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A Dynamical Characterization of Myhill's Property

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1. A motivation from the Pythagorean tuning

'Stack the perfect fifth intervals iteratively, modulo octave' — this is the most simple construction of the chromatic musical scale, called the Pythagorean tuning, and commonly used in Europe in the medieval period. Starting with the musical note F, the tuning goes

$$F \xrightarrow{+p5} C \xrightarrow{+p5} G \xrightarrow{+p5} D \xrightarrow{+p5} A \xrightarrow{+p5} E \xrightarrow{+p5} B \xrightarrow{+p5} F\# \xrightarrow{+p5} C\# \xrightarrow{+p5} G\# \xrightarrow{+p5} D\# \xrightarrow{+p5} A\# \xrightarrow{+p5} \cdots,$$

where $+p5$ means a stack of a perfect fifth interval, that is, to multiply the frequency of each tone by $3/2$. Since two notes an octave apart, i.e., one musical pitch and another with double frequency, sound 'the same', this tuning process is emulated by a circle rotation $\mathbf{R}/\mathbf{Z} \ni x \mapsto x + \log_2(3/2) \in \mathbf{R}/\mathbf{Z}$. The continued fractional approximation $\log_2(3/2) \fallingdotseq [1, 1, 2, 2] = 7/12$ induces a cyclic sequence of 12 notes

$$F \xrightarrow{+p5} C \xrightarrow{+p5} G \xrightarrow{+p5} D \xrightarrow{+p5} A \xrightarrow{+p5} E \xrightarrow{+p5} B \xrightarrow{+p5} F\# \xrightarrow{+p5} C\# \xrightarrow{+p5} G\# \xrightarrow{+p5} D\# \xrightarrow{+p5} A\# \xrightarrow{+p5} F,$$

called the *circle of fifths*, and hence the *chromatic scale* consists of 12 tones, $CC^\#DD^\#EFF^\#GG^\#AA^\#B$, by rearranging the notes in small order of frequencies modulo octaves. The Pythagorean tuning is reproduced by the translation $T : \mathbf{Z}/12\mathbf{Z} \ni x \mapsto x + 7 \in \mathbf{Z}/12\mathbf{Z}$. It is noticeable that the first 7 notes F, C, G, D, A, E, B in the circle of fifths form the *diatonic scale* $CDEFGAB$ on C, known as the collection of white keys of piano keyboard.

As is well known in the mathematical music theory, the diatonic collection has significant features, in particular, *Myhill's property* we discuss mainly in this report is an embodiment of the *maximal evenness* concept. Consider any successive n ($\not\equiv 0 \mod 12$) notes in the chromatic scale, the number of diatonic notes contained in the n notes are of just two kinds, k and $k+1$ for some k depending on n only. It is remarkable that only the collections of *successive 5 or 7 notes* in the circle of fifths fulfill Myhill's property. Myhill's property itself represents a spatial structure of the diatonic scale, while the 'successive' notes in the circle of fifths indicates a temporal feature of the translation T. So, is it a mere coincidence that the number $7 \equiv -5$ (mod12) of notes satisfying Myhill's property equals the extent of translation $T : x \mapsto x + 7 \equiv x - 5$ (mod12)? However mechanical words associated with the fraction 7/12 connect these phenomena. Indeed, when we attach each note to a number in $\mathbf{Z}/12\mathbf{Z}$ by semi-tone encoding $C = 0, C^\# = 1, \ldots, B = 11$, $\left\lfloor \frac{7}{12}(k+7) \right\rfloor - \left\lfloor \frac{7}{12}(k+6) \right\rfloor = 1$ holds if and only if k corresponds to a diatonic note. Inspired by Noll's work [11], we investigate a spatio-temporal symmetry on T (Proposition 3.2, or cf.[7] Theorem 3.4), and we elucidate the reason for the coincidence by the fact that the multiplicative inverse of 7 in $\mathbf{Z}/12\mathbf{Z}$ coincides with itself. Actually, we give a characterization of Myhill's property described in terms of dynamics of the translation T (Theorem 3.4).

2. Mechanical words and Myhill's property

This section is devoted to describe the relation between mechanical words and Myhill's property. Let $\{0,1\}^{\mathbf{Z}}$ be the set of all bi-infinite sequences over the alphabet $\{0,1\}$, and $\{0,1\}^*$ be the set of all finite sequences over $\{0,1\}$. The *transpose* ${}^t\boldsymbol{v}$ and *complement* $\overline{\boldsymbol{v}}$ of a bi-infinite sequence $\boldsymbol{v} = (v_k)$ are defined by $({}^t\boldsymbol{v})_k = v_{1-k}$ and $(\overline{\boldsymbol{v}})_k = 1 - v_k$ respectively. Let $\sigma : \{0,1\}^{\mathbf{Z}} \to \{0,1\}^{\mathbf{Z}}$ be a shift operator $\sigma(\boldsymbol{v})_k = v_{k+1}$.

The *period* of $\{0,1\}^{\mathbf{Z}}$ is, if it exists, the minimal number p satisfying $\sigma^p(\boldsymbol{v}) = \boldsymbol{v}$. Given a finite sequence $\boldsymbol{u} = u_1 \cdots u_n \in \{0,1\}^*$, the *length* $|\boldsymbol{u}|$ of $\boldsymbol{u}$ is n, and for $b = 0, 1$ we put

$$|\boldsymbol{u}|_b = \#\{k \mid u_k = b,\ k = 1, \ldots, n\},$$

where we call $|\boldsymbol{u}|_1$ the *height* of $\boldsymbol{u}$. We see $|\boldsymbol{u}| = |\boldsymbol{u}|_0 + |\boldsymbol{u}|_1$. The *language* $\mathcal{L}(\boldsymbol{v})$ of $\boldsymbol{v} = (v_k) \in \{0,1\}^{\mathbf{Z}}$ is a set of all finite subsequence of $\boldsymbol{v}$:

$$\mathcal{L}(\boldsymbol{v}) = \{v_i \cdots v_j \mid i, j \in \mathbf{Z}, i \leq j\} \cup \{\epsilon\},$$

where ϵ denotes the empty sequence with the length $|\epsilon| = 0$. Putting $\mathcal{L}_k(\boldsymbol{v}) = \{\boldsymbol{u} \in L(\boldsymbol{v}) \mid |\boldsymbol{u}| = k\}$, we see $\mathcal{L}(\boldsymbol{v}) = \bigcup_{k=0}^{\infty} \mathcal{L}_k(\boldsymbol{v})$. For any subset $S \subset \mathcal{L}(\boldsymbol{v})$, we call $|S|_1 = \{|\boldsymbol{u}|_1 \mid \boldsymbol{u} \in S\}$ the *height set.*

Definition 2.1 (Myhill's property)**.** *A bi-infinite sequence $\boldsymbol{v}$ said to be satisfying* Myhill's property *whenever $\#|\mathcal{L}_k(\boldsymbol{v})|_1 = 2$ holds for any $k \geq 1$. $\boldsymbol{v}$ said to be satisfying* Myhill's property with period p *whenever $\#|\mathcal{L}_k(\boldsymbol{v})|_1 = 2$ holds for any $k \not\equiv 0 \pmod{p}$.*

The set of bi-infinite sequences with Myhill's property is denoted by $\mathcal{M}_\infty$ and the set of periodic sequences with Myhill's property with period p is denoted by $\mathcal{M}_p$. We denote $\lfloor \alpha \rfloor = \max\{n \in \mathbf{Z} \mid n \leq \alpha\}$, $\lceil \alpha \rceil = \min\{n \in \mathbf{Z} \mid n \geq \alpha\}$ and $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ for a real number α.

Definition 2.2 (Mechanical word)**.** *For a real number $\alpha > 0$, the bi-infinite sequence $M(\alpha) = \boldsymbol{v} = (v_k) \in \{0,1\}^{\mathbf{Z}}$ defined by $v_k = \lfloor k\alpha \rfloor - \lfloor (k-1)\alpha \rfloor$ is called* lower mechanical word *of $\boldsymbol{v}$. The bi-infinite sequence $M'(\alpha) = \boldsymbol{v} = (v_k) \in \{0,1\}^{\infty}$ defined by $v_k = \lceil k\alpha \rceil - \lceil (k-1)\alpha \rceil$ is called* upper mechanical word *of $\boldsymbol{v}$.*

Note that if $\boldsymbol{v} \in \{0,1\}^{\mathbf{Z}}$ has the period $p \geq 1$, $\#|\mathcal{L}_k(\boldsymbol{v})|_1 = 1$ holds whenever $k \equiv 0 \pmod{p}$. Conversely $\#|\mathcal{L}_k(\boldsymbol{v})|_1 = 1$ implies that the period p of $\boldsymbol{v}$ is a factor of k. Then we define the *height* of the periodic sequence $\boldsymbol{v}$ by $|\boldsymbol{v}|_1 = q$ where q is the unique element of $|\mathcal{L}_p(\boldsymbol{v})|_1$. Also note that the mechanical word $M\left(\frac{q}{p}\right)$ has the period p and the height q.

It is easy to see $\lceil -\alpha \rceil = -\lfloor \alpha \rfloor$ and $\lfloor -\alpha \rfloor = -\lceil \alpha \rceil$, hence

$$M'(\alpha)_k = \lceil k\alpha \rceil - \lceil (k-1)\alpha \rceil = \lfloor (1-k)\alpha \rfloor - \lfloor -k\alpha \rfloor = M(\alpha)_{1-k} = {}^tM(\alpha)_k$$

Moreover $\lceil -\alpha \rceil = -\lceil \alpha \rceil + 1$ and $\lfloor -\alpha \rfloor = -\lfloor \alpha \rfloor - 1$ holds whenever $\alpha \notin \mathbf{Z}$. As $\lceil k(1-\alpha) \rceil = k + \lceil -k\alpha \rceil = k - \lfloor k\alpha \rfloor$, we see

$$M'(1-\alpha)_k = \lceil k(1-\alpha) \rceil - \lceil (k-1)(1-\alpha) \rceil = 1 - (\lfloor k\alpha \rfloor - \lfloor (k-1)\alpha \rfloor) = 1 - v_k = \overline{M(\alpha)_k}.$$

Hence,

Lemma 2.3. *For any real number $0 < \alpha < 1$, $M'(\alpha) = {}^tM(\alpha) = \overline{M(1-\alpha)}$ holds.*

Firstly we show that any mechanical word has Myhill's property.

Theorem 2.4. *The mechanical word of any irrational number $0 < \alpha < 1$ has Myhill's property: $M(\alpha) \in \mathcal{M}_\infty$. The mechanical word of any irreducible fraction $0 < \frac{q}{p} < 1$ has Myhill's property of the period p: $M\left(\frac{q}{p}\right) \in \mathcal{M}_p$.*

Proof. Let $\boldsymbol{v}$ be the lower mechanical word $M(\alpha)$ of a irrational number $0 < \alpha < 1$. Take any word $v_k \cdots v_{k+l-1} \in \mathcal{L}_l(\boldsymbol{v})$ of the length l. Representing $l\alpha = a + \beta$ with $a = \lfloor l\alpha \rfloor$ and $\beta = \{l\alpha\}$, the height of the word is given as

$$|v_k \cdots v_{k+l-1}|_1 = \lfloor (k-1)\alpha + l\alpha \rfloor - \lfloor (k-1)\alpha \rfloor = a + \lfloor (k-1)\alpha + \beta \rfloor - \lfloor (k-1)\alpha \rfloor.$$

As $0 \leq \beta < 1$, we see $\lfloor (k-1)\alpha + \beta \rfloor - \lfloor (k-1)\alpha \rfloor \in \{0,1\}$, showing $|v_k \cdots v_{k+l-1}|_1 \in \{a, a+1\}$, hence $\#|\mathcal{L}_l(\boldsymbol{v})|_1 \leq 2$.

For $k=1$, we have $|v_1\cdots v_l|_1 = \lfloor l\alpha\rfloor - \lfloor 0\rfloor = a$. By the fact that the fractional parts $\{m\alpha\},\ m\in\mathbf{Z}$ is dense in the unit interval $[0,1]$, we take a sequence $\{m_k\}\subset\mathbf{Z}$ such that $\{m_k\alpha\}$ is smallest in $\{\{m\alpha\}\mid 1\le m\le m_k\}$. Then taking such m_k greater than l, we see $\{m_k\alpha\} < \{(m_k-l)\alpha\}$ by definition. It comes from the decomposition

$$\lfloor m_k\alpha\rfloor + \{m_k\alpha\} = \lfloor l\alpha\rfloor + \{l\alpha\} + \lfloor (m_k-l)\alpha\rfloor + \{(m_k-l)\alpha\}$$

that $\{m_k\alpha\} = \{l\alpha\} + \{(m_k-l)\alpha\} - 1$ holds, hence $\lfloor m_k\alpha\rfloor = \lfloor l\alpha\rfloor + \lfloor (m_k-l)\alpha\rfloor + 1$. Consequently,

$$|v_{m_k-l}\cdots v_{m_k}|_1 = \lfloor m_k\alpha\rfloor - \lfloor (m_k-l)\alpha\rfloor = \lfloor l\alpha\rfloor + 1 = a+1,$$

showing that $\#|\mathcal{L}(\boldsymbol{v})|_1 = 2$. In case of a irreducible fraction $\frac{q}{p}$, only by taking $m_k = p$, we show Myhill's property of the period p for $M\left(\frac{q}{p}\right)$. □

Conversely, any periodic sequence over $\{0,1\}$ with Myhill's property coincides with a mechanical word for an irreducible fraction.

Theorem 2.5. *Let $\boldsymbol{v}\in\{0,1\}^{\mathbf{Z}}$ be a bi-infinite sequence with the period $p\ge 2$ and the height $q>0$. If $\boldsymbol{v}$ has Myhill's property with the period p, then q is prime to p and there exists $s\in\mathbf{Z}$ with $\sigma^s(\boldsymbol{v}) = M\left(\frac{q}{p}\right)$ or $M'\left(\frac{q}{p}\right)$.*

To show this fact, we prepare following lemmas.

Lemma 2.6. *For $\boldsymbol{v}\in\mathcal{M}_p$ with $p\ge 3$, $\mathcal{L}_2(\boldsymbol{v}) = \{00,01,10\}$ if and only if $2|\boldsymbol{v}|_1 < p$, and $\mathcal{L}_2(\boldsymbol{v}) = \{11,01,10\}$ if and only if $2|\boldsymbol{v}|_1 > p$. $2|\boldsymbol{v}|_1 = p$ never occurs.*

Proof. As $\#|\mathcal{L}_1(\boldsymbol{v})|_1 = 2$ by Myhill's property of $\boldsymbol{v}$, we have $\mathcal{L}_1(\boldsymbol{v}) = \{0,1\}$. Then the periodicity of $\boldsymbol{v}$ brings $01,10\in\mathcal{L}_2(\boldsymbol{v})$, hence only either $\mathcal{L}_2(\boldsymbol{v}) = \{00,01,10\}$ or $\mathcal{L}_2(\boldsymbol{v}) = \{11,01,10\}$ occurs because of Myhill's property $\#|\mathcal{L}_2(\boldsymbol{v})|_1 = 2$. When $00\in\mathcal{L}_2(\boldsymbol{v})$, take a subword $\boldsymbol{u}$ with the length p and the prefix 1. As 11 never appears in $\boldsymbol{v}$, each 1 in $\boldsymbol{u}$ has unique successor 0, showing $p\ge 2|\boldsymbol{u}|_1 = 2|\boldsymbol{v}|_1$. $p = 2|\boldsymbol{v}|_1$ implies $\boldsymbol{u} = (10)^{|\boldsymbol{v}|_1}$, which contradicts to $00\in\mathcal{L}_2(\boldsymbol{v})$, hence $p > 2|\boldsymbol{v}|_1$. Similarly, $11\in\mathcal{L}_2(\boldsymbol{v})$ brings $p < 2|\boldsymbol{v}|_1$. □

Therefore $\mathcal{M}_p$ has a decomposition $\mathcal{M}_p = \mathcal{M}_p^0 \coprod \mathcal{M}_p^1$, where $\mathcal{M}_p^b = \{\boldsymbol{v}\in\mathcal{M}_p \mid bb\in\mathcal{L}(\boldsymbol{v})\}$, $b=0,1$.

Lemma 2.7. *For $\boldsymbol{v}\in\mathcal{M}_p^0$ with the period $p\ge 3$ and $|\boldsymbol{v}|_1\ge 2$, it holds that*

$$\{z \mid 10^z1\in\mathcal{L}(\boldsymbol{v})\} = \{\left\lfloor \frac{p}{|\boldsymbol{v}|_1}\right\rfloor - 1, \left\lfloor \frac{p}{|\boldsymbol{v}|_1}\right\rfloor\}.$$

For $\boldsymbol{v}\in\mathcal{M}_p^1$ with the period $p\ge 3$ $|\boldsymbol{v}|_0\ge 2$, it holds that

$$\{z \mid 01^z0\in\mathcal{L}(\boldsymbol{v})\} = \{\left\lfloor \frac{p}{|\boldsymbol{v}|_0}\right\rfloor - 1, \left\lfloor \frac{p}{|\boldsymbol{v}|_0}\right\rfloor\}.$$

Proof. Suppose $\boldsymbol{v}\in\mathcal{M}_p^0$, meaning $11\notin\mathcal{L}(\boldsymbol{v})$ by Lemma 2.6, hence $z_0 = \min\{z \mid 10^z1\in\mathcal{L}(\boldsymbol{v})\}\ge 1$. Note that $|\boldsymbol{v}|_1\ge 2$ and $11\notin\mathcal{L}(\boldsymbol{v})$ bring $010^{z_0}1\in\mathcal{L}(\boldsymbol{v})$, hence $p\ge z_0+3$. If $010^z10\in\mathcal{L}(\boldsymbol{v})$ holds for some $z\ge z_0+2$, we see $0^{z_0+2}, 010^{z_0}\in\mathcal{L}_{z_0+2}(\boldsymbol{v})$ while $10^{z_0}1\in\mathcal{L}_{z_0+2}(\boldsymbol{v})$, which contradicts to $\#|\mathcal{L}_{z_0+2}(\boldsymbol{v})|_1 = 2$. Then, if $10^{z_0+1}1\notin\mathcal{L}(\boldsymbol{v})$, we see that $z=z_0$ is a unique integer fulfilling $10^z1\in\mathcal{L}(\boldsymbol{v})$, consequently we get $|\mathcal{L}_{z_0+1}(\boldsymbol{v})|_1 = \{1\}$, contrary to Myhill's property of $\boldsymbol{v}$. Therefore we have $\{z \mid 10^z1\in\mathcal{L}(\boldsymbol{v})\} = \{z_0, z_0+1\}$. Choosing a subword $\boldsymbol{u} = v_k\cdots v_{k+p-1}$ of $\boldsymbol{v}$ with the length p and $v_{k+p-1} = 1$, we see that $\boldsymbol{u}$ consists of words $0^{z_0}1$ and $0^{z_0+1}1$. Then equations

$$p = a(z_0+1) + b(z_0+2) \text{ and } a+b = |\boldsymbol{v}|_1,$$

hold for some positive integer a, b, hence $p = |\boldsymbol{v}|_1(z_0+1)+b$. Thus we obtain $z_0+1 = \left\lfloor \frac{p}{|\boldsymbol{v}|_1}\right\rfloor$. The argument above also works for the case $11\in\mathcal{L}(\boldsymbol{v})$ by exchanging the letters 0 and 1. □

Lemma 2.8. *Consider a morphism* $\phi_z : \mathcal{M}_p \to \{0,1\}^{\mathbf{Z}}$ *defined by*

$$\phi_z : \begin{cases} 0^z 1 \mapsto 0, \\ 0^{z+1}1 \mapsto 1 \end{cases} \quad \text{for } \boldsymbol{v} \in \mathcal{M}_p^0, \quad \text{and} \quad \phi_z : \begin{cases} 1^z 0 \mapsto 0, \\ 1^{z+1}0 \mapsto 1 \end{cases} \quad \text{for } \boldsymbol{v} \in \mathcal{M}_p^1,$$

where $z = \left\lfloor \frac{p}{|\boldsymbol{v}|_1} \right\rfloor - 1$ *for* $\boldsymbol{v} \in \mathcal{M}_p^0$ *and* $z = \left\lfloor \frac{p}{|\boldsymbol{v}|_0} \right\rfloor - 1$ *for* $\boldsymbol{v} \in \mathcal{M}_p^1$. *Then for* $\boldsymbol{v} \in \mathcal{M}_p^0$ *with* $|\boldsymbol{v}|_1 \geq 2$, $\boldsymbol{w} = \phi_z(\boldsymbol{v})$ *is contained in* $\mathcal{M}_{|\boldsymbol{v}|_1}$ *with the height* $|\boldsymbol{w}|_1 = p - |\boldsymbol{v}|_1(z+1) \equiv p \pmod{|\boldsymbol{v}|_1}$. *For* $\boldsymbol{v} \in \mathcal{M}_p^1$ *with* $|\boldsymbol{v}|_0 \geq 2$, $\boldsymbol{w} = \phi_z(\boldsymbol{v})$ *is contained in* $\mathcal{M}_{|\boldsymbol{v}|_0}$ *with the height* $|\boldsymbol{w}|_1 = p - |\boldsymbol{v}|_0(z+1) \equiv p \pmod{|\boldsymbol{v}|_0}$.

Proof. We only show the assertion in the case of $\boldsymbol{v} \in \mathcal{M}_p^0$. It comes from the definition of ϕ_z that the subword 0^z1 appears $|\boldsymbol{v}|_1$ times in each subword $\boldsymbol{u} \in \mathcal{L}_p(\boldsymbol{v})$ with the suffix 1. After removing the factors 0^z1's from $\boldsymbol{u}$, there remain 0's corresponding to the factors $0^{z+1}1$'s, of which number is $p - |\boldsymbol{v}|_1(z+1)$. As a result, we see that $\boldsymbol{w}$ has the period $|\boldsymbol{v}|_1$ and the height $|\boldsymbol{w}|_1 = p - |\boldsymbol{v}|_1(z+1)$. Thus $\#|\mathcal{L}_k(\boldsymbol{w})|_1 = 1$ holds if and only if k is a multiplier of $|\boldsymbol{v}|_1$, hence $\#|\mathcal{L}_k(\boldsymbol{w})|_1 \geq 2$ for $k < |\boldsymbol{v}|_1$. Since $\overline{\boldsymbol{w}}$ has Myhill's property if and only if $\boldsymbol{w}$ does by definition, we only consider the case $00 \in \mathcal{L}(\boldsymbol{w})$. Suppose that for some k, there exists $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{L}_k(\boldsymbol{w})$ with $|\boldsymbol{a}|_1 < |\boldsymbol{b}|_1 < |\boldsymbol{c}|_1$. By definition of ϕ_z, the inverse images $\boldsymbol{x} = \phi_z^{-1}(\boldsymbol{a})$, $\boldsymbol{y} = \phi_z^{-1}(\boldsymbol{b})$ and $\boldsymbol{z} = \phi_z^{-1}(\boldsymbol{c})$ consist of 0^z1's and $0^{z+1}1$'s, hence $|\boldsymbol{x}| < |\boldsymbol{y}| < |\boldsymbol{z}|$ and $|\boldsymbol{x}|_1 = |\boldsymbol{y}|_1 = |\boldsymbol{z}|_1$. Also note that $0^z1\boldsymbol{x}$ or $0^{z+1}1\boldsymbol{x}$ appears in $\boldsymbol{v}$ and the suffix of $\boldsymbol{z}$ is 1. Then there exists $\boldsymbol{p} \in \mathcal{L}_{|\boldsymbol{y}|-|\boldsymbol{x}|}(\boldsymbol{v})$ with $\boldsymbol{px} \in \mathcal{L}_{|\boldsymbol{y}|}(\boldsymbol{v})$ and $|\boldsymbol{p}|_1 \geq 1$, $\boldsymbol{z}' \in \mathcal{L}_{|\boldsymbol{y}|}(\boldsymbol{v})$ and $\boldsymbol{s} \in \mathcal{L}_{|\boldsymbol{z}|-|\boldsymbol{y}|}(\boldsymbol{v})$ with $\boldsymbol{z} = \boldsymbol{z}'\boldsymbol{s}$ and $|\boldsymbol{s}|_1 \geq 1$. Consequently, we have $|\boldsymbol{px}| = |\boldsymbol{y}| = |\boldsymbol{z}'|$ and $|\boldsymbol{px}|_1 > |\boldsymbol{y}|_1 > |\boldsymbol{z}'|_1$, hence we come to a contradiction $\#|\mathcal{L}_k(\boldsymbol{v})|_1 \geq 3$. Thus $\#|\mathcal{L}_k(\boldsymbol{w})|_1 = 2$ holds for any $k \not\equiv 0 \pmod{p}$. □

We come to the proof of Theorem 2.5.

Proof. (proof of Theorem 2.5) Suppose that $\boldsymbol{v} \in \mathcal{M}_2$, then we see $\boldsymbol{v} = (01)^{\mathbf{Z}}$ as $\mathcal{L}_1(\boldsymbol{v}) = \{0,1\}$, hence we see $\boldsymbol{v}$ or $\sigma(\boldsymbol{v})$ coincides with the mechanical word $M(\frac{1}{2})$. As $\boldsymbol{v} \in \mathcal{M}_p^0$ is equivalent to $\overline{\boldsymbol{v}} \in \mathcal{M}_p^1$, without loss of generality, it is sufficient to prove the assertion in the case of $\boldsymbol{v} \in \mathcal{M}_p^0$ with $p \geq 3$. Putting $\boldsymbol{w}^0 = \boldsymbol{v}$, define sequences $\boldsymbol{w}^{l+1} = \phi_{z_l}(\boldsymbol{w}^l)$ with $z_l = \left\lfloor \frac{p_l}{q_l} \right\rfloor - 1$ inductively, where p_l is the period of $\boldsymbol{w}^l$ and $q_l = |\boldsymbol{w}^l|_{1-b}$ when $\boldsymbol{w}^l \in \mathcal{M}_{p_l}^b$, $b \in \{0,1\}$. We always assume $\boldsymbol{w}^l \in \mathcal{M}_{p_l}^0$ by taking $\boldsymbol{w}^l$ for $\overline{\boldsymbol{w}^l}$ instead of $\boldsymbol{w}^l$ itself when $\boldsymbol{w}^l \in \mathcal{M}_{p_l}^1$. By definition of ϕ_{z_l}, we see $|\boldsymbol{w}^l|_1 > |\boldsymbol{w}^{l+1}|_1$. Thus $|\boldsymbol{w}^t|_1 = 1$ holds for some $l = t$, which means $\boldsymbol{w}^t = (0^{p_t-1}1)^{\mathbf{Z}}$. It is easily seen that $\boldsymbol{w}^t$ coincides with a mechanical word $M\left(\frac{1}{p_t}\right)$ up to translation by σ.

Assume that $\boldsymbol{w}^{l+1}$ coincides with a mechanical word $M\left(\frac{q_{l+1}}{p_{l+1}}\right)$, that is, each letter w_k^{l+1} in $\boldsymbol{w}^{l+1}$ is given by $\left\lfloor \frac{q_{l+1}}{p_{l+1}} k \right\rfloor - \left\lfloor \frac{q_{l+1}}{p_{l+1}}(k-1) \right\rfloor$. By definition, we see $\phi_{z_l}(0^{\nu_k}1) = w_k^{l+1}$ where $\nu_k = z_l + w_k^{l+1}$. Then we see

$$A_k := |\phi_{z_l}^{-1}(w_1^{l+1} \cdots w_k^{l+1})| = k(z_l+1) + |w_1^{l+1} \cdots w_k^{l+1}|_1 = k(z_l+1) + \left\lfloor \frac{q_{l+1}}{p_{l+1}} k \right\rfloor,$$

hence $\phi_{z_l}^{-1}(w_k^{l+1}) = w_{A_{k-1}+1}^l \cdots w_{A_k}^l = 0^{\nu_k}1$, showing that $w_m^l = 1$ if and only if $m = A_k$ for some k. (See Figure 1.) However, the inequalities $\frac{q_{l+1}}{p_{l+1}} k - 1 < \left\lfloor \frac{q_{l+1}}{p_{l+1}} k \right\rfloor \leq \frac{q_{l+1}}{p_{l+1}} k$ induce

$$k(z_l+1) + \frac{q_{l+1}}{p_{l+1}} k - 1 < A_k \leq k(z_l+1) + \frac{q_{l+1}}{p_{l+1}} k.$$

By the construction of the sequences $\boldsymbol{u}^l$'s, we see $p_{l+1} = q_l$ and $p_l = p_{l+1}(z_l+1) + q_{l+1}$. Then we have

$$\frac{q_l}{p_l} A_k = \frac{p_{l+1}}{p_{l+1}(z_l+1) + q_{l+1}} A_k \leq k < \frac{p_{l+1}}{p_{l+1}(z_l+1) + q_{l+1}} (A_k+1) = \frac{q_l}{p_l}(A_k+1),$$

hence $\left\lceil \frac{q_l}{p_l} A_k \right\rceil = k$ and $\left\lceil \frac{q_l}{p_l}(A_k+1) \right\rceil = k+1 = \left\lceil \frac{q_l}{p_l} A_{k+1} \right\rceil$. Consequently, we have

$$\left\lceil \frac{q_l}{p_l}(A+1) \right\rceil - \left\lceil \frac{q_l}{p_l} A \right\rceil = \begin{cases} 0, & A_k + 1 \leq A < A_{k+1} \text{ for some } k, \\ 1, & A = A_k \text{ for some } k, \end{cases}$$

that is, $\boldsymbol{w}^l$ also coincides with a mechanical word $M'\left(\frac{q_l}{p_l}\right)$. ☐

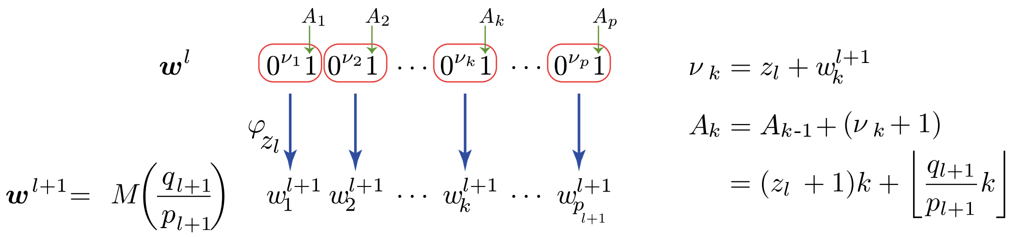

FIGURE 1. Correspondence between $\boldsymbol{w}^l$ and $\boldsymbol{w}^{l+1}$ through ϕ_{z_l}.

3. A dynamical characterization of Myhill's property

A fractional expansion of a real number $0 < \alpha < 1$, $\alpha = \frac{1}{\sigma_0+}\frac{1}{\sigma_1+}\frac{1}{\sigma_2+\cdots}$ is denoted by $\alpha = [\sigma_0, \sigma_1, \sigma_2, \dots]$ for short. We start with the following well known facts on fractional expansions.

Proposition 3.1. *For a real number $0 < \alpha < 1$, consider sequences $\{\sigma_k\}$, $\{\alpha_k\}$ and $\{\boldsymbol{P}_k = \begin{pmatrix} e_k \\ f_k \end{pmatrix}\}$ defined inductively by*

$$\begin{aligned} \sigma_k &= \lfloor \alpha_{k-1}/\alpha_k \rfloor, \quad \alpha_{k+1} = \alpha_{k-1} - \sigma_k\alpha_k, \quad \boldsymbol{P}_{k+1} = \sigma_k \boldsymbol{P}_k + \boldsymbol{P}_{k-1}, \\ \alpha_{-1} &= 1, \quad \alpha_0 = \alpha, \quad \boldsymbol{P}_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{P}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.1}$$

for $k \geq 0$. Then we have

1. *$\alpha = [\sigma_0, \sigma_1, \sigma_2, \dots]$ and $\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \alpha_{k-1}\boldsymbol{P}_k + \alpha_k \boldsymbol{P}_{k-1}$ for $k \geq 0$.*
2. *f_{k+1}/e_{k+1} gives the $k+1$-th continuant $[\sigma_0, \dots, \sigma_k]$ of α and $f_{k+1}e_k - f_k e_{k+1} = \det(\boldsymbol{P}_{k+1}\ \boldsymbol{P}_k) = (-1)^{k+1}$. Particularly, $a \in \{f_{k+1}, e_k\}$ and $b \in \{f_k, e_{k+1}\}$ are prime to each other.*
3. *It holds that $e_k/e_{k+1} = [\sigma_k, \dots, \sigma_0]$ and $f_k/f_{k+1} = [\sigma_{k-1}, \dots, \sigma_0]$.*

Proof. (1) The assertion comes from the definition of continued fractional expansion and induction. (2) As $(\boldsymbol{P}_{k+1}\ \boldsymbol{P}_k) = (\boldsymbol{P}_0\ \boldsymbol{P}_{-1})\begin{pmatrix} \sigma_0 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \sigma_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_0 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \sigma_k & 1 \\ 1 & 0 \end{pmatrix}$, we have $\det(\boldsymbol{P}_{k+1}\ \boldsymbol{P}_k) = (-1)^{k+1}$. (3) Taking the transpose, we have $\begin{pmatrix} e_{k+1} & f_{k+1} \\ e_k & f_k \end{pmatrix} = \begin{pmatrix} \sigma_k & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \sigma_0 & 1 \\ 1 & 0 \end{pmatrix}$, hence the assertion. ☐

Let us consider the circle rotations $\rho : \mathbf{R}/\mathbf{Z} \ni x \mapsto x + \alpha \in \mathbf{R}/\mathbf{Z}$ associated with a finite continued fraction $\alpha = [\sigma_0, \dots, \sigma_{n-1}] = f_n/e_n$ and its transpose ${}^t\rho(x) = x + {}^t\alpha$ where ${}^t\alpha = [\sigma_{n-1}, \dots, \sigma_0] = e_{n-1}/e_n$. These maps have the same period e_n, as f_n and e_n, or e_{n-1} and e_n are prime to each other respectively by Proposition 3.1 (2). Thus the circle rotations ρ and ${}^t\rho$ are regarded as translations on $\mathbf{Z}/e_n\mathbf{Z}$, that is, $\mathbf{Z}/e_n\mathbf{Z} \ni k \mapsto k + f_n \in \mathbf{Z}/e_n\mathbf{Z}$ and $\mathbf{Z}/e_n\mathbf{Z} \ni k \mapsto k + e_{n-1} \in \mathbf{Z}/e_n\mathbf{Z}$ respectively. Since $e_{n-1}f_n \equiv (-1)^n \pmod{e_n}$, we note that $(-1)^n e_{n-1}$ is nothing but the multiplicative inverse $f_n^{-1} \in (\mathbf{Z}/e_n\mathbf{Z})^\times$ of f_n. Consequently, the spatio-temporal symmetry of circle rotations are stated as follows.

Proposition 3.2 (Spatio-temporal symmetry. cf.[7] Theorem 3.4)**.** *Take an element $f \in (\mathbf{Z}/e\mathbf{Z})^\times$ and consider the translations on $(\mathbf{Z}/e\mathbf{Z})^2$,*

$$\eta(x, y) = (x + 1, y + f) \quad \text{and} \quad \theta(x, y) = (x + f^{-1}, y + 1).$$

Then we have $\eta^{f^{-1}} = \theta$ and $\theta^f = \eta$.

Proof. The proof is straightforward, e.g., $\theta^f(x, y) = (x + f \cdot f^{-1}, y + f) = (x + 1, y + f) = \eta(x, y)$. ☐

Definition 3.3. *For any subset $D \subset \mathbf{Z}/e\mathbf{Z}$, we associate a periodic sequence $\boldsymbol{v}(D) \in \{0,1\}^{\mathbf{Z}}$ as*

$$\boldsymbol{v}(D)_k = 1 \text{ if and only if } k \equiv l \pmod{e} \text{ for some } l \in D.$$

A subset D is called a Myhill set *of period e whenever $\boldsymbol{v}(D) \in \mathcal{M}_e$.*

For a subset $D \subset \mathbf{Z}/e\mathbf{Z}$, $\boldsymbol{v}(D) \in \mathcal{M}_e$ describes a spatial feature how D is embedded in $\mathbf{Z}/e\mathbf{Z}$. However, the theorem below states that the Myhill set D has a temporal characterization in terms of a dynamics on $\mathbf{Z}/e\mathbf{Z}$.

Theorem 3.4 (Dynamical characterization of Myhill's property)**.** *Consider a subset $D \subset \mathbf{Z}/e\mathbf{Z}$ with the cardinality $\#|D| = f$ relatively prime to e, and take a translation*

$$T : \mathbf{Z}/e\mathbf{Z} \ni x \mapsto x + f^{-1} \in \mathbf{Z}/e\mathbf{Z},$$

where f^{-1} is a multiplicative inverse of $f \in (\mathbf{Z}/e\mathbf{Z})^{\times}$. Then D is a Myhill set of the period e if and only if D is a collection of successive f images of some element $g \in \mathbf{Z}/e\mathbf{Z}$ by T, namely

$$D = \{g, T(g), T^2(g), \ldots, T^{f-1}(g)\} = \{g, g+f^{-1}, g+2f^{-1}, \ldots, g+1-f^{-1}\}.$$

Proof. Suppose $\boldsymbol{v}(D) \in \mathcal{M}_e$. We see $|\boldsymbol{v}(D)|_1 = f$ by definition. Theorem 2.5 shows that $\boldsymbol{v}(D)$ coincides with $M\left(\frac{f}{e}\right)$ or $M'\left(\frac{f}{e}\right)$ up to translation by σ. Assume $\boldsymbol{v}(D) = M\left(\frac{f}{e}\right)$. The equation $\boldsymbol{v}(D)_k = \left\lfloor \frac{f}{e}k \right\rfloor - \left\lfloor \frac{f}{e}(k-1) \right\rfloor = 1$ is equivalent to $fk \geq \left\lfloor \frac{fk}{e} \right\rfloor e > f(k-1)$, that is, $k \in D$ if and only if $f > \{fk\}_e \geq 0$ holds as an inequality on $\mathbf{Z}$, where we put $\{a\}_e = \left\lfloor \frac{a}{e} \right\rfloor e$. By the definition of θ stated in Proposition 3.2, we see that $0 \leq \pi_2(\theta^b(0,0)) = b < f$ holds on $\mathbf{Z}$ if and only if $b = 0, 1, \ldots, f-1$, where π_2 denotes a projection $(\mathbf{Z}/e\mathbf{Z})^2 \ni (x,y) \mapsto y \in \mathbf{Z}/e\mathbf{Z}$. Consequently $0 \leq \{fk\}_e = \pi_2(\eta^k(0,0)) = \pi_2(\theta^{fk}(0,0)) < f$ is equivalent to $fk \in \{0, 1, \ldots, f-1\}$, as an element of $\mathbf{Z}/e\mathbf{Z}$, hence $k \in \{0, f^{-1}, 2f^{-1}, \ldots, (f-1)f^{-1}\} = \{0, T(0), \ldots, T^{f-1}(0)\}$. Theorem 2.4 induces the converse. □

Since $g + kf^{-1} \equiv g + 1 - f^{-1} - (f-k-1)f^{-1} \pmod{e}$, by putting $g' = g + 1 - f^{-1}$, the Myhill set is also represented as $D = \{g', g'-f^{-1}, g'-2f^{-1}, \ldots, g'-(f-1)f^{-1}\} = \{g', T^{-1}(g'), T^{-2}(g'), \ldots, T^{1-f}(g')\}$. Therefore any Myhill set is symmetric in $\mathbf{Z}/e\mathbf{Z}$. Consider a continued fraction $f/e = [\sigma_0 \ldots, \sigma_{n-1}]$ and its transpose $e'/e = {}^t[\sigma_{n-1}, \ldots, \sigma_0]$. As is stated above, $f^{-1} \equiv (-1)^n e' \pmod{e}$. Therefore we obtain another representation of Myhill sets.

Corollary 3.5. *Any Myhill set D in $\mathbf{Z}/e\mathbf{Z}$ with cardinality $\#D = f$ is described as*

$$D = \{g + ke' \mid k = 0, \ldots, f-1\}, \quad \text{for some } g \in \mathbf{Z}/e\mathbf{Z}$$

where $e'/e = [\sigma_{n-1}, \ldots, \sigma_0]$ is a transpose of $f/e = [\sigma_0, \ldots, \sigma_{n-1}]$.

4. Observation and discussion

To illustrate the results stated in the section 3, we consider a Myhill set $D \subset \mathbf{Z}/30\mathbf{Z}$ with the cardinality $\#D = 13$. In this situation, $e = 30$, $f = 13$ and $f^{-1} = 7 \in (\mathbf{Z}/30\mathbf{Z})^{\times}$, and we may assume $\boldsymbol{v}(D)$ coincides with the lower mechanical word $M\left(\frac{13}{30}\right) = (001010100101010010101010010101)^{\mathbf{Z}}$ (upper in Figure 2), which represents the spatial arrangement of D in $\mathbf{Z}/30\mathbf{Z}$: the green polygon in lower circle in Figure 2. The middle illustrates the dynamics of $\eta : (x,y) \mapsto (x+1, y+13)$ and $\theta : (x,y) \mapsto (x+7, y+1)$ on $(\mathbf{Z}/30\mathbf{Z})^2$. It is observed that the orbit of $(0,0)$ by η (blue lines) is traced by θ (pink lines), and vice versa. Theorem 3.4 states that the Myhill set D is reconstructed by the dynamics $T : \mathbf{Z}/30\mathbf{Z} \ni x \mapsto x + f^{-1} = x + 7 \in \mathbf{Z}/30\mathbf{Z}$: the collection of the first $f = 13$ plots in the orbit of 0 by T coincides with D, ans its orbit is represented by pink arrows in lower circle in Figure 2. Figure 3 illustrates the case $e = 12$ and $f = 7$, namely the 12 tone system derived from Pythagorean tuning. In this case, we have $f = f^{-1}$ since $7^2 \equiv 1 \pmod{12}$, hence

n	fraction	Mechanical word	Myhill set
3	$\frac{13}{30} = [2,3,4]$	001010100101010010101001010101	$D_3 = \{3,5,7,10,12,14,17,19,21,24,26,28,0\}$
2	$\frac{4}{13} = [3,4]$	1 0 0 1 0 0 1 0 0 1 0 0 0	$D_2 = \{3,10,17,24\}$
1	$\frac{1}{4} = [4]$	0 0 0 1	$D_1 = \{24\}$

TABLE 1. Myhill resolution associated with $[2,3,4]$.

the dynamics of η and θ are symmetric. The Myhill set D of cardinality 7 is constructed as the first $f = 7$ plots in the orbit of 0 by $T : \mathbf{Z}/12\mathbf{Z} \ni x \mapsto x + f^{-1} = x + 7 \in \mathbf{Z}/12\mathbf{Z}$, which coincides with the diatonic set in musical set theory.

In [7], we have introduced a nested structure of 'diatonic' sets, which we call the *sub-diatonic resolution* associated with a real number. Consider a real number $0 < \alpha < 1$ and its continued fractional expansion $\alpha = [\sigma_0, \sigma_1, \ldots]$. The transpose $e_{n-1}/e_n = [\sigma_{n-1}, \sigma_{n-2}, \ldots, \sigma_0]$ of the n-th continuant $f_n/e_n = [\sigma_0, \ldots, \sigma_{n-1}]$ induces a Myhill set D_n with cardinality $\#D_n = e_{n-1}$ defined by $\boldsymbol{v}(D_n) = M\left(\frac{e_{n-1}}{e_n}\right)$, and the associated dynamics in the sense of Theorem 3.4 is given by $\mathbf{Z}/e_n\mathbf{Z} \ni x \mapsto x + f_n \in \mathbf{Z}/e_n\mathbf{Z}$. As is seen in the proof of Theorem 3.4, mechanical words have a nested structure. It comes from Lemma 2.8 and Theorem 2.5 that the morphism $\phi_z : \mathcal{M}_{e_n} \to \{0,1\}^{\mathbf{Z}}$ with $z = \left\lfloor \frac{e_n}{e_{n-1}} \right\rfloor - 1$ given in Lemma 2.8 induces a mechanical word $M'\left(\frac{e_{n-2}}{e_{n-1}}\right) = {}^tM\left(\frac{e_{n-2}}{e_{n-1}}\right)$, where $e_{n-2}/e_{n-1} = [\sigma_{n-2}, \ldots, \sigma_0]$. The morphism ϕ_z induces a natural bijection $D_n \to \mathbf{Z}/e_{n-1}\mathbf{Z}$, hence a Myhill set $D_{n-1} \subset \mathbf{Z}/e_{n-1}\mathbf{Z}$ is obtained by way of $\boldsymbol{v}(D_{n-1}) = M'\left(\frac{e_{n-2}}{e_{n-1}}\right)$ with cardinality $\#D_{n-1} = e_{n-2}$. In this manner, we obtain a sequence of embedding of Myhill sets

$$D_1 \hookrightarrow D_2 \hookrightarrow \cdots \hookrightarrow D_n \tag{4.1}$$

associated with continued fractions $1/e_1 = [\sigma_0], e_1/e_2 = [\sigma_1, \sigma_0], \ldots, e_{n-1}/e_n = [\sigma_{n-1}, \ldots, \sigma_0]$, which we call a *Myhill resolution* associated with a continued fraction $[\sigma_{n-1}, \ldots, \sigma_0]$. For instance, Table 1 gives a Myhill resolution $D_1 \hookrightarrow D_2 \hookrightarrow D_3$ induced by $[2,3,4]$. It is noticeable that Myhill sets D_2 is represented as the successive 4 plots of an orbit by a translation not only on $\mathbf{Z}/13\mathbf{Z}$ but also on $\mathbf{Z}/30\mathbf{Z}$, namely

$$D_2 = \{0, S(0) = 0, S^2(0) = 20 \equiv 7, S^3(0) = 30 \equiv 4\} \subset \mathbf{Z}/13\mathbf{Z}$$

where $S(x) = x + 10$ is a translation by $10 \equiv 4^{-1}$ on $\mathbf{Z}/13\mathbf{Z}$, and

$$D_2 = \{3, T(3) = 10, T^2(3) = 17, T^3(3) = 24\} \subset \mathbf{Z}/30\mathbf{Z}$$

where $T(x) = x + 7$ is a translation by $7 \equiv 13^{-1}$ on $\mathbf{Z}/30\mathbf{Z}$.

The Myhill resolution induced from $7/12 = [1,1,2,2]$ has a music theoretical meaning (Table 2). Myhill sets D_k's embedded in the chromatic scale $\mathbf{Z}/12\mathbf{Z}$ form historic named scales: the diatonic scale D_4, the major pentatonic scale D_3, the perfect 5th D_2 and the unison D_1. Also we note that two dynamical representations exist for D_3 and D_2:

$$D_3 = \{2, S(2) = 5, S^2(2) = 8 \equiv 1, S^3(2) = 11 \equiv 4, S^4(2) = 14 \equiv 0\} \subset \mathbf{Z}/7\mathbf{Z}$$

where $S(x) = x + 3$ is a translation by $3 \equiv 5^{-1}$ on $\mathbf{Z}/7\mathbf{Z}$, while

$$D_3 = \{0, T(0) = 7, T^2(0) = 14 \equiv 2, T^3(0) = 21 \equiv 9, T^4(0) = 28 \equiv 4\} \subset \mathbf{Z}/12\mathbf{Z}$$

where $T(x) = x + 7$ is a translation by $7 \equiv 7^{-1}$ on $\mathbf{Z}/12\mathbf{Z}$, and $D_2 = \{0, R(0) = 3\} \subset \mathbf{Z}/5\mathbf{Z}$ where $R(x) = x + 3$ is a translation by $3 \equiv 2^{-1}$ on $\mathbf{Z}/5\mathbf{Z}$, while $D_2 = \{0, T(0) = 7\} \subset \mathbf{Z}/12\mathbf{Z}$.

In light of Theorem 2.4, 2.5 and 3.4, we may understand, in the context of the Pythagorean tuning, the reason not only why the 'seven' notes are selected as a diatonic scale from the twelve notes but also why the

n	fraction	Mechanical word	Myhill set	Note name	Scale name
4	$\frac{7}{12} = [1,1,2,2]$	101010110101	$D_4 = \{0,2,4,6,7,9,11\}$	$\{F,G,A,B,C,D,E\}$	Diatonic
3	$\frac{5}{7} = [1,2,2]$	1 1 1 01 1 0	$D_3 = \{0,2,4,7,9\}$	$\{F,G,A,C,D\}$	Pentatonic
2	$\frac{2}{5} = [2,2]$	1 0 0 1 0	$D_2 = \{0,7\}$	$\{F,C\}$	Perfect 5th
1	$\frac{1}{2} = [2]$	1	$D_1 = \{0\}$	$\{F\}$	Unison

Table 2. Myhill resolution associated with $[1,1,2,2]$. For inessential but technical reasons, we adopt a morphism $\phi_0 : 10 \mapsto 1, 1 \mapsto 0$ as an embedding $D_3 \hookrightarrow \mathbf{Z}/12\mathbf{Z}$, and a morphism $\phi_0 : 01 \mapsto 1, 1 \mapsto 0$, as an embedding $D_2 \hookrightarrow \mathbf{Z}/12\mathbf{Z}$.

'successive' seven notes are done as follows. The chromatic scale might be too complicated to sing, play or tune musical instruments for the ancients; collections of less number of notes might be preferred. Because of its simplicity, the Pythagorean tuning was widely adopted until medieval period. It was natural to select musical tones by stacking the perfect fifth 'successively': iterations of $x \mapsto x + 7$ on $\mathbf{Z}/12\mathbf{Z}$. However, when the stacking process should be stopped? To spin a melody smoothly, it might be preferred that the selected musical notes are more evenly arranged in the octave, called the *maximal evenness*. For instance, all adjacent two notes in the selected collection should have less kinds of ratios of frequencies. Myhill's property is a realization of the maximal evenness concept, and owing to Theorem 3.4, Myhill set is obtained by just $7^{-1} \equiv 7 \pmod{12}$ times of the perfect fifth stacking. This is why the 'successive seven' notes in the circle of fifth are selected as the diatonic scale.

Remark. The concept of spatio-temporal analysis on the Pythagorean tuning is based on the *cut and project method* which widely applied to studies on aperiodic tilings and quasi-crystals[2][12][10].

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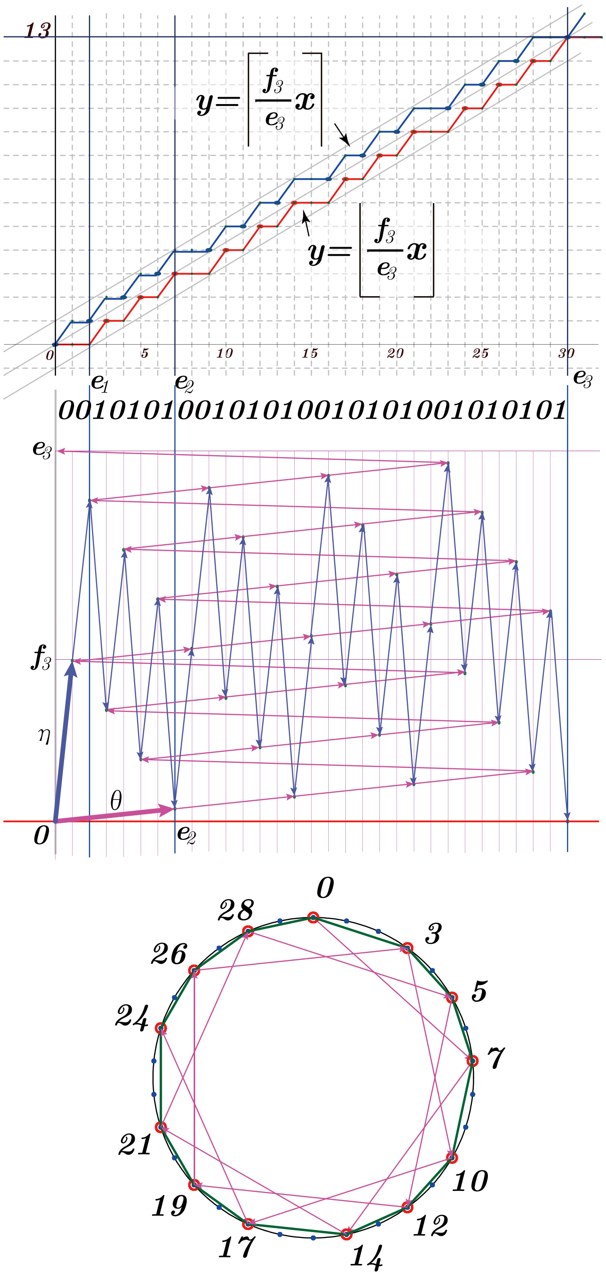

FIGURE 2. Spatio-temporal structure for $13/30 = [2, 3, 4]$, with $1/e_1 = 1/4$, $e_1/e_2 = 4/13 = [3, 4]$, $e_2/e_3 = 13/30 = [2, 3, 4]$.

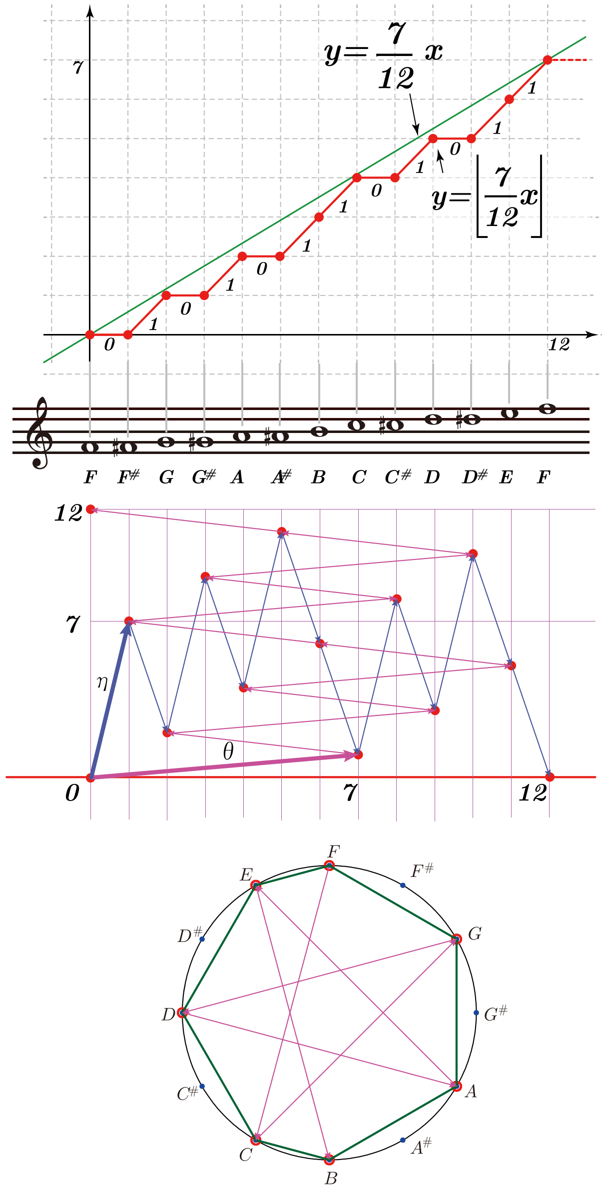

FIGURE 3. Mechanical word $M\left(\frac{7}{12}\right)$ and chromatic scale. The diatonic notes correspond to letter 1's in $M\left(\frac{7}{12}\right)$.

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