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Spatio-temporal Symmetry on Circle Rotations and a Notion on Diatonic Set Theory

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1. Introduction

The most simple way to construct the 12-tone musical scale is the Pythagorean tuning, by *"stacking the perfect fifth intervals iteratively, modulo octave"*. Mathematically, we just consider a circle rotation $\mathbf{R}/\mathbf{Z} \ni x \mapsto x + \log_2(3/2) \in \mathbf{R}/\mathbf{Z}$. Generally, the dynamics of a circle rotation $\rho(x) = x + \gamma$ is completely described by the substitution dynamics associated with the continued fractional expansion of γ, and firstly we give the detail in Theorem 2.3, where the subdivision process of the interval $[0, 1]$ are described by substitutions on some combinatorial words. Inspired by Noll's work [8] on mathematical music theory, we found a spatio-temporal symmetry on circle rotations (Theorem 3.4), which gives the 1:1 correspondence between mechanical words associated with the line $y = \gamma x$ and the combinatorial words induced from subdivisions of $[0, 1]$. Moreover, this symmetry gives a notion of the diatonic set theory from the viewpoint of rotation dynamics.

2. Substitution dynamics associated with circle rotations: revisited

Let us take a real number $0 < \gamma < 1$, whose continued fractional expansion is given by

$$\gamma = \frac{1}{\sigma_0 +}\frac{1}{\sigma_1 +}\frac{1}{\sigma_2 +}\cdots,$$

denoted by $\gamma = [0; \sigma_0, \sigma_1, \sigma_2, \dots]$. It is convenient to introduce a geometrical interpretation of the continued fractional expansion.

Proposition 2.1. *Let us take a real number $0 < \gamma < 1$. Consider sequences $\{\sigma_k\}$, $\{\gamma_k\}$ and $\{\boldsymbol{P}_k = \begin{pmatrix} e_k \\ f_k \end{pmatrix}\}$ defined inductively by*

$$(2.1)\qquad \begin{gathered} \sigma_k = \lfloor \gamma_{k-1}/\gamma_k \rfloor, \quad \gamma_{k+1} = \gamma_{k-1} - \sigma_k\gamma_k, \quad \boldsymbol{P}_{k+1} = \sigma_k \boldsymbol{P}_k + \boldsymbol{P}_{k-1}, \\ \gamma_{-1} = 1, \quad \gamma_0 = \gamma, \quad \boldsymbol{P}_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{P}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{gathered}$$

for $k \geq 0$, where $\lfloor x \rfloor$ stands for the greatest integer not greater than x. Then we have

(1) *$\gamma = [0; \sigma_0, \sigma_1, \sigma_2, \dots]$ and $\begin{pmatrix} 1 \\ \gamma \end{pmatrix} = \gamma_{k-1}\boldsymbol{P}_k + \gamma_k \boldsymbol{P}_{k-1}$ for $k \geq 0$.*

(2) *f_{k+1}/e_{k+1} gives the $k+1$-th continuant $[0; \sigma_0, \dots, \sigma_k]$ of γ and $f_{k+1}e_k - f_k e_{k+1} = \det(\boldsymbol{P}_{k+1}\ \boldsymbol{P}_k) = (-1)^{k+1}$. Particularly, $a \in \{f_{k+1}, e_k\}$ and $b \in \{f_k, e_{k+1}\}$ are prime to each other.*

(3) *It holds that $e_k/e_{k+1} = [0; \sigma_k, \dots, \sigma_0]$ and $f_k/f_{k+1} = [0; \sigma_{k-1}, \dots, \sigma_0]$.*

Proof. (1) It can be seen by the definition of continued fractional expansion and induction.

(2) As $(\boldsymbol{P}_{k+1}\ \boldsymbol{P}_k) = (\boldsymbol{P}_0\ \boldsymbol{P}_{-1})\begin{pmatrix} \sigma_0 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \sigma_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_0 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \sigma_k & 1 \\ 1 & 0 \end{pmatrix}$, we have $\det(\boldsymbol{P}_{k+1}\ \boldsymbol{P}_k) = (-1)^{k+1}$.

(3) Taking the transpose, we have $\begin{pmatrix} e_{k+1} & f_{k+1} \\ e_k & f_k \end{pmatrix} = \begin{pmatrix} \sigma_k & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \sigma_0 & 1 \\ 1 & 0 \end{pmatrix}$, hence the assertion. □

A trajectory of a circle rotation $\rho : \mathbf{R}/\mathbf{Z} \ni x \mapsto x + \gamma \in \mathbf{R}/\mathbf{Z}$ gives a partition of the unit interval $[0, 1]$ (identifying 1 with 0): as increasing the iteration of ρ, the interval $[0, 1]$ is subdivided more and more. The mechanism of the dynamics is already observed in [6], however we describe it in terms of substitution dynamics as follows.

Proposition 2.2. *For a circle rotation $\rho : \mathbf{R}/\mathbf{Z} \ni x \mapsto x + \gamma \in \mathbf{R}/\mathbf{Z}$ with $0 < \gamma < 1$, consider sequences $\{\sigma_k\}$, $\{\gamma_k\}$ and $\{e_k\}$ given by (2.1). Then we have*

(1) *Any natural number N has a unique expression*

$$N = h_n e_n + h_{n-1} e_{n-1} + \cdots + h_0 e_0$$

by integers $h_0, h_1, \ldots, h_n$ fulfilling

$$0 \le h_0 \le \sigma_0 - 1,\ 0 \le h_k \le \sigma_k\ (k \ge 1) \text{ and } h_k = 0 \text{ whenever } h_{k+1} = \sigma_{k+1}. \tag{2.2}$$

As a result, $e_{k+1} > h_k e_k + \cdots + h_0 e_0$ holds for any $k \ge 0$. We call the n-tuple $(h_n, h_{n-1}, \ldots, h_0)_\gamma$ the greedy expansion *of N associated with γ.*

(2) *$\rho^{e_k}(0) = (-1)^k \gamma_k$, hence for a greedy expansion $N = (h_n, h_{n-1}, \ldots, h_0)_\gamma$, we have*

$$\rho^N(0) = \sum_{k=0}^{n} (-1)^k h_k \gamma_k.$$

Proof. (1) Given a greedy expansion $(h_n, \ldots, h_0)_\gamma$, we see $h_0 e_0 < \sigma_0 e_0 = e_1$. Suppose that $h_l e_l + \cdots + h_0 e_0 < e_{l+1}$ for $l = 0, \ldots, k-1$. If $h_k < \sigma_k$, then $h_k e_k + h_{k-1} e_{k-1} + \cdots + h_0 e_0 < (h_k + 1) e_k \le \sigma_k e_k < e_{k+1}$. If $h_k = \sigma_k$, then $h_{k-1} = 0$ by definition and hence $h_k e_k + h_{k-1} e_{k-1} + \cdots + h_0 e_0 = \sigma_k e_k + h_{k-2} e_{k-2} + \cdots + h_0 e_0 < \sigma_k e_k + e_{k-1} = e_{k+1}$. Thus we see $h_k e_k + \cdots + h_0 e_0 < e_{k+1}$ for $k > 0$. For a natural number N, we find n with $e_n \le N < e_{n+1}$ as $e_{k+1} > e_k$ for any k. Taking $h_n = \lfloor N/e_n \rfloor$, we see $N' = N - h_n e_n < e_n$ and $0 \le h_n \le \lfloor e_{n+1}/e_n \rfloor = \lfloor \sigma_n + e_{n-1}/e_n \rfloor = \sigma_n$. If $h_n = \sigma_n$, as $\sigma_n e_n + e_{n-1} = e_{n+1} > N$ we have $e_{n-1} > h_{n-1} e_{n-1} + \cdots + h_0 e_0$, implying $h_{n-1} = 0$. Applying the same procedure to N' and its successors, we come to a greedy expansion $(h_n, \ldots, h_0)_\gamma$. Suppose $(h'_m, \ldots, h'_0)_\gamma$ be an another expansion of N with the property (2.2). It is shown that $e_k > h'_{k-1} e_{n-1} + \cdots + h'_0 e_0$ for any $k > 0$ as $(h'_n, \ldots, h'_0)_\gamma$ fulfills (2.2). Then we see $e_m \le h'_m e_m \le N = h'_m e_m + \cdots + h'_0 e_0 < e_{m+1}$, hence $m = n$ and $h_n = \lfloor N/e_n \rfloor = h'_n$. Applying same argument to the greedy expansions $(h_{n-1}, \ldots, h_0)_\gamma$ and $(h'_{n-1}, \ldots, h'_0)_\gamma$, $h_k = h'_k$ holds for any $k \ge 0$.
(2) We see $\rho^{e_0}(0) = \gamma_0$ by definition. Suppose $\rho^{e_k}(0) = (-1)^k \gamma_k$ for $0 \le k \le l$, then we have $\rho^{e_{l+1}}(0) = (\rho^{e_l})^{\sigma_l} \circ \rho^{e_{l-1}}(0) = (-1)^l (\sigma_l \gamma_l - \gamma_{l-1}) = (-1)^{l+1} \gamma_{l+1}$. Therefore for a greedy expansion $N = (h_n, \ldots, h_0)_\gamma$, $\rho^N(0) = (\rho^{e_n})^{h_n} \circ \cdots \circ (\rho^{e_0})^{h_0}(0) = \sum_{k=0}^{n} (-1)^k h_k \gamma_k$. □

Let $\mathcal{I} = \{[0, l] \subset [0, 1]\}$ be a set of closed intervals with 0 as the left end. The length $|A|$ of $A \in \mathcal{I}$ is given by l whenever $A = [0, l]$. For $A_i = [0, a_i] \in \mathcal{I},\ i = 1, \ldots, n$, we define the *direct sum* of intervals A_i's as

$$A_1 \oplus A_2 \oplus \cdots \oplus A_n = \{[0, a_1], [a_1, a_1 + a_2], \ldots, [a_1 + \cdots + a_{n-1}, a_1 + \cdots + a_n]\},$$

labeled by $a_1 a_2 \cdots a_n$. We say $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ gives a *partition* of $B = [0, b] \in \mathcal{I}$ whenever $b = a_1 + \cdots + a_n$. Conversely, for a direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_n$, $[A_1 \oplus A_2 \oplus \cdots \oplus A_n]$ stands for a *merge* of intervals A_i's:

$$[A_1 \oplus A_2 \oplus \cdots \oplus A_n] = [0, a_1 + a_2 + \cdots + a_n].$$

Since each label $a_1 a_2 \cdots a_n$ uniquely corresponds to a partition $\oplus_{k=1}^n [0, a_k]$ of $[0, 1]$ whenever $a_1 + \cdots + a_n = 1$, a bijection Φ between labels and direct sums is defined:

$$\begin{aligned} \Phi : \bigcup_{n=1}^{\infty} \{a_1 a_2 \cdots a_n \mid a_k > 0, a_1 + \cdots + a_n = 1\} \ni a_1 \cdots a_n \\ \mapsto \bigoplus_{k=1}^{n} [0, a_k] \in \bigcup_{n=1}^{\infty} \{A_1 \oplus \cdots \oplus A_n \mid A_k \in \mathcal{I},\ [A_1 \oplus \cdots \oplus A_n] = [0, 1]\} \end{aligned} \tag{2.3}$$

We abbreviate the k-times direct sum $A \oplus \cdots \oplus A$ of $A \in \mathcal{I}$ as $A^{\oplus k}$ and k-repetition $a \cdots a$ of a as a^k. $a \pm S$ denotes a set $\{a \pm s \mid s \in S\}$ for $S \subset \mathbf{R}/\mathbf{Z}$ and $a \in \mathbf{R}/\mathbf{Z}$. Given any subset of positive real numbers

$\Sigma \subset \mathbf{R}_{>0}$, Σ^* stands for a free semigroup

$$\Sigma^* = \bigcup_{n=1}^{\infty} \{a_1 \cdots a_n \mid a_k \in \Sigma\}$$

generated by the alphabet Σ with a concatenation as a semigroup operation. For alphabets Σ_1 and Σ_2 with same finite cardinality, we call a word $\boldsymbol{a} \in \Sigma_1^*$ is *isomorphic* to a word $\boldsymbol{b} \in \Sigma_2^*$ whenever there exists a bijection between the alphabets $\pi : \Sigma_1 \to \Sigma_2$ which extends to a semigroup isomorphism $\pi^* : \Sigma_1^* \to \Sigma_2^*$, and $\pi^*(\boldsymbol{a}) = \boldsymbol{b}$ holds.

Theorem 2.3 (cf.[6], **Theorem 3.7**). *Let ρ be a circle rotation $\mathbf{R}/\mathbf{Z} \ni x \mapsto x + \gamma \in \mathbf{R}/\mathbf{Z}$ with $0 < \gamma < 1$ and take the sequences given by* (2.1). *Consider substitutions $\theta_k : \{\gamma_{k-1}, \gamma_k\}^* \to \{\gamma_k, \gamma_{k+1}\}^*$ and $\eta_k : \{\gamma_{k-1}, \delta_{k-1}\}^* \to \{\gamma_k, \delta_k\}^*$ defined by*

$$\theta_k : \begin{cases} \gamma_{k-1} \mapsto \gamma_k^{\sigma_k} \gamma_{k+1}, & \text{whenever } k \text{ is even,} \\ \gamma_{k-1} \mapsto \gamma_{k+1} \gamma_k^{\sigma_k}, & \text{whenever } k \text{ is odd,} \\ \gamma_k \mapsto \gamma_k, \end{cases}$$

and

$$\eta_k : \begin{cases} \gamma_{k-1} \mapsto \gamma_k^{\sigma_k - 1} \delta_k, & \\ \delta_{k-1} \mapsto \gamma_k^{\sigma_k} \delta_k, & \text{whenever } k \text{ is even,} \\ \gamma_{k-1} \mapsto \delta_k \gamma_k^{\sigma_k - 1}, & \\ \delta_{k-1} \mapsto \delta_k \gamma_k^{\sigma_k}, & \text{whenever } k \text{ is odd,} \end{cases}$$

where $\delta_k = \gamma_k + \gamma_{k+1}$. Then

(1) *$\rho^{D_n}(0) := \{\rho^k(0) \mid k \in D_n\}$ generate a partition $\Phi(\theta_{n-1} \circ \theta_{n-2} \circ \cdots \circ \theta_0(\gamma_{-1}))$ of $[0, 1]$, where $D_n = \{0, 1, \ldots, e_n + e_{n-1} - 1\}$.*

(2) *$\rho^{C_n}(0) := \{\rho^k(0) \mid k \in C_n\}$ generate a partition $\Phi(\eta_n \circ \eta_{n-1} \circ \cdots \circ \eta_0(\gamma_{-1}))$ of $[0, 1]$, where $C_n = \{0, 1, \ldots, e_{n+1} - 1\}$.*

Proof. (1) For $n = 1$, the first $e_1 + e_0 = \sigma_0 + 1$ points $0, \rho(0), \ldots, \rho^{\sigma_0}(0)$ induce a partition $[0, \gamma_0]^{\oplus \sigma_0} \oplus [0, \gamma_1]$, which corresponds to the label $\theta_0(\gamma_{-1}) = \gamma_0^{\sigma_0} \gamma_1$. Suppose that the statement holds for $n \geq 1$, then we see that the partition $P_n = \Phi(\theta_{n-1} \circ \theta_{n-2} \circ \cdots \circ \theta_0(\gamma_{-1}))$ consists of intervals $[0, \gamma_n]$ and $[0, \gamma_{n-1}]$, where $\rho^{D_n}(0)$ gives the set of division points of the partition P_n. We also see that each component I of the partition P_n with $|I| = \gamma_{n-1}$ is represented as $I = \rho^N(0) + [0, \gamma_{n-1}] = [\rho^N(0), \rho^{N+e_{n-1}}(0)]$ if n is odd, and $I = \rho^N(0) + [-\gamma_{n-1}, 0] = [\rho^{N+e_{n-1}}(0), \rho^N(0)]$ if n is even, as $\rho^{e_{n-1}}(0) = (-1)^{n-1}\gamma_{n-1}$, where $0 \leq N < e_n$. It follows from Proposition 2.2 (2) that $I \cap (D_{n+1} \setminus D_n) = \{\rho^{N+e_{n-1}+ke_n}(0) \mid k = 1, \ldots, \sigma_n\}$, showing that, if n is even, the increasing sequence $\rho^{N+e_{n-1}+e_n}(0) < \cdots < \rho^{N+e_{n-1}+\sigma_n e_n}(0) = \rho^{N+e_{n+1}}(0)$ yields a subdivision $\rho^N(0) + ([0, \gamma_n]^{\oplus \sigma_n} \oplus [0, \gamma_{n+1}])$ of I, and if n is odd, the decreasing sequence $\rho^{N+e_{n-1}+e_n}(0) > \cdots > \rho^{N+e_{n+1}}(0)$ yields a subdivision $\rho^N(0) + ([0, \gamma_{n+1}] \oplus [0, \gamma_n]^{\oplus \sigma_n})$ of I. In any case, the subdivision is represented as $\rho^N(0) + \Phi(\eta_n(\gamma_{n-1}))$. It follows from the decomposition

$$D_{n+1} \setminus D_n = \bigcup_{N=0}^{e_n - 1} \{N + e_{n-1} + ke_n \mid k = 1, \ldots, \sigma_n\},$$

that no component I' of the partition P_n with $|I'| = \gamma_n$ contains $\rho^k(0)$ for $k \in D_{n+1} \setminus D_n$. Consequently we prove that the set $\rho^{D_{n+1}}(0)$ generates $\Phi(\theta_n \circ \theta_{n-1} \circ \cdots \circ \theta_0(\gamma_{-1}))$. Similar argument shows (2). $\square$

Definition 2.4. *We call the label $\theta_{n-1} \circ \theta_{n-2} \circ \cdots \circ \theta_0(\gamma_{-1})$ associated with the set $\rho^{D_n}(0)$ the n-th D-word, and the label $\eta_n \circ \eta_{n-1} \circ \cdots \circ \eta_0(\gamma_{-1})$ associated with the set $\rho^{C_n}(0)$ the n-th C-word respectively.*

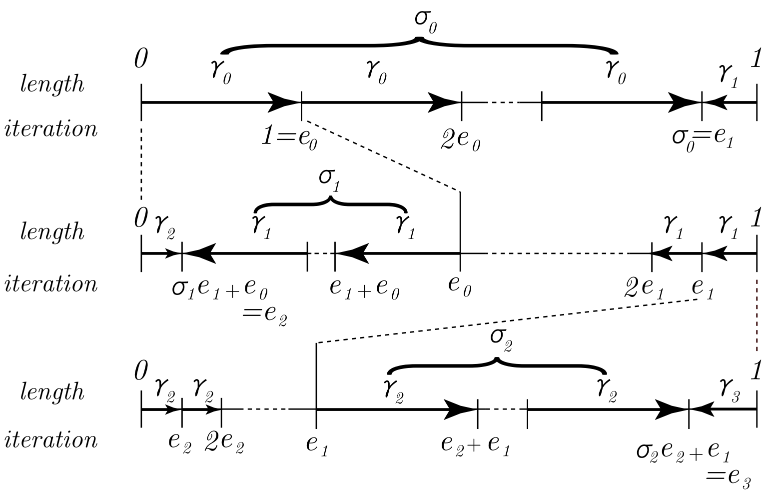

FIGURE 1. The division process of $\mathbf{R}/\mathbf{Z}$ induced by the circle rotation ρ. The right end 1 is identified with 0.

Note that the C- and D-words are finite whenever γ is a rational number; finally $\gamma_{n+1}=0$, hence $\gamma_n=\delta_n$ for some n, however, we distinguish between γ_n and δ_n as letters for technical reasons. Those 'particular' δ_n's have the following characterization, which implies the spatio-temporal symmetry on circle rotations.

Corollary 2.5. *Let $\boldsymbol{c}=c_1c_2\cdots c_{e_{n+1}}\in\{\gamma_n,\delta_n\}^*$ be the n-th C-word associated with a rational number $0<\gamma=[0;\sigma_0,\dots,\sigma_n]<1$. Then $c_k=\delta_n$ for even n (resp. $c_{e_{n+1}-k+1}=\delta_n$ for odd n) if and only if $\{k\lambda\}<\lambda$ where $\lambda={}^t[0;\sigma_0,\dots,\sigma_n]:=[0;\sigma_n,\dots,\sigma_0]$.*

Proof. We firstly note that ρ has the period e_{n+1}: $\rho^{e_{n+1}}(x)=x$. Suppose that n is even, then we see $\rho^{e_n}(0)=\gamma_n$ is positive by Proposition 2.2. Thus each letter c_k in $\boldsymbol{c}$ corresponds to the interval $[\rho^{(k-1)e_n}(0),\rho^{ke_n}(0)]$ the partition $\Phi(\boldsymbol{c})$ contains. Consider the $(n-1)$-th C-word $\boldsymbol{c'}=c'_1\cdots c'_{e_n}$. It comes from Theorem 2.3 (2) that $\eta_n(c'_l)=\gamma_n^{\sigma_n-1}\delta_n$, or $\gamma_n^{\sigma_n}\delta_n$ holds for any c'_l, while c'_l corresponds to an interval $[\rho^a(0),\rho^b(0)]\in\Phi(\boldsymbol{c'})$ with $0\le a,b<e_n$. As a result, we see that $\rho^{ke_n}(0)=\rho^b(0)$ for some $0\le b<e_n$ whenever $c_k=\delta_n$, that is, $ke_n\equiv b \bmod e_{n+1}$. Therefore we see $\{k\lambda\}<\lambda$ since $e_n/e_{n+1}=[0;\sigma_n,\dots,\sigma_0]$ by Proposition 2.2 (3). Note that, as $\rho^{e_n}(0)=-\gamma_n$ is negative when n is odd, c_k corresponds to the interval $[1+\rho^{(e_{n+1}-k+1)e_n}(0),1+\rho^{(e_{n+1}-k)e_n}(0)]$. Then, the similar argument shows the rest. $\square$

Example 2.1. Consider $\gamma=13/30=[0;2,3,4]$. The D-words grow as

$$\gamma_{-1}\overset{\theta_0}{\mapsto}\gamma_0^2\gamma_1\overset{\theta_1}{\mapsto}(\gamma_2\gamma_1^3)(\gamma_2\gamma_1^3)\gamma_1\overset{\theta_2}{\mapsto}(\gamma_2(\gamma_2^4\gamma_3)(\gamma_2^4\gamma_3)(\gamma_2^4\gamma_3))(\gamma_2(\gamma_2^4\gamma_3)(\gamma_2^4\gamma_3)(\gamma_2^4\gamma_3))(\gamma_2^4\gamma_3),$$

using γ_3 in spite of $\gamma_3=0$, while the C-words grow as

$$\gamma_{-1}\overset{\eta_0}{\mapsto}\gamma_0\delta_0\overset{\eta_1}{\mapsto}(\delta_1\gamma_1^2)(\delta_1\gamma_1^3)\overset{\eta_2}{\mapsto}((\gamma_2^4\delta_2)(\gamma_2^3\delta_2)(\gamma_2^3\delta_2))((\gamma_2^4\delta_2)(\gamma_2^3\delta_2)(\gamma_2^3\delta_2)(\gamma_2^3\delta_2)),$$

which is obtained by replacing $\gamma_k\gamma_{k+1}$ for even k, and $\gamma_{k+1}\gamma_k$ for odd k, with δ_k in the D-words. Again here, in spite of $\gamma_2=\delta_2$ as real numbers, we distinguish between γ_2 and δ_2 as letters.

Example 2.2. Consider $\gamma=\log_2(3/2)=[0;1,1,2,2,3,1,5,2,23,\dots]$. The D-words grow as

$$\gamma_{-1}\overset{\theta_0}{\mapsto}\gamma_0\gamma_1\overset{\theta_1}{\mapsto}(\gamma_2\gamma_1)\gamma_1\overset{\theta_2}{\mapsto}\gamma_2(\gamma_2\gamma_2\gamma_3)(\gamma_2\gamma_2\gamma_3)\overset{\theta_3}{\mapsto}\cdots,$$

while the C-words grow as

$$\gamma_{-1}\overset{\eta_0}{\mapsto}\delta_0\overset{\eta_1}{\mapsto}\delta_1\gamma_1\overset{\eta_2}{\mapsto}(\gamma_2\gamma_2\delta_2)(\gamma_2\delta_2)\overset{\eta_3}{\mapsto}((\delta_3\gamma_3)(\delta_3\gamma_3)(\delta_3\gamma_3\gamma_3))((\delta_3\gamma_3)(\delta_3\gamma_3\gamma_3))\overset{\eta_4}{\mapsto}\cdots.$$

These words are deeply connected with the musical scales, stated in Section 4.

3. **Spatio-temporal symmetry on circle rotations**

As we see, a circle rotation ρ induces a sequence of subdivisions of the circle $\mathbf{R}/\mathbf{Z}$, which represents the spatial structure of the dynamics, and described by C- and D-words, while the temporal sequence $0, \rho(0), \rho^2(0), \ldots$ induces another kinds of words. In this section, we state the one-to-one correspondence between the spatial words and the temporal ones, which shows that circle rotations have a kind of spatio-temporal symmetry.

Definition 3.1. *Let us take a real number* $0 < \gamma < 1$. *The* lower mechanical word *associated with* γ *is a (in)finite sequence of* $\{0,1\}$ *given by*

$$\boldsymbol{v} = v_1 v_2 \cdots, \quad v_k = \lfloor k\gamma \rfloor - \lfloor (k-1)\gamma \rfloor .$$

Similarly, the upper mechanical word *is a (in)finite sequence of* $\{0,1\}$ *given by*

$$\boldsymbol{u} = u_1 u_2 \cdots, \quad u_k = \lceil k\gamma \rceil - \lceil (k-1)\gamma \rceil ,$$

where $\lceil x \rceil$ *stands for the smallest integer not less than* x. *If a finite lower (resp. upper) mechanical word* $\boldsymbol{w}$ *has a factorization* $\boldsymbol{w} = \boldsymbol{ab}$ *where both* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are finite lower (resp. upper) mechanical words, we call* $\boldsymbol{w} = \boldsymbol{ab}$ *a* standard factorization.

It is noted that the lower (resp. upper) *Christoffel* word is obtained from the lower (resp. upper) mechanical word by replacing 1 with 01 (resp. 10). Thus the palindromic characterization and the unique factorization property is inherited from Christoffel words as follows. See [1] for details.

Lemma 3.2. *Let* $0 < \gamma < 1$ *be a real number with fractional expansion* $[0; \sigma_0, \sigma_1, \sigma_2, \ldots]$, *whose* n*-th continuant* $[0; \sigma_0, \sigma_1, \ldots, \sigma_n]$ *equals to* f_{n+1}/e_{n+1}, *where* f_k*'s and* e_k*'s are defined by* (2.1). *Let* $\boldsymbol{v} = v_1 \cdots v_{e_{n+1}}$ *and* $\boldsymbol{u} = u_1 \cdots u_{e_{n+1}}$ *be the lower and upper mechanical words associated with the continuant* $\alpha = f_{n+1}/e_{n+1}$ *respectively.*

(1) $v_k = u_k$ *holds for* $2 \le k \le e_{n+1} - 1$, *thus let us put* $\boldsymbol{w} = w_2 \cdots w_{e_{n+1}-1}$ *where* $w_k = v_k = u_k$. *Then we have* $\boldsymbol{v} = 0\boldsymbol{w}1$ *and* $\boldsymbol{u} = 1\boldsymbol{w}0$. *Moreover we see* $\boldsymbol{w}$ *is a palindrome:* $\boldsymbol{w} = \widetilde{\boldsymbol{w}} = w_{e_{n+1}-1} \cdots w_2$.

(2) *Let us put* $\boldsymbol{w}_p = w_2 \cdots w_{e_n - 1}$ *and* $\boldsymbol{w}_s = w_{e_n+2} \cdots w_{e_{n+1}-1}$. *Suppose that* n *is even. Then* $0\boldsymbol{w}_p1$ *and* $0\boldsymbol{w}_s1$ *coincide with the lower mechanical words associated with* $\alpha_p = f_n/e_n$ *and* $\alpha_s = (f_{n+1} - f_n)/(e_{n+1} - e_n)$ *respectively, hence* $\boldsymbol{w}_p$ *and* $\boldsymbol{w}_s$ *are also palindromes. Moreover, we have factorizations* $\boldsymbol{v} = (0\boldsymbol{w}_p1)(0\boldsymbol{w}_s1)$ *and* $\boldsymbol{u} = (1\boldsymbol{w}_s0)(1\boldsymbol{w}_p0)$, *which are unique standard factorizations. For the case of odd* n, $1\boldsymbol{w}_p0$ *and* $1\boldsymbol{w}_s0$ *coincide with the upper mechanical words, and it holds that unique standard factorizations* $\boldsymbol{u} = (1\boldsymbol{w}_p0)(1\boldsymbol{w}_s0)$ *and* $\boldsymbol{v} = (0\boldsymbol{w}_s1)(0\boldsymbol{w}_p1)$.

Proof. (1) We see $\lfloor k\alpha \rfloor + \lceil (e_{n+1} - k)\alpha \rceil = \lfloor k\alpha \rfloor + f_{n+1} + \lceil -k\alpha \rceil = f_{n+1}$ for $k = 0, \ldots, e_{n+1}$. Hence for $k = 1, \ldots, e_{n+1}$, we have

$$v_k = \lfloor k\alpha \rfloor - \lfloor (k-1)\alpha \rfloor = (e_n - \lceil (e_{n+1} - k)\alpha \rceil) - (e_n - \lceil (e_{n+1} - k + 1)\alpha \rceil) = u_{e_{n+1}-k+1},$$

while we have

$$u_k = \lceil k\alpha \rceil - \lceil (k-1)\alpha \rceil = (\lfloor k\alpha \rfloor + 1) - (\lfloor (k-1)\alpha \rfloor + 1) = v_k$$

for $k = 2, \ldots, e_{n+1} - 1$, showing that $\boldsymbol{w}$ is a palindrome.

(2) Suppose that n is even. Then, it can be seen from Proposition 2.2 and Theorem 2.3 that $0 < \{e_n\alpha\} < \{k\alpha\}$ holds for $k \in \{1, \ldots, e_{n+1} - 1\} \setminus \{e_n\}$, which shows $\lfloor k\alpha \rfloor = \lfloor k\alpha_p \rfloor$ for $k = 0, \ldots, e_n$ and $\lfloor k\alpha \rfloor = \lfloor k\alpha_s \rfloor$ for $k = e_n, \ldots, e_{n+1}$. As $0 < \alpha = e_n\alpha - (e_n - 1)\alpha = v_{e_n} + \{e_n\alpha\} - \{(e_n - 1)\alpha\}$ and $\{e_n\alpha\} < \{(e_n - 1)\alpha\}$, we have $v_{e_n} = 1$. We also see $1 > \alpha = v_{e_n+1} + \{(e_n + 1)\alpha\} - \{e_n\alpha\} > v_{e_n+1}$, hence $v_{e_n+1} = 0$. Consequently, we have the factorizations $\boldsymbol{v} = (0\boldsymbol{w}_p1)(0\boldsymbol{w}_s1)$ and $\boldsymbol{u} = \tilde{\boldsymbol{v}} = (1\boldsymbol{w}_s0)(1\boldsymbol{w}_p0)$. The palindromic property (1) induces the statements for odd n. $\square$

The following simple fact is significant in two meanings: connecting the spatio and temporal structure of circle rotations, and suggesting a general notion of diatonic scales in mathematical music theory.

Lemma 3.3. *Let $\boldsymbol{v} = v_1 v_2 \cdots$ be the lower mechanical word associated with $0 < \gamma < 1$. Then $v_k = 1$ holds if and only if $\{k\gamma\} < \gamma$.*

Proof. Because $\gamma = k\gamma - (k-1)\gamma = v_k + \{k\gamma\} - \{(k-1)\gamma\}$, $\gamma > \{k\gamma\}$ holds if and only if $v_k = 1$. □

Combining Corollary 2.5 and Lemma 3.3, we show that the spatio structure of rotation dynamics stated in Theorem 2.3 is also described by the temporal structure of its 'transpose' rotation as follows.

Theorem 3.4. *Consider a rational number $0 < \gamma = [0; \sigma_0, \ldots, \sigma_n] < 1$ and take the n-th C-word $\boldsymbol{c} = c_1 \cdots c_{e_{n+1}} \in \{\gamma_n, \delta_n\}^*$ associated with γ. Let $\boldsymbol{v}$ be the lower mechanical words associated with the rational number $0 < \lambda = {}^t[0; \sigma_0, \ldots, \sigma_n] < 1$. Then, it holds that $\boldsymbol{c} = \pi_n(\boldsymbol{v})$ if n is even, and $\boldsymbol{c} = \pi_n(\tilde{\boldsymbol{v}})$ if n is odd, where π_n is a semigroup isomorphism $\pi_n : \{0,1\}^* \to \{\gamma_n, \delta_n\}^*$ with $\pi_n(0) = \gamma_n$ and $\pi_n(1) = \delta_n$.*

Proof. For even n, $c_k = \delta_n$ means $\{k\lambda\} < \lambda$ by Corollary 2.5, hence $v_k = 1$ by Lemma 3.3. For odd n, as $\lambda = e_n/e_{n+1}$ by Proposition 2.1 (3), $\boldsymbol{v}$ has length e_{n+1}, hence $v_{e_{n+1}-k+1} = \tilde{v}_k$. Thus $c_k = \delta_n$ holds if and only if $\tilde{v}_k = 1$. □

Roughly speaking, as a dynamics on $\mathbf{Z}/e_{n+1}\mathbf{Z}$, the temporal behavior of the circle rotation ρ with the rotation number $\gamma = [0; \sigma_0, \ldots, \sigma_n] = f_{n+1}/e_{n+1}$ is simulated by the dynamics $x \mapsto x + f_{n+1}$, while the spatio structure is described by the dynamics $x \mapsto x + e_n$ for even n and $x \mapsto x - e_n$ for odd n. Theorem 3.4 shows that, by replacing the rotation number $\lambda = [0; \sigma_n, \ldots, \sigma_0]$, the transpose of γ, the temporal and spatial roles are interchanged, which implies that circle rotations have a spatio-temporal symmetry.

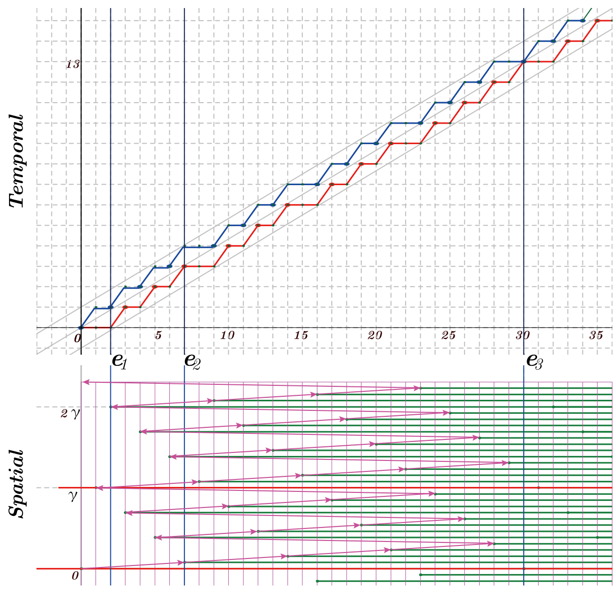

FIGURE 2. Spatio-temporal structure for $\gamma = [0; 2, 3, 4]$, with $f_1/e_1 = 1/3$, $f_2/e_2 = 3/7$, $f_3/e_3 = 13/30$.

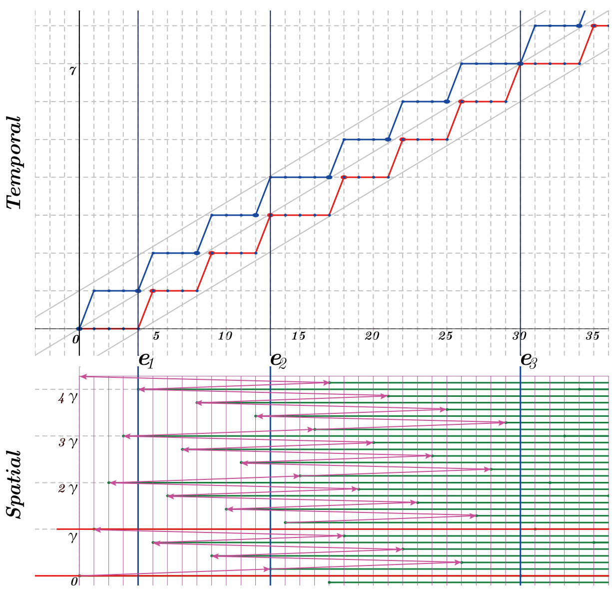

FIGURE 3. Spatio-temporal structure for $\gamma = [0; 4, 3, 2]$, with $f_1/e_1 = 1/4$, $f_2/e_2 = 3/13$, $f_3/e_3 = 7/30$.

4. Pythagorean tuning and circle rotations: a notion of general diatonic/chromatic scale

Musical tuning based on the frequency ratio 3 : 2, called *Pythagorean tuning*, was widely used and developed by medieval music theorists. The pitch height ratio 3 : 2 of two pure tones is called the *perfect* *fifth*, which is known as one of the most consonant ratio after the *unison* $1:1$ and the *octave* $2:1$. The tuning goes by *"stacking the perfect fifth intervals iteratively, modulo octave"*. That is, given a base note with frequency f_0, the tuning proceeds

$$f_0,\ \frac{3}{2}f_0,\ \left(\frac{3}{2}\right)^2 f_0 \equiv \frac{9}{8}f_0,\ \left(\frac{3}{2}\right)^3 f_0 \equiv \frac{27}{16}f_0,\ \left(\frac{3}{2}\right)^4 f_0 \equiv \frac{81}{64}f_0, \ldots.$$

Equivalently, taking $\log_2$ and eliminating the common term $\log_2 f_0$, we are to consider

$$0,\ \log_2\left(\frac{3}{2}\right),\ \left\{\ 2\log_2\left(\frac{3}{2}\right)\right\},\ \left\{\ 3\log_2\left(\frac{3}{2}\right)\right\},\ \left\{\ 4\log_2\left(\frac{3}{2}\right)\right\}, \ldots,$$

that is, we are tracing the orbit $0, \rho(0), \rho^2(0), \ldots$ of the circle rotation $\rho(x) = x + \log_2(3/2)$. When we start with the musical note F, stacking perfect fifth intervals repeatedly gives a sequence of notes (called the *cycle of fifth*) F, C, G, D, A, E, B, and rearranging these notes in ascending order, we obtain the usual diatonic scale (the *Lydian mode* on C)

$$F \xrightarrow{w} G \xrightarrow{w} A \xrightarrow{w} B \xrightarrow{h} C \xrightarrow{w} D \xrightarrow{w} E \xrightarrow{h} (\dot{F}),$$

where w and h on each arrow stand for the distance of pitches between adjacent notes: w means the *whole* step and h the *half* step. We note that the word $wwwhwwh$ corresponds to the 3rd D-word c while the 3rd C-word $\delta_3\gamma_3\delta_3\gamma_3\delta_3\gamma_3\gamma_3\delta_3\gamma_3\delta_3\gamma_3\gamma_3$ corresponds to the successive half steps of the 12-tones chromatic scale

$$F \to F\# \to G \to G\# \to A \to A\# \to B \to C \to C\# \to D \to D\# \to E \to (\dot{F}).$$

Thus one may think that the correspondence of the diatonic scale and the D-word derives a general notion of the diatonic scale, however it is a particular case. It should be noted that the 4-th continuant $7/12 = [0; 1, 1, 2, 2]$ of $\log_2(3/2)$ has a striking spatio-temporal symmetry, as follows.

Proposition 4.1. *For a rational number $0 < \gamma = [0; \sigma_0, \ldots, \sigma_n] < 1$, let $\boldsymbol{c}$ and $\boldsymbol{v}$ be the n-th C-word and the lower mechanical word respectively. Then it holds that, for even n,*

$$\boldsymbol{c} = \begin{cases} \pi_n(\boldsymbol{v}), & \text{if } [0;\sigma_0,\ldots,\sigma_n] = {}^t[0;\sigma_0,\ldots,\sigma_n], \\ \pi_n(\tilde{\boldsymbol{v}}), & \text{if } \sigma_0 = 1 \text{ and } [0;\sigma_1+1,\ldots,\sigma_n] = {}^t[0;\sigma_1+1,\ldots,\sigma_n], \end{cases}$$

and for odd n,

$$\boldsymbol{c} = \begin{cases} \pi_n(\tilde{\boldsymbol{v}}), & \text{if } [0;\sigma_0,\ldots,\sigma_n] = {}^t[0;\sigma_0,\ldots,\sigma_n], \\ \pi_n(\boldsymbol{v}), & \text{if } \sigma_0 = 1 \text{ and } [0;\sigma_1+1,\ldots,\sigma_n] = {}^t[0;\sigma_1+1,\ldots,\sigma_n], \end{cases}$$

where π_n is a semigroup isomorphism $\pi_n : \{0,1\}^* \to \{\gamma_n, \delta_n\}^*$ with $\pi_n(0) = \gamma_n$ and $\pi_n(1) = \delta_n$.

Proof. When $\sigma_0 \neq 1$, the assertion comes from Theorem 3.4. As $[0; 1, \sigma_1, \ldots, \sigma_n] = 1 - [0; 1+\sigma_1, \ldots, \sigma_n]$, the case $\sigma_0 = 1$ is reduced to the case $\sigma_0 \neq 1$. □

The concept of diatonic sets is well studied in musical set theory, which has significant features such as maximal evenness, Myhill's property, well-formedness, the deep scale property, cardinality equals variety, and structure implies multiplicity(cf. [10]). Here, we adopt the following definition.

Definition 4.2. *Consider a (ir)rational number $0 < \gamma < 1$, and take the lower mechanical word $\boldsymbol{v}$ associated with its n-th continuant $[0; \sigma_0, \ldots, \sigma_n]$. We call the subset $\mathcal{D}_\gamma^n = \{k \in \mathbf{Z}/e_{n+1}\mathbf{Z} \mid v_k = 1\}$ the n-th* diatonic set*, while $\mathcal{C}_\gamma^n = \mathbf{Z}/e_{n+1}\mathbf{Z}$ itself the n-th* chromatic set.

We note that, by eliminating the first σ_0 in the continuant $\gamma = [0; \sigma_0, \ldots \sigma_n]$, the interval $[0, \gamma]$ is renormalized to $[0, 1]$, and the circle rotation with rotation number $\gamma' = [0; \sigma_1, \ldots, \sigma_n]$ conserves the temporal behavior of the diatonic set $\mathcal{D}_\gamma^n$. In another words, the temporal behavior of γ'-rotation is embedded in γ-rotation dynamics as a diatonic set. We prove this fact as an application of the spatio-temporal symmetry on circle rotations.

Proposition 4.3. *Consider the n-th diatonic set $\mathcal{D}^n_\gamma \subset \mathbf{Z}/e_{n+1}\mathbf{Z}$ associated with $\gamma = [0; \sigma_0, \sigma_1, \ldots]$. Then,*

(1) $\#\mathcal{D}^n_\gamma = f_{n+1}$.

(2) *Let $k_0 = 0 < k_1 < \cdots < k_{f_{n+1}-1} < k_{f_{n+1}} = e_{n+1} (\equiv 0 \bmod e_{n+1})$ be arrangement of elements in $\mathcal{D}^n_\gamma$ in ascending order, and consider a word $\boldsymbol{d} = d_1 d_2 \cdots d_{f_{n+1}}$, where $d_l = k_l - k_{l-1}$. Then $\boldsymbol{d} \in \{\sigma_0, \sigma_0+1\}^*$.*

(3) *Let $\boldsymbol{v}'$ be the lower mechanical word associated with $[0; \sigma_1, \sigma_2, \ldots, \sigma_n]$. Then $\psi_n(\tilde{\boldsymbol{v}}') = \boldsymbol{d}$ holds, where the semigroup isomorphism $\psi_n : \{0,1\}^* \to \{\sigma_0, \sigma_0+1\}^*$ is given by $\psi_n(0) = \sigma_0$ and $\psi_n(1) = \sigma_0 + 1$.*

Proof. (1) As $\gamma = [0; \sigma_0, \ldots, \sigma_n] = f_{n+1}/e_{n+1}$, the definition of mechanical words shows the identity $\#\mathcal{D}^n_\gamma = \sum_{k=1}^{e_{n+1}} v_k = f_{n+1}$.

(2) For $k_l \in \mathcal{D}^n_\gamma$, suppose $\gamma_1 < \{k_l\gamma\} < \gamma$. Then $\{k_l\gamma\} + m\gamma < 1$ for $0 \le m < \sigma_0$ and $\{k_l\gamma\} + \sigma_0\gamma > \gamma_1 + \sigma_0\gamma = 1$, hence $k_{l+1} = k_l + \sigma_0$. Suppose $\{k_l\gamma\} < \gamma_1$, then $\{k_l\gamma\} + \sigma_0\gamma < \gamma_1 + \sigma_0\gamma = 1$, hence $k_{l+1} = k_l + \sigma_0 + 1$.

(3) Suppose n is even. Note that $\gamma = f_{n+1}\gamma_n$ and $\gamma_1 = (e_{n+1} - \sigma_0 f_{n+1})\gamma_n$ hold. We have seen that $d_{l+1} = \sigma_0 + 1$ if and only if $\{k_l\gamma\} < \gamma_1$, hence $k_l \equiv m \bmod e_{n+1}$ with $0 \le m < e_{n+1} - \sigma_0 f_{n+1}$. By Theorem 3.4, the C-word $\boldsymbol{c}$ associated with γ is isomorphic to the lower mechanical word $\boldsymbol{v}$ associated with the transpose $\lambda = [0; \sigma_n, \ldots, \sigma_0]$. Since $\gamma = f_{n+1}\gamma_n$, the diatonic set $\mathcal{D}^n_\gamma$ corresponds to the prefix $\boldsymbol{c}_p = c_1 \cdots c_{f_{n+1}}$, which is isomorphic to the prefix $\boldsymbol{v}_p = v_1 \cdots v_{f_{n+1}}$ of $\boldsymbol{v}$. By Lemma 3.2 (2), the prefix $\boldsymbol{v}_p$ coincides with the upper mechanical word associated with $f_n/f_{n+1} = [0; \sigma_n, \ldots, \sigma_1]$, which is isomorphic to the C-word $\boldsymbol{c}'$ associated with $\gamma' = [0; \sigma_1, \ldots, \sigma_n]$. Since

$$\begin{pmatrix} \sigma_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \sigma_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} e_{n+1} & e_n \\ f_{n+1} & f_n \end{pmatrix} = \begin{pmatrix} f_{n+1} & f_n \\ e_{n+1} - \sigma_0 f_{n+1} & e_n - \sigma_0 f_n \end{pmatrix},$$

we have $\gamma' = (e_{n+1} - \sigma_0 f_{n+1})/f_{n+1}$. As a result, $d_l = \sigma_0 + 1$ holds if and only if $\{l\gamma'\} < \gamma'$, while the condition $\{l\gamma'\} < \gamma'$ is equivalent to $u'_l = 1$, the l-th letter in the upper mechanical word $\boldsymbol{u}' = u'_1 \cdots u'_{f_{n+1}} = \tilde{\boldsymbol{v}}'$. The assertion is proved. $\square$

From these observations, we come to a notion of the *sub-diatonic resolution*.

Definition 4.4. *For a rational number $0 < \gamma = [0; \sigma_0, \ldots, \sigma_n]$, take continuants $\gamma^{(t)} = [0; \sigma_t, \sigma_{t+1}, \ldots, \sigma_n]$ for $0 \le t \le n$, and associated diatonic sets $\mathcal{D}^{n-t}_{\gamma^{(t)}}$. Proposition 4.3 (3) brings a sequence of natural inclusions*

$$\mathcal{D}^n_{\gamma^{(0)}} \supset \mathcal{D}^{n-1}_{\gamma^{(1)}} \supset \mathcal{D}^{n-2}_{\gamma^{(2)}} \supset \cdots \supset \mathcal{D}^0_{\gamma^{(n)}}. \tag{4.1}$$

We call (4.1) *the* sub-diatonic resolution *associated with γ.*

Let us apply the notion of the sub-diatonic resolution to $\gamma = 7/12 = [0; 1, 1, 2, 2]$. As $\gamma^{(1)} = [0; 1, 2, 2] = 5/7$, $\gamma^{(2)} = [0; 2, 2] = 2/5$ and $\gamma^{(3)} = [0; 2] = 1/2$, the corresponding diatonic sets are given below.

t	diatonic set $\mathcal{D}^{n-t}_{\gamma^{(t)}}$	usual name
0	$\{0, 2, 4, 6, 7, 9, 11\}$	diatonic scale
1	$\{0, 2, 4, 7, 9\}$	pentatonic scale
2	$\{0, 7\}$	perfect fifth
3	$\{0\}$	unison

Remark. We note that, as $7/12 = [0; 1, 1, 2, 2]$, Proposition 4.1 shows that the 3rd D-word coincides with the image of the lower mechanical word by π_3. Thus the features of diatonic set $\mathcal{D}^3_{7/12}$ can be discussed from the viewpoint of the distribution on the unit interval induced from the circle rotation, with the rotation number 7/12, however for a general $\gamma = [0; \sigma_0, \ldots]$, the diatonic set $\mathcal{D}^n_\gamma$ is to be analyzed by the circle rotation with its transpose $[0; \sigma_n, \ldots, \sigma_0]$ as the rotation number.

5. **Remarks**

It may be said that the history of the development of the musical temperament was that of conflicts of the 'warp' and the 'weft', that is, *the harmony and the melody.* For various art of the melody, including the modulation (changing key), it is convenient that the musical scale consists of a series of musical tones whose frequencies are constant multiples of the frequency of a fundamental tone, modulo octave. Particularly, as the Pythagorean scale is defined by a geometric series with integer geometric ratio, the Pythagorean tuning was widely used in Europe from ancient to medieval eras. However, such a scale contains multiple tones which sound unpleasant to most people if they sound simultaneously, called *dissonant.* It is empirically known that, to obtain pleasant sounds of multiple tones, the frequencies of the tones forms a simple integer ratio. As a result, simple integer ratios are needed for the musical scale from the harmonic aspect, which is incompatible with the geometric series construction, unfortunately. The incompatibility has brought a various kind of musical temperaments, such as just intonation, meantone temperament, well temperament and equal temperament. With these points, every musical tone (or note) in a piece of music should be interpreted in various means, at least the horizontal line (melody) and the vertical line (harmony).

In this note, we study the structure of musical scales from the viewpoint of the weft, but I hope the spatio-temporal approach induced from circle rotation dynamics serves studies on the diatonic set theory as an mathematical tool.

References

[1] J. Berstel, A. Lauve, C. Reutenauer and F. V. Saliola, *Combinatorics on Words: Christoffel Words and Repetitions in Words*, CRM Monograph Series, AMS, 2009.

[2] P. Bleher, *The energy level spacing for two harmonic oscillators with golden mean ratio of frequencies,* J. Statistical Physics **61**, pp 869-876, 1990.

[3] J. Clough and J. Douthett, *Maximal Even Sets*, Journal of Music Theory **35**, pp. 93-173, 1991.

[4] J. Clough N. Engebretsen and J. Kochavi, *Scales, Sets, and Interval Cycles: A Taxonomy*, Music Theory Spectrum **21**, no. 1, pp. 74-104, 1999.

[5] K. Dajani and C. Kraaikamp, *Ergodic theory of numbers*, Carus Mathematical Monographs **29**, Mathematical Association of America, 2002.

[6] Y. Hashimoto, *A renormalization approach to level statistics on 1-dimensional rotations,* Bull. of Aichi Univ. of Education, Natural Science **58**, pp. 5-11, 2009.

[7] M. Lothaire, *Algebraic combinatorics on words*, Encyclopedia of mathematics and its applications **90**, Cambridge University Press, 2002.

[8] T. Noll, *Sturmian sequence and morphisms a musical-theoretical application*, Mathematique et musique, Journee annuelle de la SMF, pp. 79-102, 2008.

[9] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8**, pp. 477-493, 1957.

[10] J. Timothy, *Foundations of Diatonic Theory: A Mathematically Based Approach to Music Fundamentals*, Key College Publishing, 2003.

[11] D. Wright, *Mathematics and Music*, Mathematical world **28**, AMS, 2009.

[12] J. C. Yoccoz, *Continued Fraction Algorithms for Interval Exchange Maps: an Introduction*, in '*Frontiers in Number Theory, Physics, and Geometry* **I**', pp. 403-438, 2005.

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