGURE 1. 8-card trick (the number 1 is the target.)

As you see, this trick is a variation of the well-known *21-card trick* or *Gergonne's trick*, which was observed firstly by J. D. Gergonne[6]. It may be natural to understand this trick as an existence of an attracting fixed point of some discrete dynamical system. Actually the behavior of the target card is emulated by the map $T : \{0, 1, \ldots, 7\} \to \{0, 1, \ldots, 7\}$,

$$T(x) = \begin{cases} 7 - \frac{x}{2}, & \text{if the target is in the left pile, i.e., its position } x \text{ is even,} \\ 7 - \frac{x-1}{2}, & \text{if the target is in the right pile, i.e., its position } x \text{ is odd,} \end{cases}$$
$$= 7 - \left\lfloor \frac{x}{2} \right\rfloor,$$

which has the unique stable fixed point $x = 5$, third from the bottom of the deck. Gergonne's trick is generalized to p-pile version, where the number n of cards is multiple of p [3][7], however, in this paper we prove the existence of a unique stable fixed point or a unique doubly periodic point for arbitrary n and p (Theorem 2.7): it is not necessary that the p piles are of the same number of cards. For instance, in the case of 11-card with two piles, the target is fixed at fourth from the bottom, while in the case of 9-card with two piles, the target appears at third and fourth from the bottom alternately.

Such a kind of card tricks are often treated as a combinatorial matter or a phenomenon caused by some discrete dynamics. In [7], p-pile problem with $n = hp$ cards has been translated into a dynamics of $f : x \mapsto \left\lfloor \dfrac{x+r}{p} \right\rfloor$, which has been analysed in combinatorial manner. In contrast to their approach, we introduce a suitable continuous map which approximates the piecewise constant map associated with the p-pile problem like $T(x)$ stated above, so that the orbit of each x under the continuous map accompanies the orbit under the piecewise constant one. As a result, the global stability of the fixed point of the continuous map is inherited to the fixed point of the piecewise constant map.

2. Continuous approximation

Let $p < n$ be natural numbers with $n = Ap + B$, $0 \leq B < p$. We deal n pieces of cards into p piles and stack them again in following way: hold the deck face-up, and deal it face-up from the top to the bottom; deal each p pieces of cards (and B pieces of cards at the last turn) into p piles in random order. Then pick the pile face-up which contains the target card and stack the remaining piles face-up on it. We call this process *p-shuffle*. p-shuffle causes a permutation on $\{0, \ldots, n-1\}$, where we assign $k-1$ to the k-th card from the top of the deck. Then the target is traced by a map $T : \{0, \ldots, n-1\} \to \{0, \ldots, n-1\}$ as follows.

Proposition 2.1. *p-shuffle moves the target at the position $x \in \{0, 1, \ldots, n-1\}$ to the position at*

$$T(x) = n - 1 - \left\lfloor \frac{x}{p} \right\rfloor. \tag{2.1}$$

Proof. Since any $x \in \{0, \ldots, n-1\}$ is written as $x = kp + q$ where $k = \left\lfloor \frac{x}{p} \right\rfloor$ and $0 \leq q < p$, the target appears after k times repetition of dealing p cards to p piles. Thus the target is stacked at the $k+1$-th from the bottom of the pile. As p-shuffle arranges the pile containing the target at the bottom of the deck, the target is at the $(k+1)$-th from the bottom of the deck. Thus $T(x) = n - 1 - k$. □

It is convenient for our discussion to change coordinates as

$$S(x) = n - 1 - T(n - 1 - x) = \left\lfloor \frac{n - 1 - x}{p} \right\rfloor, \tag{2.2}$$

and we introduce a 'continuous approximation'

$$F(x) = \frac{n - 1 - x}{p} \tag{2.3}$$

of $S(x)$.

Let us observe how the trajectory of $F(x)$ escorts that of $S(x)$. For any $x \in \mathbf{R}$, $\overline{x}$ denotes the maximum

$$\overline{x} = \max\{x' \mid S(x') = S(x)\}.$$

By definition, for any $\epsilon > 0$ we have

$$S(\overline{x} + \epsilon) = \left\lfloor \frac{n - 1 - (\overline{x} + \epsilon)}{p} \right\rfloor < \left\lfloor \frac{n - 1 - \overline{x}}{p} \right\rfloor = S(\overline{x}),$$

which means $S(\overline{x}) = \left\lfloor \frac{n - 1 - \overline{x}}{p} \right\rfloor = \frac{n - 1 - \overline{x}}{p} = F(\overline{x})$ and $\overline{x} \in \mathbf{Z}$. Also we see

$$\{x' \mid S(x') = S(x)\} = (\overline{x} - p, \overline{x}] \tag{2.4}$$

and $S(\overline{x} - p) = S(\overline{x}) + 1 = F(\overline{x} - p)$.

Proposition 2.2. *Take $a, b \in \mathbf{R}$ with $|a - b| < 1$, and put $a_k = F^k(a), a_0 = a$ and $b_k = S^k(b), b_0 = b$. Then $|a_k - b_k| < 1$ holds for any $k \in \mathbf{N}$. In particular,*

$$|F^k(x) - S^k(x)| < 1$$

holds for any $x \in \mathbf{R}$ and $k \in \mathbf{N}$.

Proof. Suppose that $|a_k - b_k| < 1$ holds for some $k \geq 0$. We see $S(b_k) = S(\overline{b}_k) = F(\overline{b}_k)$ by definition. It follows from $b_k \in (\overline{b}_k - p, \overline{b}_k]$ and $b_k, \overline{b}_k \in \mathbf{Z}$ that $|b_k - \overline{b}_k| \leq p - 1$. Thus we have

$$\begin{aligned} |a_{k+1} - b_{k+1}| &= |F(a_k) - S(b_k)| = |F(a_k) - F(\overline{b}_k)| = \frac{1}{p}|a_k - \overline{b}_k| \\ &\leq \frac{1}{p}\left(|a_k - b_k| + |b_k - \overline{b}_k|\right) < \frac{1}{p}(1 + p - 1) = 1, \end{aligned}$$

hence the assertion for a_k's and b_k's. Since $F(x) - S(x)$ gives the fractional part of $F(x)$, the assertion holds in the case of $a = b = x$. $\square$

The dynamics $F(x)$ has a unique fixed point $x^* = \dfrac{n-1}{p+1}$. As $F(x)$ is decreasing, $x^* < x$ (resp. $x < x^*$) leads $F(x) < F(x^*) = x^* < x$ (resp. $x < x^* < F(x)$). It comes from the equation $|F(x) - F(x^*)| = \dfrac{1}{p}|x - x^*|$ for any x that

$$|F^k(x) - F^k(x^*)| = \frac{1}{p}|F^{k-1}(x) - F^{k-1}(x^*)| = \cdots = \frac{1}{p^k}|x - x^*|,$$

which shows x^* is globally stable. Similar assertion holds on $S(x)$, however, whose discreteness causes a little bit modification as follows.

Proposition 2.3. *For $x^\circ = \left\lfloor \dfrac{n-1}{p+1} \right\rfloor$, $S(x^\circ) = x^\circ$ holds when $n \not\equiv 0 \pmod{p+1}$. When $n \equiv 0 \pmod{p+1}$, we have $S(x^\circ) = x^\circ + 1$ and $S(x^\circ + 1) = x^\circ$.*

Proof. As $S(x) \in \mathbf{Z}$, $x = S(x)$ means $x \in \mathbf{Z}$ and

$$\frac{n-1-x}{p} = x + \frac{t}{p}, \quad i.e., \quad (p+1)x = n - 1 - t$$

holds for some $t = 0, \ldots, p-1$, hence $n \equiv t + 1 \not\equiv 0 \pmod{p+1}$. Thus for $n = h(p+1) + t$ with $h \in \mathbf{Z}$ and $t = 1, 2 \ldots, p$, we see

$$x^\circ = \left\lfloor \frac{n-1}{p+1} \right\rfloor = h + \left\lfloor \frac{t-1}{p+1} \right\rfloor = h$$

and

$$S(x^\circ) = \left\lfloor \frac{n-1-h}{p} \right\rfloor = h + \left\lfloor \frac{t-1}{p} \right\rfloor = h,$$

hence $S(x^\circ) = x^\circ$. When $n = h(p+1)$, we see

$$x^\circ = \left\lfloor \frac{n-1}{p+1} \right\rfloor = h + \left\lfloor -\frac{1}{p+1} \right\rfloor = h - 1.$$

Therefore we have

$$S(x^\circ) = \left\lfloor \frac{n-1-(h-1)}{p} \right\rfloor = h = x^\circ + 1 \text{ and } S^2(x^\circ) = \left\lfloor \frac{n-1-h}{p} \right\rfloor = h + \left\lfloor -\frac{1}{p} \right\rfloor = h - 1 = x^\circ.$$

$\square$

Since Proposition 2.2 shows that the dynamics of $F(x)$ approximates that of $S(x)$, and $F(x)$ has the unique globally stable fixed point x^*, we also expect that x° is a unique globally stable fixed or doubly periodic point of $S(x)$. Actually it is true. To show this, we need following technical results.

Proposition 2.4. *For any real numbers a, b, consider sequences $a_{k+1} = F(a_k)$, $a_0 = a$ and $b_{k+1} = S(b_k)$, $b_0 = b$, and suppose that $|a_k - b_k| < 1$ holds for any $k \in \mathbf{N}$. If $b_k - a_k \geq \dfrac{1}{p+1}$ or $a_k - b_k \geq \dfrac{p}{p+1}$ holds for some k, then $|a_l - b_l| \geq \dfrac{1}{p+1}$ for all $l \leq k$.*

Proposition 2.4 is a consequence of following lemma.

Lemma 2.5. *Under the notations and the assumptions stated in Proposition 2.4, we see followings.*

(1) *If $a_{k+1} \leq b_{k+1}$, then it holds that*

$$a_k \geq b_k = \overline{b}_k \quad \text{and} \quad 0 \leq b_{k+1} - a_{k+1} < \frac{1}{p}.$$

(2) *If $b_{k+1} - a_{k+1} \geq \dfrac{1}{p+1}$, then $a_k - b_k \geq \dfrac{p}{p+1}$ holds.*

(3) *If* $a_{k+1} - b_{k+1} \geq \dfrac{p}{p+1}$, *then* $b_k - a_k \geq \dfrac{1}{p+1}$ *holds.*

Proof. (1)Since $F(x)$ is decreasing and bijective, the inequality $F(a_k) \leq S(b_k) = S(\bar{b}_k) = F(\bar{b}_k)$ means $b_k \leq \bar{b}_k \leq a_k$. Then the assumption $|a_k - b_k| < 1$ derives $\bar{b}_k = b_k$ as $b_k, \bar{b}_k \in \mathbf{Z}$. Hence

$$b_{k+1} - a_{k+1} = S(\bar{b}_k) - F(a_k) = F(\bar{b}_k) - F(a_k) = \frac{1}{p}(a_k - \bar{b}_k) = \frac{1}{p}(a_k - b_k) < \frac{1}{p}.$$

(2) Suppose $b_{k+1} - a_{k+1} \geq \dfrac{1}{p+1}$, then (1) shows $b_k = \bar{b}_k$ and $a_k > b_k$. Thus

$$\frac{1}{p+1} \leq S(\bar{b}_k) - F(a_k) = F(\bar{b}_k) - F(a_k) = \frac{a_k - \bar{b}_k}{p} = \frac{a_k - b_k}{p},$$

hence the inequality.

(3) As $b_{k+1} = S(\bar{b}_k) = F(\bar{b}_k)$, we have

$$\frac{p}{p+1} \leq a_{k+1} - b_{k+1} = F(a_k) - F(\bar{b}_k) = \frac{\bar{b}_k - a_k}{p},$$

hence

$$a_k \leq \bar{b}_k - p + 1 - \frac{1}{p+1}. \tag{2.5}$$

Combining the assumption $|a_k - b_k| < 1$ and the fact $b_k \in (\bar{b}_k - p, \bar{b}_k] \cap \mathbf{Z}$, we see $b_k = \bar{b}_k - p + 1 > a_k$. Substituting $\bar{b}_k = b_k + p - 1$ in (2.5), we get the inequality $b_k - a_k \geq \dfrac{1}{p+1}$. □

Proof. (proof of Proposition 2.4)

It comes from Lemma 2.5 that $b_k - a_k \geq \dfrac{1}{p+1}$ induces $a_{k-1} - b_{k-1} \geq \dfrac{p}{p+1} > \dfrac{1}{p+1}$, hence $b_{k-2} - a_{k-2} \geq \dfrac{1}{p+1}$ and so on. Inductively we obtain $b_l - a_l \geq \dfrac{1}{p+1}$ for all $l \leq k$. □

Proposition 2.6. *Suppose*

$$|F^{k+1}(x) - F^k(x)| \leq 1 \tag{2.6}$$

holds for some $k \in \mathbf{N}$. *Then* $S^{k+1}(x) \in \{x^\circ, x^\circ + 1\}$ *holds whenever* $n \equiv 0 \pmod{p+1}$, *and* $S^{k+1}(x) = x^\circ$ *holds whenever* $n \not\equiv 0 \pmod{p+1}$.

Proof. Firstly we show no integer exists between x_{k+1} and x^* whenever $|x_{k+1} - x_k| \leq 1$ holds, where x_k denotes $F^k(x)$. Suppose that some integer u fulfills $x_{k+1} < u < x^* < x_k$. Taking $v = n - 1 - pu$, we see $u = S(v) = F(v)$. As $F(x)$ is bijective and decreasing, we get $x_{k+1} < u < x^* < v < x_k$ and hence a contradiction $1 \leq |v - u| < |x_k - x_{k+1}| \leq 1$. The case $x_k < x^* < u < x_{k+1}$ is shown by similar argument.

Thus we assume the inequality $u \leq x_{k+1} < x^* \leq u + 1$ holds for some $u \in \mathbf{Z}$. As $|x_{k+1} - y_{k+1}| < 1$ by Proposition 2.2, where y_{k+1} denotes $S^{k+1}(x) \in \mathbf{Z}$, we see $y_{k+1} = u$ or $u + 1$. Suppose $y_{k+1} = u + 1$. The inequality $x_{k+1} < x^* \leq u + 1 = y_{k+1} = S(y_k) = S(\bar{y}_k) = F(\bar{y}_k)$ derives $y_k \leq \bar{y}_k \leq x^* < x_k$, then we have $y_k = \bar{y}_k = u$ on account of $|x_k - y_k| < 1$. Thus we have $F(y_k) = S(y_k) = y_k + 1$ and hence $n = (p+1)(y_k + 1)$, showing that $x^\circ = y_k + \left\lfloor \dfrac{p}{p+1} \right\rfloor = y_k$ and then $S^{k+1}(x) = x^\circ + 1$.

Suppose $y_{k+1} = u$. If $u \leq x^* < u + 1$ then $u = x^\circ$, hence $S^{k+1}(x) = x^\circ$. If $x^* = u + 1 = y_{k+1} + 1$, meaning that x^* is an integer, we see $x^\circ = x^* = \dfrac{n-1}{p+1}$, hence

$$\bar{y}_k = F^{-1}(y_{k+1}) = F^{-1}(x^\circ - 1) = n - 1 - px^\circ + p = x^\circ + p.$$

As the integer y_k is contained in the interval $(\overline{y}_k - p, \overline{y}_k] = (x^\circ, x^\circ + p]$, we see $y_k \geq x^\circ + 1$, while the inequality

$$1 \geq x_k - F(x_k) = \frac{(p+1)x_k - (n-1)}{p} = \frac{p+1}{p}(x_k - x^\circ)$$

holds. Then we obtain

$$y_k - x_k \geq x^\circ + 1 - \frac{p}{p+1} - x^\circ = \frac{1}{p+1}.$$

Applying Proposition 2.4, we obtain $|y_l - x_l| \geq \dfrac{1}{p+1}$ for any $0 \leq l \leq k$, which contradicts to our situation $x_0 = y_0 = x$. Therefore the case $x^* = y_{k+1} + 1$ does not occur.

Assume the inequality $u \leq x^* < x_{k+1} \leq u+1$ for some $u \in \mathbf{Z}$, then we have $u = x^\circ$. The same argument above induces $y_{k+1} = u$ or $u+1$. The case $y_{k+1} = u = x^\circ$ shows the assertion. Thus suppose $y_{k+1} = u+1 = x^\circ+1$. Applying Lemma 2.5 (1) as $x_{k+1} \leq y_{k+1}$, we have $y_k = \overline{y}_k \leq x_k < x^* < x_{k+1} \leq y_{k+1}$. The assumption $|x_{k+1} - x_k| \leq 1$ induces

$$1 \geq x_{k+1} - x_k = \frac{n-1-x_k}{p} - x_k = \frac{p+1}{p}(x^* - x_k),$$

that is, $x^* - x_k \leq \dfrac{p}{p+1}$, while we need $x_k - y_k < \dfrac{p}{p+1}$ by Proposition 2.4, hence $x^* - y_k < \dfrac{2p}{p+1}$. If $y_{k+1} - y_k \geq 2$ holds, we have

$$2 \leq y_{k+1} - y_k = F(\overline{y}_k) - \overline{y}_k = \frac{p+1}{p}(x^* - y_k),$$

hence a contradiction $x^* - y_k \geq \dfrac{2p}{p+1}$. Thus we see $0 < y_{k+1} - y_k \leq 1$, that is, $y_k = y_{k+1} - 1 = x^\circ$. The equation $x^\circ + 1 = y_{k+1} = S(y_k) = F(\overline{y}_k) = F(x^\circ)$ gives $n = (p+1)x^\circ$. Therefore $y_{k+1} = x^\circ$ occurs if and only if $n \equiv 0 \pmod{p+1}$. The assertion is proved. $\square$

Theorem 2.7. *For any $x \in \mathbf{R}$ and $k \geq \dfrac{\log\{(p+1)|x-x^*|\}}{\log p}$, where $x^* = \dfrac{n-1}{p+1}$, we have*

$$S^k(x) = x^\circ = \left\lfloor \frac{n-1}{p+1} \right\rfloor \tag{2.7}$$

whenever $n \not\equiv 0 \pmod{p+1}$, and

$$S^k(x) \in \{x^\circ, x^\circ + 1\} \tag{2.8}$$

whenever $n \equiv 0 \pmod{p+1}$. In particular, (2.7) *or* (2.8) *holds for all $x \in [0, n-1]$ whenever*

$$k \geq 1 + \frac{\log(n-1)}{\log p} \tag{2.9}$$

holds. x° is a globally fixed or doubly periodic point of $S(x)$.

Proof. By definition, we have

$$|F^{k+1}(x) - F^k(x)| = \frac{1}{p}|F^k(x) - F^{k-1}(x)| = \frac{1}{p^k}|F(x) - x| = \frac{(p+1)|x - x^*|}{p^{k+1}}.$$

Thus the condition (2.6) is equivalent to the inequality $k+1 \geq \dfrac{\log\{(p+1)|x-x^*|\}}{\log p}$. Therefore the assertion comes from Proposition 2.6 and Proposition 2.3. In particular, since

$$\max\{|x - x^*| \mid x \in [0, n-1]\} = |n - 1 - x^*| = \frac{(n-1)p}{p+1}$$

holds, we obtain the assertion for all $x \in [0, n-1]$. $\square$

Consequently, Theorem 2.7 shows that if we use the p-shuffle, we can perform p-pile version of Gergonne's tricks for n cards, whenever $n \not\equiv 0 \pmod{p+1}$.

3. Illustrations and improvements

3.1. **The case** $n \equiv 0 \pmod{p+1}$**.** In this case, $S(x)$ has doubly periodic points $x^\circ = \left\lfloor \dfrac{n-1}{p+1} \right\rfloor$ and $x^\circ + 1$, which are globally attracting.

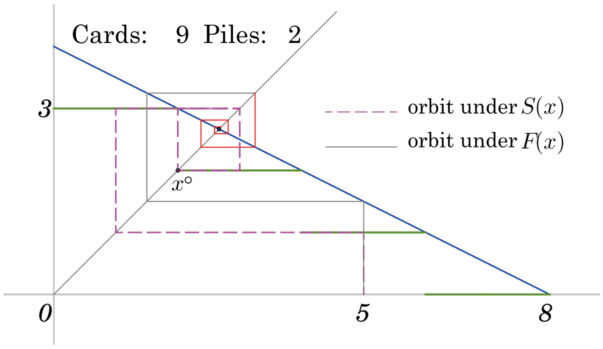

FIGURE 2. Orbits of $x = 5$ for 9-card and 2-pile trick.

3.2. **The case** $n \not\equiv 0 \pmod{p+1}$**.** In this case, $S(x)$ has a unique globally stable fixed point $x^\circ = \left\lfloor \dfrac{n-1}{p+1} \right\rfloor$. The left figure of Figure 3 illustrates the orbits for $n = 8$, $p = 2$ and $x = 6$ under $F(x)$ and $S(x)$. As one see, $S^3(x) = x^\circ = 2$ holds for all $x = 0, 1, \dots, 7$, while the condition (2.9) for k stated in Theorem 2.7 is $k \geq 1 + \dfrac{\log 7}{\log 2} > 3$. In contrast, in the case of 17-card and 2-pile trick, $k \geq 5$ is required for $S^k(13) = x^\circ$, which coincides with $k \geq 1 + \dfrac{\log 16}{\log 2} = 5$ (Figure 3 right). Therefore the condition (2.9) may not necessary give the minimal value of k satisfying (2.7) for all $x \in [0, n-1]$. This is because that our criterion (2.6) which ensures $S^{k+1}(x) = x^\circ$ is too strong.

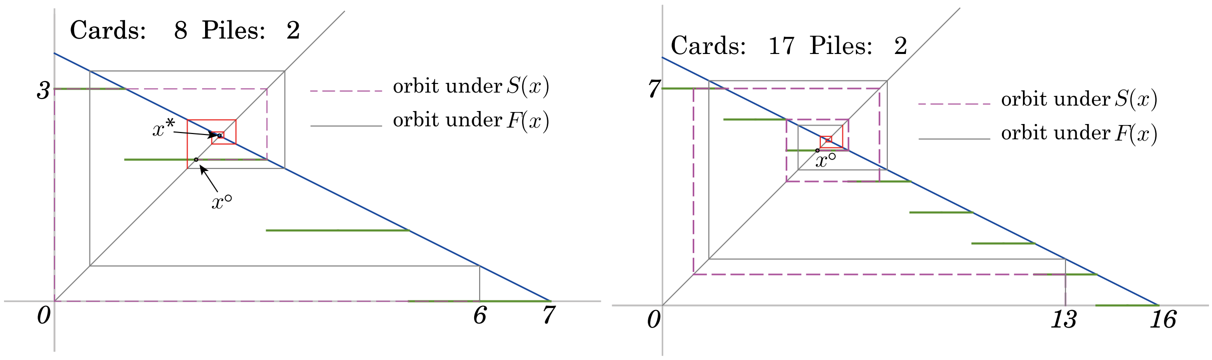

FIGURE 3. Left: orbits of $x = 6$ for 8-card and 2-pile trick. Right: orbits of $x = 13$ for 17-card and 2-pile trick. The orbits under $F(x)$ is drawn in red where (2.6) holds.

The original Gergonne's trick corresponds to $n = 21$ and $p = 3$: as $21 \not\equiv 0 \pmod 4$, there exists a globally stable fixed point $x = 5$. In contrast to our p-shuffle, however, the performer in the original trick puts the pile containing the target card between the rest ones when he stacks three piles. Then instead of $T(x)$, we consider

$$M(x) = \frac{n}{p} - 1 - \left\lfloor \frac{x}{p} \right\rfloor + \frac{n}{p} = \frac{2n}{p} - 1 - \left\lfloor \frac{x}{p} \right\rfloor = 13 - \left\lfloor \frac{x}{3} \right\rfloor .$$

One see $M(10) = 10$, which means the middle of 21 cards is the fixed point.

3.3. **Improvement.** Figure 4 illustrates an orbit under $M(x)$ for the original Gergonne trick. We take

$$G(x) = 13 + \frac{1}{3} - \frac{x}{3} = \frac{40 - x}{3}$$

as a continuous approximation for $M(x)$, so that the fixed point x^* of $G(x)$ accords with $x^\circ = 10$.

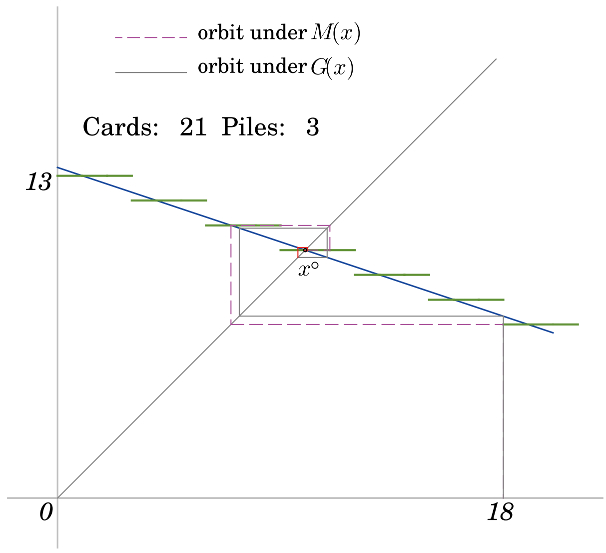

FIGURE 4. Orbits of $x = 18$ for Gergonne's trick.

By similar argument in Proposition 2.2, we see that the orbit of each x under $G(x)$ accompanies with the orbit under $M(x)$. This property is described in a little bit general situation. For $p \geq 2$ and $n = Ap + B, 1 \leq B \leq p$ (hence $A = \left\lfloor \frac{n-1}{p} \right\rfloor$), after dealing n pieces of cards into p piles, choose q $(\leq B - 1)$ piles that consists of $A + 1$ pieces of cards where the target card is not contained, and stack up them. Then stack the pile containing the target card on the top of the piles, and stack up the remaining all piles on the above. We call this process (p, q)-shuffle. In the case of original Gergonne's trick, we repeat $(3, 1)$-shuffle. When (p, q)-shuffle is applied, the target card at the position $x \in \{0, \ldots, n-1\}$, meaning that the card is arranged at the $(x+1)$-th from the top of the deck, is moved to the position at

$$M(x) = n - 1 - \left((A+1)q + \left\lfloor \frac{x}{p} \right\rfloor \right) = n - 1 - \left\lfloor \frac{n+p-1}{p} \right\rfloor q - \left\lfloor \frac{x}{p} \right\rfloor. \tag{3.1}$$

Again we change the coordinate as

$$L(x) = m - 1 - M(m - 1 - x) = \left\lfloor \frac{m - 1 - x}{p} \right\rfloor$$

where $m = n - (A+1)q$. Note that the range of x is also changed to $m - n \leq x \leq m - 1$. The same argument in Proposition 2.3 shows $L(x^\circ) = x^\circ$ where $x^\circ = \left\lfloor \frac{m-1}{p+1} \right\rfloor$.

Suppose $m \not\equiv 0 \pmod{p+1}$, then we put $m = (p+1)x^\circ + r + 1$ with $0 \leq r \leq p - 1$. In this situation, we introduce a linear map

$$G(x) = \frac{m - 1 - (x + r)}{p},$$

so as to satisfy the equation $G(x^\circ) = x^\circ$. For $m-n \le x \le m$, $\overline{x}$ stands for the unique solution of the equation $L(x) = L(\overline{x}) = G(\overline{x})$. As

$$G(\overline{x}) = \frac{m-1-\overline{x}-r}{p} = \frac{1}{p}\left((p+1)x^\circ - \overline{x}\right) = x^\circ + \frac{1}{p}(x^\circ - \overline{x}) \in \mathbf{Z}$$

and

$$L(x) = \left\lfloor \frac{m-1-x}{p} \right\rfloor = \left\lfloor \frac{(p+1)x^\circ + r - x}{p} \right\rfloor = x^\circ + \frac{1}{p}(x^\circ - \overline{x}) + \left\lfloor \frac{\overline{x} - x + r}{p} \right\rfloor,$$

the equation $L(x) = G(\overline{x})$ brings $0 \le \overline{x} - x + r \le p-1$, that is, $r - p + 1 \le x - \overline{x} \le r$. Then we improve Proposition 2.2 as follows.

Proposition 3.1. *Put* $\alpha = \max\left\{\frac{r+1}{p}, \frac{p-r}{p}\right\} \le 1$, *where* $m = (p+1)x^\circ + r + 1$ *with* $0 \le r \le p-1$. *Take* $a, b \in \mathbf{R}$ *with* $|a-b| \le \alpha$, *and put* $a_k = G^k(a), a_0 = a$ *and* $b_k = L^k(b), b_0 = b$. *Then* $|a_k - b_k| \le \alpha$ *holds for any* $k \in \mathbf{N}$. *In particular,*

$$|G^k(x) - L^k(x)| \le \alpha$$

holds for any $x \in \mathbf{R}$ *and* $k \in \mathbf{N}$.

Proof. As is stated above, the inequality $|x - \overline{x}| \le \max\{r, p-1-r\}$ holds. Then we have

$$\begin{aligned} |a_{k+1} - b_{k+1}| &= |G(a_k) - G(\overline{b}_k)| = \frac{1}{p}|a_k - \overline{b}_k| \\ &\le \frac{1}{p}\left(|a_k - b_k| + |b_k - \overline{b}_k|\right) \le \frac{1}{p}\left(1 + \max\{r, p-1-r\}\right) = \alpha. \end{aligned}$$

□

In the case of $(3,1)$-shuffle, we see $m = 21 - 7 = 14$, $x^\circ = \lfloor 13/4 \rfloor = 3$ and hence $r = 14 - 12 - 1 = 1$. Therefore we have an inequality

$$|G^k(x) - L^k(x)| \le \frac{2}{3}$$

for any $x \in \mathbf{R}$ and $k \in \mathbf{N}$.

We also improve the estimation of minimum number k satisfying $L^k(x) = x^\circ$. We see by Proposition 3.1 that $y_{k+1} = L^{k+1}(x) = x^\circ$ holds if

$$x_k = G^k(x) \in (x^\circ - p + 1 + r - \beta, x^\circ + r + \beta), \tag{3.2}$$

where $\beta = \min\{1-\alpha, \alpha\}$. We also have inductively

$$|x_{k+1} - x^\circ| = |G(x^k) - G(x^\circ)| = \frac{|x^k - x^\circ|}{p} = \cdots = \frac{|x - x^\circ|}{p^{k+1}}.$$

To fulfill (3.2), we need $|x_{k+1} - x^\circ| \le \max\{r + \beta, p - 1 - r + \beta\}$, hence the following.

Theorem 3.2. *Let us consider* (p,q)*-shuffle on* n *pieces of cards, and put* $m = n - \left\lfloor \frac{n-1+p}{p} \right\rfloor q$. *Suppose* $m \not\equiv 0 \pmod{p+1}$. *Then for any* $x \in \mathbf{R}$ *and*

$$k \ge \frac{\log|x - x^\circ| - \log\max\{r+\beta, p-1-r+\beta\}}{\log p}, \tag{3.3}$$

we have

$$L^k(x) = x^\circ,$$

where we put $x^\circ = \left\lfloor \frac{m-1}{p+1} \right\rfloor$, $r = m - 1 - (p+1)x^\circ$, $\beta = \min\{\alpha, 1-\alpha\}$ *and* $\alpha = \max\left\{\frac{r+1}{p}, \frac{p-r}{p}\right\}$.

In the case of $(3,1)$-shuffle on 21 pieces of cards, we see $x^\circ = 3$, $m = 14$, $r = 1$, $\alpha = \dfrac{2}{3}$ and $\beta = \dfrac{1}{3}$, hence the inequality becomes

$$k \geq \frac{\log|x-3| - \log\dfrac{4}{3}}{\log 3}.$$

In particular, as $x \in [m-n, m-1] = [-7, 13]$, $L^k(x) = x^\circ$ holds whenever $k \geq \dfrac{\log 10 \times \dfrac{3}{4}}{\log 3} > 2$, that is, $k \geq 3$. This result reproduce Gergonne's trick.

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