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The Form of Hanging Slinky

Kenzi ODANI

Department of Mathematics Education, Aichi University of Education, Kariya 448-8542, Japan

1 Introduction

Slinky is a spring toy. If it hangs down, it extends under its own weight. In this paper, we consider the form of Slinky which extends under its own weight. The following is our result.

Theorem. *Consider a spring which extends under the following conditions:*

(i) *The spring is uniform along the axis, hanging down as the axis to be vertical.*

(ii) *The gravity acts each point P of the spring vertically down, and its magnitude is proportional to the mass locating blow P.*

(iii) *The spring extends under the local Hooke's law, given in Subsection 2.1.*

(iv) *When no forces act on the spring, its form is a helix of winding number at least one.*

(v) *Though the spring extends, it preserves its curvature.*

Under the above conditions, the spring lies on a one-sheeted hyperboloid of revolution.

Though the above theorem says so, a hanging Slinky seems not to be a hyperboloid but a cylinder. This is because its eccentricity is very large. See Subsection 3.2. The most important idea of this paper is the assumption (v). We will see the validity of it in Subsection 2.3. The author got the idea from the textbook of Frank Morgan [1].

Figure 1: Slinky

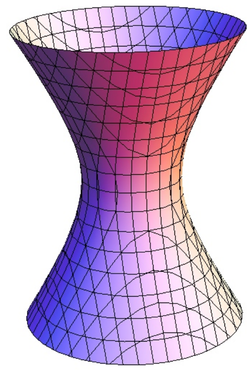

Figure 2: One-Sheeted Hyperboloid of Revolution

2 Preparation

2.1 Extention of Spring under Its Own Weight

Throughout this paper, we denote by g be the acceleration of gravity, and by m, k and L respectively the mass, the spring constant and the natural length of the spring. We also denote by A and B the endpoints of the spring. In this subsection, we prove the following well-known lemma. The author does not know who got this result first.

Lemma. *Consider a spring under the conditions* (i), (ii) *and* (iii). *Take any point P on the spring AB. We denote by h the natural length between P and A. After hanging down from A, the length comes to $h + \frac{mg}{kL^2}\Big(Lh - \frac{1}{2}h^2\Big)$. Especially, the length of the spring comes to $L + \frac{mg}{2k}$.*

Before proving the lemma, we explain local Hooke's Law. Consider a spring with no other forces acting. We assume that the spring pulled with a force F extends its length by ΔL. By Hooke's Law, we have that $F = k\Delta L$. The spring constant k is inversely proportional to the length L. For example, if we arrange two same springs in series, then the coupled spring pulled with the force F extends its length by $2\Delta L$. That is, the spring constant of the coupled spring is considered as $F/(2\Delta L) = k/2$.

Put the spring horizontally, fix an endpoint A, and take two points P, Q on the spring. We denote by h, $h+\delta h$ respectively the distances of P, Q from A. We assume that after pulling another endpoint B by the force F in the axial direction, the distances h, $h+\delta h$ change to z, $z+\delta z$ respectively. See Figure 3. Since the spring constant is inversely proportional to the length, the spring constant of the sub-spring PQ comes to $k(L/\delta h)$. Therefore, by Hooke's Law, we have that $F = k\frac{L}{\delta h}\big(\delta z - \delta h\big)$. By letting $\delta h \to 0$, we obtain that

$$F = kL\Big(\frac{dz}{dh} - 1\Big) \ . \tag{2.1}$$

We call it the *local Hooke's Law.*

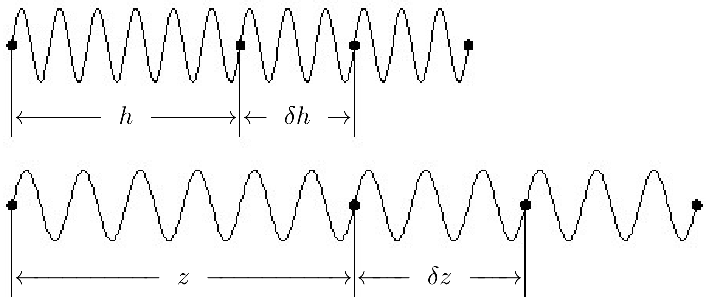

Figure 3: Derivation of Local Hooke's Law

Proof of Lemma. Since the spring is uniform, the mass of sub-spring PB comes to $m(L-h)/L$. By the local Hooke's Law (2.1), we have that

$$kL\Big(\frac{dz}{dh} - 1\Big) = m\frac{L-h}{L}g \ , \quad \text{that is,} \quad \frac{dz}{dh} = 1 + \frac{mg}{kL^2}(L-h) \ . \tag{2.2}$$

By integrating both sides, we have that

$$z = h + \frac{mg}{kL^2}\Big(Lh - \frac{1}{2}h^2\Big) \ . \tag{2.3}$$

Especially, by putting $h = L$, we obtain that the total length of the spring $L + \frac{mg}{2k}$.

2.2 Helical Springs

Consider a helix represented by a position vector

$$\vec{x}(s) = \Big(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}}\ ,\ \frac{bs}{\sqrt{a^2+b^2}}\Big) \ , \qquad 0 \leqq s \leqq l \ , \tag{2.4}$$

where a and b are positive constants. By Eq. (2.4), we have that

$$\big|\,\vec{x}\,'(s)\,\big| = 1 \ , \qquad \big|\,\vec{x}\,''(s)\,\big| = \frac{a}{a^2+b^2} \ . \tag{2.5}$$

The former identity indicates that s is an arc-length parameter, and the latter indicates that $\lambda = a/(a^2+b^2)$ is the curvature of the helix (2.4). We assume that the points A, B, P respectively correspond to the parametric values 0, l, s. Then we have that

$$h = \frac{bs}{\sqrt{a^2+b^2}} \ , \qquad L = \frac{bl}{\sqrt{a^2+b^2}} \ . \tag{2.6}$$

Notice that l is the "arc-length" though L is the "length". They are different notions. By putting Eq. (2.6) into Eq. (2.2), we have that

$$z'(s) = \mu + \lambda\varepsilon(l-s) \ , \quad \text{where} \quad \mu = \frac{b}{\sqrt{a^2+b^2}}, \ \varepsilon = \frac{mg}{kl^2\lambda}. \tag{2.7}$$

By integrating both sides of Eq. (2.7), we have that

$$z(s) = \mu s + \lambda\varepsilon\Big(ls - \frac{1}{2}s^2\Big) \ . \tag{2.8}$$

2.3 Preservation of Curvatures

Frank Morgan [1] introduces an interesting story on curvatures of space curves.

There was a company constructing a huge smokestack, which requires the attachment of spiraling strip for structural support. An engineer aimed to cut an annulus out of a flat metal, and to attach it to the smokestack along a helix. See Figures 4 and 5. A problem was what choice of inner radius r of the annulus would make it fit on the smokestack best? The size of the helix were: $a = 3.75$ feet, $2\pi b = 31.5$ feet.

After some trial and error, the engineer found that annulus cut with $r \approx 10.5$ feet fit well. However, we can compute the inner radius r mathematically. The way to compute r is to make the circle and the helix to have the same curvature. That is,

$$r = \frac{1}{\lambda} = \frac{a^2+b^2}{a} \approx 10.45 \text{ feet.} \tag{2.9}$$

It is close to the engineer's experiment.

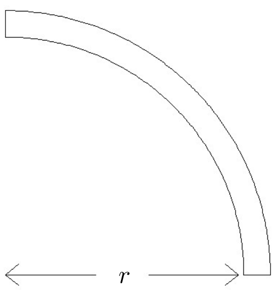

Figure 4: Annulus Cut out of a Flat Metal

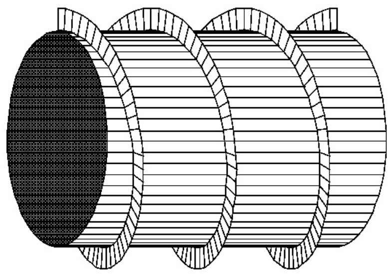

Figure 5: A Smokestack Attached with an Annulus

3 Form of Spring Extending under Its Own Weight

3.1 Finding the Equation of the Curve

Consider a spring extending under its own weight. We represent the curve by a position vector

$$\vec{x}(s) = \big(\, x(s),\, y(s),\, z(s)\,\big) \ , \tag{3.1}$$

where $z(s)$ is a function obtained in Eq. (2.8). Since s is an arc-length parameter, we have that

$$\big|\, \vec{x}\,'(s)\,\big|^2 = \big(\,x'\,\big)^2 + \big(\,y'\,\big)^2 + \varphi^2 = 1 \ , \tag{3.2}$$

where $\varphi(s) = \mu + \lambda\varepsilon(l-s)$. Since the curvature λ is invariant, we have that

$$\left|\,\vec{x}\,''(s)\,\right|^2 = \big(x''\big)^2 + \big(y''\big)^2 + (-\lambda\varepsilon)^2 = \lambda^2 \ . \tag{3.3}$$

So the spring satisfies the differential equations (3.2) and (3.3).

By Eq. (3.2), the spring must satisfy $\varphi(s) \leqq 1$. The restriction comes from the fact that the length must be smaller than the arc-length. By Eq. (3.2), we can put x' and y' as follows:

$$x' = \sqrt{1-\varphi^2}\cos\alpha(s) \ , \quad y' = \sqrt{1-\varphi^2}\sin\alpha(s) \ . \tag{3.4}$$

By differentiating both sides of Eq. (3.4), we have that

$$x'' = \frac{\lambda\varepsilon\varphi}{\sqrt{1-\varphi^2}}\cos\alpha - \sqrt{1-\varphi^2}\,\alpha'\sin\alpha \ , \qquad y'' = \frac{\lambda\varepsilon\varphi}{\sqrt{1-\varphi^2}}\sin\alpha + \sqrt{1-\varphi^2}\,\alpha'\cos\alpha \ . \tag{3.5}$$

By putting them into Eq. (3.3), we have that

$$\frac{\lambda^2\varepsilon^2\varphi^2}{1-\varphi^2} + (1-\varphi^2)\big(\alpha'\big)^2 + \lambda^2\varepsilon^2 = \lambda^2 \ , \quad \text{and so} \quad \big(\alpha'\big)^2 = \lambda^2\frac{1-\varepsilon^2-\varphi^2}{(1-\varphi^2)^2} \ . \tag{3.6}$$

By Eq. (3.6), the spring must satisfy $1-\varepsilon^2-\varphi^2 \geqq 0$. The restriction comes from the assumption that the spring preserves its curvature.

We assume that the spring winds counter-clockwise, that is, $\alpha' \geqq 0$. By taking square root of Eq. (3.6), and by integrating both sides, we have that

$$\alpha(s) = \lambda\int\frac{\sqrt{1-\varepsilon^2-\varphi^2}}{1-\varphi^2}\,ds \ . \tag{3.7}$$

To find this integral, we introduce a new variable u as follows:

$$\varphi(s) = \sqrt{1-\varepsilon^2}\cos u \ , \quad \text{and so} \quad \frac{ds}{du} = \frac{\sqrt{1-\varepsilon^2}}{\lambda\varepsilon}\sin u \ . \tag{3.8}$$

Then Eq. (3.7) comes to

$$\begin{aligned}\alpha(s) &= \frac{1}{\varepsilon}\int\frac{(1-\varepsilon^2)\sin^2 u}{1-(1-\varepsilon^2)\cos^2 u}\,du = \frac{1}{\varepsilon}\int\left\{1 - \frac{\varepsilon^2}{1-(1-\varepsilon^2)\cos^2 u}\right\}du \\ &= \frac{u}{\varepsilon} - \varepsilon\int\frac{1}{\varepsilon^2+\tan^2 u}\cdot\frac{1}{\cos^2 u}\,du \ .\end{aligned} \tag{3.9}$$

To find this integral, we introduce a new variable v by $\tan u = \varepsilon\tan v$. Then we have that

$$\begin{aligned}\cos v &= \frac{1}{\sqrt{1+\tan^2 v}} = \frac{\varepsilon\cos u}{\sqrt{\varepsilon^2\cos^2 u + \sin^2 u}} = \frac{\varepsilon\cos u}{\sqrt{1-\varphi^2}} \ , \\ \sin v &= \sqrt{1-\cos^2 v} = \sqrt{\frac{\varepsilon^2\cos^2 u + \sin^2 u - \varepsilon^2\cos^2 u}{1-\varphi^2}} = \frac{\sin u}{\sqrt{1-\varphi^2}} \ .\end{aligned} \tag{3.10}$$

Then Eq. (3.9) comes to

$$\alpha(s) = \frac{u}{\varepsilon} - \varepsilon\int\frac{1}{\varepsilon^2(1+\tan^2 v)}\cdot\frac{\varepsilon}{\cos^2 v}\,dv = \frac{u}{\varepsilon} - v + c \ , \tag{3.11}$$

where c is an integrating constant. By putting Eq. (3.11) into Eq. (3.4), and by using Eq. (3.10), we have that

$$\begin{aligned}x'(s) &= \sqrt{1-\varphi^2}\cos\left(\frac{u}{\varepsilon}+c-v\right) = \sqrt{1-\varphi^2}\left\{\cos\left(\frac{u}{\varepsilon}+c\right)\cos v + \sin\left(\frac{u}{\varepsilon}+c\right)\sin v\right\} \\ &= \varepsilon\cos\left(\frac{u}{\varepsilon}+c\right)\cos u + \sin\left(\frac{u}{\varepsilon}+c\right)\sin u \ .\end{aligned} \tag{3.12}$$

Similarly, we have that

$$y'(s) = \varepsilon \sin\left(\frac{u}{\varepsilon} + c\right)\cos u - \cos\left(\frac{u}{\varepsilon} + c\right)\sin u. \tag{3.13}$$

By Eq. (3.12), we have that

$$\begin{aligned} x'(s)\,ds &= \left\{\varepsilon \cos\left(\frac{u}{\varepsilon} + c\right)\cos u + \sin\left(\frac{u}{\varepsilon} + c\right)\sin u\right\} \cdot \frac{\sqrt{1-\varepsilon^2}}{\lambda\varepsilon}\sin u\,du \\ &= \frac{\sqrt{1-\varepsilon^2}}{2\lambda\varepsilon}\left\{\varepsilon\cos\left(\frac{u}{\varepsilon} + c\right)\sin 2u + \sin\left(\frac{u}{\varepsilon} + c\right)(1 - \cos 2u)\right\} du \\ &= -\frac{\sqrt{1-\varepsilon^2}}{4\lambda\varepsilon}\left\{(1-\varepsilon)\sin\left(\frac{u}{\varepsilon} + 2u + c\right) + (1+\varepsilon)\sin\left(\frac{u}{\varepsilon} - 2u + c\right) - 2\sin\left(\frac{u}{\varepsilon} + c\right)\right\} du\ . \end{aligned} \tag{3.14}$$

Similarly, we have that

$$y'(s)\,ds = \frac{\sqrt{1-\varepsilon^2}}{4\lambda\varepsilon}\left\{(1-\varepsilon)\cos\left(\frac{u}{\varepsilon} + 2u + c\right) + (1+\varepsilon)\cos\left(\frac{u}{\varepsilon} - 2u + c\right) - 2\cos\left(\frac{u}{\varepsilon} + c\right)\right\} du\ . \tag{3.15}$$

By integrating both sides of Eqs. (3.14), (3.15), we have that

$$\begin{aligned} x(s) &= \frac{\sqrt{1-\varepsilon^2}}{4\lambda}\left\{\frac{1-\varepsilon}{1+2\varepsilon}\cos\left(\frac{u}{\varepsilon} + 2u + c\right) + \frac{1+\varepsilon}{1-2\varepsilon}\cos\left(\frac{u}{\varepsilon} - 2u + c\right) - 2\cos\left(\frac{u}{\varepsilon} + c\right)\right\} + x_0\ , \\ y(s) &= \frac{\sqrt{1-\varepsilon^2}}{4\lambda}\left\{\frac{1-\varepsilon}{1+2\varepsilon}\sin\left(\frac{u}{\varepsilon} + 2u + c\right) + \frac{1+\varepsilon}{1-2\varepsilon}\sin\left(\frac{u}{\varepsilon} - 2u + c\right) - 2\sin\left(\frac{u}{\varepsilon} + c\right)\right\} + y_0\ , \end{aligned} \tag{3.16}$$

where x_0 and y_0 are integrating constants. Eq. (3.16) represents the form of the spring which is extending under its own weight.

3.2 The Restriction to the Value of ε

The denominator of Eq. (3.16) vanishes when $\varepsilon = 1/2$. However, it is not problem. In this subsection, we prove that the spring must satisfy $\varepsilon < 1/2$. Since we have that $1 - \varepsilon^2 - \varphi^2 \geqq 0$, we have that

$$1 - \varepsilon^2 - \big(\varphi(0)\big)^2 = 1 - \varepsilon^2 - (\mu + \lambda\varepsilon l)^2 \geqq 0\ . \tag{3.17}$$

By solving the quadratic inequality (3.17) for ε, we have that

$$\varepsilon \leqq \frac{1-\mu^2}{\mu\lambda l + \sqrt{(\lambda l)^2 + (1-\mu^2)}}\ . \tag{3.18}$$

The curvature and the winding number of the helix are respectively given by

$$\lambda = \frac{a}{a^2+b^2}\ , \qquad n = \frac{l}{2\pi\sqrt{a^2+b^2}}\ . \tag{3.19}$$

By combining them, we have that

$$\lambda l = 2\pi n \frac{a}{\sqrt{a^2+b^2}} = 2\pi n\sqrt{1-\mu^2}\ . \tag{3.20}$$

By substituting it into Eq. (3.18), we have that

$$\varepsilon \leqq \frac{\sqrt{1-\mu^2}}{2\pi n\mu + \sqrt{(2\pi n)^2+1}} < \frac{1}{2\pi n}. \tag{3.21}$$

Therefore, a spring of winding number $n \geqq 1$ must satisfy $\varepsilon < 1/(2\pi) \doteqdot 0.16$. (It is a mathematical restriction. Physical conditions may require the value of ε much smaller.)

3.3 Extending Spring Lies on a Hyperboloid

By Eq. (3.16), we have that

$$\begin{aligned}
&(x-x_0)^2+(y-y_0)^2 \\
&\quad = \frac{1-\varepsilon^2}{16\lambda^2}\left\{\left(\frac{1-\varepsilon}{1+2\varepsilon}\right)^2+\left(\frac{1+\varepsilon}{1-2\varepsilon}\right)^2+4+\frac{1-\varepsilon^2}{1-4\varepsilon^2}\cos 4u-2\frac{1-\varepsilon}{1+2\varepsilon}\cos 2u-2\frac{1+\varepsilon}{1-2\varepsilon}\cos 2u\right\} \\
&\quad = \frac{1-\varepsilon^2}{\lambda^2}\left\{\frac{1-\varepsilon^2}{1-4\varepsilon^2}\cos^4 u-\frac{2+\varepsilon^2}{1-4\varepsilon^2}\cos^2 u+\frac{(1-\varepsilon^2)^2}{(1-4\varepsilon^2)^2}\right\} \\
&\quad = \frac{1}{\lambda^2(1-4\varepsilon^2)}\left\{(1-\varepsilon^2)\cos^2 u-1-\frac{\varepsilon^2}{2}\right\}^2+\frac{27\varepsilon^4}{\lambda^2(1-4\varepsilon^2)^2}\ .
\end{aligned} \tag{3.22}$$

We can arrange Eq. (2.8) into

$$z(s)=-\frac{1}{2\lambda\varepsilon}\left\{\varphi(s)^2-(\mu+\lambda\varepsilon l)^2\right\}\ . \tag{3.23}$$

By combining Eqs. (3.8) and (3.23), we have that

$$(1-\varepsilon^2)\cos^2 u=\varphi(s)^2=-2\lambda\varepsilon z(s)+(\mu+\lambda\varepsilon l)^2\ . \tag{3.24}$$

By substituting it into Eq. (3.22), we have that

$$(x-x_0)^2+(y-y_0)^2=\frac{4\varepsilon^2}{1-4\varepsilon^2}(z-z_0)^2+\frac{27\varepsilon^4}{\lambda^2(1-4\varepsilon^2)^2}\ , \tag{3.25}$$

where

$$z_0=\frac{(\mu+\lambda\varepsilon l)^2-1-\varepsilon^2/2}{2\lambda\varepsilon}\ . \tag{3.26}$$

By the result of Subsection 3.2, the constant $\varepsilon<1/2$. So Eq. (3.25) represents a one-sheeted hyperboloid of revolution. Hence a spring extending under its own weight lies on a one-sheeted hyperboloid of revolution.

The eccentricity of the hyperbola is given by

$$\sqrt{1+\frac{1-4\varepsilon^2}{4\varepsilon^2}}=\frac{1}{2\varepsilon}\ . \tag{3.27}$$

Since the constant ε is very small, the eccentricity (3.27) is very large. So the hyperboloid seems like a cylinder. Moreover, by applying Eq. (3.17) to Eq. (3.26), we have that

$$z_0=\frac{(\mu+\lambda\varepsilon l)^2-1-\varepsilon^2/2}{2\lambda\varepsilon}\leqq-\frac{3\varepsilon}{4\lambda}<0. \tag{3.28}$$

Therefore the center of symmetry of the hyperboloid is located above the spring. That is, the upper part of the extending spring is thin though the lower part is thick.

References

[1] F. Morgan, Riemannian geometry: a beginner's guide, 2nd Ed., Routledge, 2009.

[2] J. D. Serna and A. Joshi, The center of mass of a soft spring, College Math. J. **42** (2011), 389–394.

[3] `https://en.wikipedia.org/wiki/Slinky#Physical_properties`

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