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k-summability of Formal Solutions for Linear PDEs with Constant Coefficients

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Abstract

We shall study the k-summability of formal solutions for the Cauchy problem of linear partial differential equations with constant coefficients. We employ the method of successive approximation for the analysis of summability.

1 Results

We consider the following Cauchy problem for linear partial differential equations with constant coefficients

$$\text{(E)} \qquad \begin{cases} \partial_t U(t,x) = \displaystyle\sum_{j=0}^{m} a_j \partial_x^{m-j} U(t,x), \\ U(0,x) = \varphi(x) \in \mathcal{O}_x, \end{cases}$$

where $(t,x) \in \mathbb{C}^2$, $m \geq 2$ and $a_j \in \mathbb{C}$ $(j = 0, 1, \ldots, m)$, $a_0 \neq 0$. Here the symbol $\mathcal{O}_x$ denotes the set of holomorphic functions in a neighborhood of the origin $x = 0$.

This Cauchy problem has a unique formal solution of the form

$$\hat{U}(t,x) = \sum_{n\geq 0} U_n(x) \frac{t^n}{n!}, \qquad U_0(x) = \varphi(x). \tag{1.1}$$

From the assumptions that $m \geq 2$ and $a_0 \neq 0$, this formal solution is divergent in general.

We shall study the summability of this formal solution.

Before stating our result, we define a sector $S = S(d, \beta, \rho)$ by

$$S(d,\beta,\rho) := \left\{ t \in \mathbb{C}; |d - \arg t| < \frac{\beta}{2}, 0 < |t| < \rho \right\}, \tag{1.2}$$

where d, β and ρ are called the direction, the opening angle and the radius of S, respectively. We write $S(d, \beta, \infty) = S(d, \beta)$ for short.

Now, our result is stated as follows.

Theorem 1 *For a fixed* $d \in \mathbb{R}$*, we define* $d_i = \frac{d+\arg a_m+2\pi i}{m}$ *for* $i = 0, 1, \ldots, m-1$*. Let*

$$\kappa = \frac{1}{m-1}$$

and for some $\varepsilon > 0$ *and* $r > 0$*,*

$$\Omega_x := \cup_{i=0}^{m-1} S(d_i, \varepsilon) \bigcup B(r).$$

We assume that the Cauchy data can be analytically continued in the region Ω_x *and has the exponential growth estimate of order at most* $m/(m-1)$*, that is*

$$|\varphi(x)| \le C \exp\left(\delta |x|^{m/(m-1)}\right), \quad x \in \Omega_x.$$

Then the formal solution $\hat{U}(t,x)$ *of the Cauchy problem (E) is* κ*-summable in* d *direction.*

We remark that this theorem is not a new result. In fact, k-summability of formal solutions for linear partial differential equations with constant coefficients was studied by many mathematicians. The first result of the k-summability was given by Lutz-Miyake-Schäfke [8] for the heat equation case. In Miyake [11], he treated the operator $\partial_t^p - \partial_x^q$ ($p < q$), which is a generalization of the heat operator. The results on more general operator for the constant coefficients case were given by Balser-Miyake [5], Balser [3] and Michalik [10]. The results for multisummability were given by Blaser [4], Michalik [9] (for moment partial differential equations) and Ōuchi [12], [13] (for linear and nonlinear partial differential equations). The t-variable coefficients case was treated by Ichinobe [6] and [7].

In this paper, we shall give the proof of Theorem 1 by employing the method of successive approximation, which is a new approach. This paper consists of the following sections. We shall give the review of k-summability in section 2. In section 3, we shall construct the formal solution of the original Cauchy problem (E) by the method of successive approximation. Moreover, we shall prepare some results for the series associated with the formal solution of (E) in section 4. In section 5, we shall prove Theorem 1. We shall give the proof of lemma 6 which is needed for proof of Theorem 1 in the final section 6.

2 Review of summability

In this section, we give some notations and definitions (cf. W. Balser [1], [2]).

Let $\kappa > 0$, $S = S(d, \beta)$ and $B(r) := \{x \in \mathbb{C}; |x| \le r\}$. Let $u(t,x) \in \mathcal{O}(S \times B(r))$ which means that $u(t,x)$ is holomorphic in $S \times B(r)$. Then we define that $u(t,x) \in \mathrm{Exp}_t(\kappa, S \times B(r))$, if for any closed subsector S' of S, there exist some positive constants C and δ such that

$$\max_{|x| \le r} |u(t,x)| \le C e^{\delta |t|^{\kappa}}, \quad t \in S'. \tag{2.1}$$

For $\kappa > 0$, we define that $\hat{u}(t,x) = \sum_{n=0}^{\infty} u_n(x)t^n$ is the formal power series of Gevrey order $1/\kappa$, if $u_n(x)$ are holomorphic on a common closed disk $B(r)$ for some $r > 0$ and there exist some positive constants C and K such that for any n,

$$\max_{|x|\le r} |u_n(x)| \le CK^n \Gamma\left(1+\frac{n}{\kappa}\right). \tag{2.2}$$

In this case, we write $\hat{u}(t,x) \in \mathcal{O}_x[[t]]_{1/\kappa}$.

Let $\kappa > 0$, $\hat{u}(t,x) = \sum_{n=0}^{\infty} u_n(x)t^n \in \mathcal{O}_x[[t]]_{1/\kappa}$ and $u(t,x)$ be an analytic function on $S(d,\beta,\rho) \times B(r)$. Then we define that

$$u(t,x) \cong_\kappa \hat{u}(t,x) \quad \text{in } S = S(d,\beta,\rho), \tag{2.3}$$

if for any closed subsector S' of S, there exist some positive constants C and K such that for any N, we have

$$\max_{|x|\le r} \left| u(t,x) - \sum_{n=0}^{N-1} u_n(x)t^n \right| \le CK^N |t|^N \Gamma\left(1+\frac{N}{\kappa}\right), \quad t \in S'. \tag{2.4}$$

For $\kappa > 0$, $d \in \mathbb{R}$ and $\hat{u}(t,x) \in \mathcal{O}[[t]]_{1/\kappa}$, we define that $\hat{u}(t,x)$ is *κ-summable* in d direction if there exist a sector $S = S(d,\beta,\rho)$ with $\beta > \pi/\kappa$ and an analytic function $u(t,x)$ on $S \times B(r)$ such that $u(t,x) \cong_\kappa \hat{u}(t,x)$ in S.

In this case, we write $\hat{u}(t,x) \in \mathcal{O}\{t\}_{\kappa,d}$.

We remark that the function $u(t,x)$ above for a *κ-summable* $\hat{u}(t,x)$ is unique if it exists. Therefore such a function $u(t,x)$ is called the *κ-sum* of $\hat{u}(t,x)$ in d direction.

3 Construction of the formal solution

We construct the formal solution by using the successive approximation method.

3.1 The sequence of Cauchy problems

First, we consider the sequence of the Cauchy problems.

$$(E_0) \qquad \begin{cases} \partial_t u_0(t,x) = a_0 \partial_x^m u_0(t,x), \\ u_0(0,x) = \varphi(x). \end{cases}$$

For $k \ge 1$,

$$(E_k) \qquad \begin{cases} \partial_t u_k(t,x) = \displaystyle\sum_{j=0}^{\min\{m,k\}} a_j \partial_x^{m-j} u_{k-j}(t,x), \\ u_k(0,x) = 0. \end{cases}$$

For each k, the Cauchy problem (E_k) has a unique formal power series solution of the form

$$(Sol_k) \qquad \hat{u}_k(t,x) = \sum_{n\geq 0} u_{k,n}(x)\frac{t^n}{n!}.$$

Then $\hat{U}(t,x) = \sum_{k\geq 0} \hat{u}_k(t,x)$ is the formal power series solution of the original Cauchy problem (E).

3.2 Construction of formal solutions $\hat{u}_k(t,x)$

In this subsection, we give a construction of the formal solutions $\hat{u}_k(t,x)$ of the Cauchy problem (E_k).

Lemma 2 *Let $k \geq 0$. For each k, the Cauchy problems (E_k) has a unique formal power series solution $\hat{u}_k(t,x)$ which is given by*

$$(3.1) \qquad \hat{u}_k(t,x) = \sum_{n\geq 0} u_{k,n}(x)\frac{t^n}{n!} = \sum_{n\geq 0} A_k(n)\varphi^{(mn-k)}(x)\frac{t^n}{n!},$$

where we take the sum of n for $mn-k \geq 0$, and $\{A_k(n)\}$ satisfy the following recurrence formula:

When $k = 0$,

$$(3.2) \qquad \begin{cases} A_0(n+1) = a_0 A_0(n) \quad (n \geq 0), \\ A_0(0) = 1. \end{cases}$$

When $k \geq 1$,

$$(3.3) \qquad \begin{cases} A_k(n+1) = \displaystyle\sum_{j=0}^{\min\{m,k\}} a_j A_{k-j}(n) \quad (n \geq 0), \\ A_k(0) = 0. \end{cases}$$

Proof.

We consider the case where $k = 0$. By substituting the formal solution (Sol_0) into the equation (E_0), we obtain the recurrence formula of $\{u_{0,n}(x)\}$

$$(3.4) \qquad \begin{cases} u_{0,n+1}(x) = a_0 u_{0,n}^{(m)}(x) \quad (n \geq 0), \\ u_{0,0}(x) = \varphi(x). \end{cases}$$

From the construction of the above recurrence formula (3.4), we can put

$$(3.5) \qquad u_{0,n}(x) = A_0(n)\varphi^{(mn)}(x) \quad (n \geq 0),$$

where $\{A_0(n)\}$ satisfy the recurrence formula (3.2)

$$\begin{cases} A_0(n+1) = a_0 A_0(n) \quad (n \geq 0), \\ A_0(0) = 1. \end{cases}$$

Indeed, we get $A_0(n) = a_0^n$.

We consider the case where $k \geq 1$. By substituting (Sol_k) into the equation (E_k), we obtain the recurrence formula of $\{u_{k,n}(x)\}$

$$\begin{cases} u_{k,n+1}(x) = \displaystyle\sum_{j=0}^{\min\{m,k\}} a_j u_{k-j,n}^{(m-j)}(x) \quad (n \geq 0), \\ u_{k,0}(x) \equiv 0. \end{cases} \tag{3.6}$$

From the construction of the above recurrence formula (3.6), we can put

$$u_{k,n}(x) = A_k(n)\varphi^{(mn-k)}(x) \quad (mn-k \geq 0), \tag{3.7}$$

and $u_{k,n}(x) \equiv 0$ for $mn - k < 0$, where $\{A_k(n)\}$ satisfy the recurrence formula (3.3)

$$\begin{cases} A_k(n+1) = \displaystyle\sum_{j=0}^{\min\{m,k\}} a_j A_{k-j}(n) \quad (n \geq 0), \\ A_k(0) = 0, \end{cases}$$

where $A_k(n) = 0$ for $mn - k < 0$.

Remark. We put $k = m\ell - r(k)$ $(\ell \geq 0)$ and $r(k) = 0, 1, \ldots, m-1$. When $k \equiv 0 \pmod m$, $r(k) = 0$. When $k \not\equiv 0 \pmod m$, the number $m - r(k)$ represents the remainder which divided k by m.

In this case, the expression (3.7) holds for $n \geq \ell$. Especially, it holds $u_{k,n}(x) \equiv 0$ for $n < \ell$, in other words, $A_k(n) = 0$ for $n < \ell$, which will be proved in next section (see Lemma 3).

4 Preliminaries for proof of Theorem 1

In this section, we shall give the property of generating functions of $\{A_k(n)\}$ and the associated moment series.

4.1 Generating functions of $\{A_k(n)\}$

We put

$$f_k(t) = \sum_{n \geq 0} A_k(n) t^n.$$

When $k = 0$, we obtain from the recurrence formula (3.2)

$$f_0(t) = \frac{1}{1 - a_0 t}, \tag{4.1}$$

which has a singular point at $t = 1/a_0$.

Similarly, from the recurrence formula (3.3) we have

$$f_k(t) = \frac{1}{1-a_0 t} \sum_{j=1}^{\min\{m,k\}} a_j t f_{k-j}(t), \tag{4.2}$$

which has a singular point at $t = 1/a_0$.

From the expressions (4.1) and (4.2), we can get the following result.

Lemma 3 *We write $k = m\ell - r(k)$, $0 \le r(k) \le m-1$. Then the order of zeros of $f_k(t)$ at $t = 0$ is at least ℓ, that is, $f_k(t) = O(t^\ell)$.*

Proof. The proof is done by the induction. We assume that it holds until $k-1$.

i) The case where $0 \le r(k) \le m-2$. Since $1 \le j \le \min\{m,k\} \le m$, we have

$$1 \le r(k) + j \le 2m - 2.$$

i-1) The case where $0 \le r(k) + j \le m-1$. In this case, we can represent

$$k - j = m\ell - r(k-j), \quad r(k-j) = r(k) + j.$$

Therefore from the assumption of induction, we have $f_{k-j}(t) = O(t^\ell)$.

i-2) The case where $m \le r(k) + j \le 2m-2$. In this case, we can represent

$$k - j = m(\ell - 1) - r(k-j), \quad r(k-j) = r(k) + j - m.$$

Therefore we have $f_{k-j}(t) = O(t^{\ell-1})$.

Hence in case i), we have $f_k(t) = O(t^\ell)$ from the recurrence formula (4.2).

ii) The case where $r(k) = m-1$.

Since $m \le r(k) + j \le 2m-1$, we can represent

$$k - j = m(\ell - 1) - r(k-j), \quad r(k-j) = r(k) + j - m.$$

Therefore we have $f_{k-j}(t) = O(t^{\ell-1})$ for all j with $1 \le j \le \min\{m,k\}$.

Hence in case ii), we have $f_k(t) = O(t^\ell)$ from the recurrence formula (4.2).

Moreover, from the expressions (4.1) and (4.2), we see that $|f_k(t)| \le C_\ell |t|^\ell$ ($|t| \to \infty$) with $C_\ell \le CK^\ell$ by some positive constants C and K.

We put

$$F_k(t) := \sum_{n \ge 0} A_k(n+\ell) t^n = f_k(t)/t^\ell.$$

Then we see that $F_k(t)$ has the same singular point as the one of $f_k(t)$.

4.2 Moment series of $F_k(t)$

For $p \in \mathbb{N}$ and $k = m\ell - r(k)$, we define the moment series $G_{k,p}(t)$ of $F_k(t)$ as follows.

$$G_{k,p}(t) := (M^w F_k)(t) = \sum_{n\geq 0} A_k(n+\ell) w_{k,p}(n) t^n, \quad w_{k,p}(n) = \frac{(\alpha_{k1})_n \cdots (\alpha_{kp})_n}{(\gamma_{k1})_n \cdots (\gamma_{kp})_n}, \tag{4.3}$$

where $\gamma_{ki} > \alpha_{ki} > 0$ for all k and i. Here the symbol $(\alpha)_n$ denotes the Pochhammer symbol which is given by

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

The function $G_{k,p}(t)$ has the same singular point as the one of $F_k(t)$ and $f_k(t)$.

Lemma 4 *$G_{k,p}(t)$ is analytic in $\mathbb{C} \setminus \{1/a_0\}$. Moreover, $G_{k,p}(t)$ is bounded as $|t| \to \infty$.*

Proof. We remark that when $\mathrm{Re}\gamma > \mathrm{Re}\alpha > 0$,

$$\begin{aligned} \frac{(\alpha)_n}{(\gamma)_n} &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} B(\alpha+n, \gamma-\alpha) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 x^{\alpha-1+n}(1-x)^{\gamma-\alpha-1} dx. \end{aligned}$$

By using the above formula, we have

$$G_{k,p}(t) = \prod_{i=1}^{p} C_{ki} \int_{[0,1]^p} x_i^{\alpha_i - 1}(1-x_i)^{\gamma_i - \alpha_i - 1} F_k(\mathbf{x}t) d\mathbf{x},$$

where $C_{ki} = \Gamma(\gamma_{ki})/\Gamma(\alpha_{ki})\Gamma(\gamma_{ki} - \alpha_{ki})$, $\mathbf{x} = \prod_{i=1}^p x_i$ and $d\mathbf{x} = dx_1 dx_2 \cdots dx_p$. Therefore $G_{k,p}(t)$ has the same singular point as the one of $F_k(t)$.

Moreover, we see that since $F_k(t)$ is bounded as $|t| \to \infty$, from the above integral representation, $G_{k,p}(t)$ is also bounded as $|t| \to \infty$. Exactly, we have $|G_{k,p}(t)| \leq C_\ell$.

5 Proof of Theorem 1

First, we give the important lemma for the summability theory (cf. Balser [1] and Lutz-Miyake-Schäfke [8]).

Lemma 5 *Let $\kappa > 0$, $d \in \mathbb{R}$ and $\hat{u}(t,x) = \sum_{n\geq 0} u_n(x) t^n \in \mathcal{O}_x[[t]]_{1/\kappa}$. Then the following statements are equivalent:*

i) $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{\kappa,d}$.

ii) We put

$$u_B(s,x) = (\hat{\mathcal{B}}_\kappa \hat{u})(s,x) := \sum_{n\geq 0} \frac{u_n(x)}{\Gamma(1+n/\kappa)} s^n, \tag{5.1}$$

which is a formal κ-Borel transformation of $\hat{u}(t,x)$, which is convergent in a neighborhood of the origin $(s,x) = (0,0)$. Then $u_B(s,x) \in \mathrm{Exp}_s(\kappa, S(d,\varepsilon) \times B(\sigma))$ for some $\varepsilon > 0$ and $\sigma > 0$.

Under the assumptions for the Cauchy data, we shall prove Theorem 1. By using Lemmas 4 and 5, we obtain the following results, which will be proved in next section.

Lemma 6 *Let $k = m\ell - r(k)$ and $\hat{u}_k(t,x)$ be the formal solutions for the Cauchy problem (E_k). Then for a fixed $d \in \mathbb{R}$, we have $(u_k)_B(s,x) \in \mathrm{Exp}_s(\kappa, S(d,\varepsilon) \times B(\sigma))$ for some positive ε and σ. Exactly, we have*

$$\max_{|x|\le\sigma} |(u_k)_B(s,x)| \le C\frac{K^\ell |s|^\ell}{\ell!((m-1)\ell)!}\exp\left(\delta|s|^\kappa\right), \quad \left(\kappa = \frac{1}{m-1}\right) \tag{5.2}$$

where C, K and δ are independent of k (therefore ℓ).

Lemma 6 implies the following result.

Corollary 7 *For all k, we have $\hat{u}_k(t,x) \in \mathcal{O}_x\{t\}_{\kappa,d}$.*

Proof of Theorem 1. By Lemmas 5 and 6, we have the following desired exponential growth estimate for $U_B(s,x) = \sum_{k\ge0}(u_k)_B(s,x)$, which means that the proof of Theorem 1 is completed.

$$\begin{aligned}
\max_{|x|\le\sigma}|U_B(s,x)| &\le \sum_{k\ge0}\max_{|x|\le\sigma}|(u_k)_B(s,x)| \\
&= \sum_{\ell\ge0}\sum_{r(k)=0}^{m-1}\max_{|x|\le\sigma}|(u_{m\ell-r(k)})_B(s,x)| \\
&\le C\exp\left(\delta|s|^\kappa\right)\sum_{r(k)=0}^{m-1}\sum_{\ell\ge0}\frac{K^\ell|s|^\ell}{\ell!((m-1)\ell)!} \quad \left(\kappa=\frac{1}{m-1}\right) \\
&\le \widetilde{C}\exp\left(\delta|s|^{1/(m-1)}\right)\exp((K|s|)^{1/m}) \quad (C<\widetilde{C}) \\
&\le \widetilde{C}\exp\left(\tilde{\delta}|s|^{1/(m-1)}\right) \quad (\tilde{\delta}>\delta).
\end{aligned}$$

6 Proof of Lemma 6

We give the proof of Lemma 6 in the final section.

We put $k = m\ell - r(k)$, $r(k) = 0, 1, \cdots, m-1$. We have

$$(u_k)_B(s,x) = (\hat{\mathcal{B}}_\kappa u_k)(s,x) = \sum_{n\ge0}A_k(n)\varphi^{(mn-k)}(x)\frac{s^n}{n!((m-1)n)!} \quad \left(\kappa = \frac{1}{m-1}\right).$$

Since the order of zeros of $(u_k)_B(s,x)$ at $s=0$ is at least ℓ, we have

$$\begin{aligned}
(u_k)_B(s,x) &= \sum_{n\ge\ell}A_k(n)\varphi^{(mn-k)}(x)\frac{s^n}{n!((m-1)n)!} \\
&= \sum_{n\ge0}A_k(n+\ell)\varphi^{(mn+r(k))}(x)\frac{s^{n+\ell}}{(n+\ell)!((m-1)(n+\ell))!} \\
&= \frac{s^\ell}{2\pi i}\oint\frac{\varphi(x+\zeta)}{\zeta^{r(k)+1}}\sum_{n\ge0}A_k(n+\ell)\frac{(mn+r(k))!}{(n+\ell)!((m-1)(n+\ell))!}\left(\frac{s}{\zeta^m}\right)^n d\zeta.
\end{aligned}$$

By using the relation

$$(pn+q)! = q!p^{pn}\left(\frac{q+1}{p}\right)_n \cdot \left(\frac{q+2}{p}\right)_n \cdots \left(\frac{q+p}{p}\right)_n,$$

we have

$$\begin{aligned}
(u_k)_B(s,x) &= \frac{s^\ell}{2\pi i}\frac{r(k)!}{\ell!((m-1)\ell)!}\oint \frac{\varphi(x+\zeta)}{\zeta^{r(k)+1}} \\
\times \sum_{n\geq 0} A_k(n+\ell) & \frac{m^{mn}\left(\frac{r(k)+1}{m}\right)_n \cdots \left(\frac{r(k)+m}{m}\right)_n}{(\ell+1)_n (m-1)^{(m-1)n}\left(\frac{(m-1)\ell+1}{m-1}\right)_n \cdots \left(\frac{(m-1)(\ell+1)}{m-1}\right)_n}\left(\frac{s}{\zeta^m}\right)^n d\zeta \\
&= \frac{s^\ell}{2\pi i}\frac{r(k)!}{\ell!((m-1)\ell)!}\oint \frac{\varphi(x+\zeta)}{\zeta^{r(k)+1}} G_{k,m}\left(\frac{m^m}{(m-1)^{m-1}}\frac{s}{\zeta^m}\right) d\zeta,
\end{aligned}$$

where we put

$$G_{k,m}(X) = \sum_{n\geq 0} A_k(n+\ell)\frac{\left(\frac{r(k)+1}{m}\right)_n \cdots \left(\frac{r(k)+m}{m}\right)_n}{(\ell+1)_n\left(\frac{(m-1)\ell+1}{m-1}\right)_n \cdots \left(\frac{(m-1)(\ell+1)}{m-1}\right)_n} X^n.$$

Here we remark that $G_{k,m}(X)$ has a singular point at $X = 1/a_0$. Therefore $G_{k,m}(cs/\zeta^m)$ has m singular points in ζ complex plane, which are given by $\zeta = (ca_0 s)^{1/m}\omega_m^i$ $(i = 0, 1, \dots, m-1)$, where $\omega_m = e^{2\pi i/m}$.

From the assumption of Cauchy data and Lemma 4, we obtain

$$\max_{|x|\leq\sigma} |(u_k)_B(s,x)| \leq C\frac{K^\ell |s|^\ell}{\ell!((m-1)\ell)!}\exp\left(\delta|s|^{1/(m-1)}\right),$$

where C, K and δ are independent of k (therefore ℓ). We omit the detail (cf. Balser [3], Balser and Miyake [5], Ichinobe [6], Lutz, Miyake and Schäfke [8], Michalik [9], [10] and Miyake [11]).

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