Bulletin of Aichi Univ. of Education, 66 (Natural Sciences), pp. 7 − 16, March, 2017

Maximal Evenness Ansatz and Diatonic System

Yukihiro HASHIMOTO

Department of Mathematics Education, Aichi University of Education, Kariya 448-8542, Japan

1. Can mathematics tell about tonality?

Transformational approaches to music have been a large movement in the communities of music theorists, analysts and composers, so, our set-theoretical approach examined in this paper may look old-fashioned. However the *maximal evenness* ansatz seems to brings us a clue to create a purely mathematical model for tonal music theory. In this research, we give an attempt to understand tonality established on diatonic system, in terms of maximal evenness which has been firstly introduced into mathematical music theory by Clough and Douthett[1]. The concept gives a systematic perception for tone system in diatonic system, such as pentatonic or diatonic scales, triads, 7-th chords and so on. We focus our argument on tonal music, of which pitches or chords are systematically arranged around a single central tone called the *key* of a piece. Usual tonal music are constructed by chords stacking in thirds, which are described systematically under the concept of 'second-order maximally even sets' introduced by Douthett[3]. These descriptions also brings a sort of representation for several chord progressions, such as translations, substitutions, modulations and cadences. In particular, cadences, progressions of several chords which create a sense of resolution, indicate the central pitch of a passage strongly, and thus establish the tonality of a piece, typically the chord progression from the dominant to the tonic (so called the *authentic cadence*). It is considerable feature of tonal music that all these mechanisms are described by J-functions (Definition 2.1).

2. Maximal evenness ansatz in mathematical music theory

2.1. **A short story on diatonic scales and maximal evenness.** Classical European music (or Western popular music even today) is established on 12-tone musical scale (so-called the *chromatic scale*). The scale is obtained by stacking perfect fifth intervals, i.e., multiplying the frequency of successive tones by $3/2$, called the *Pythagorean tuning*. As the perception of octave equivalence, the tuning process is emulated by a circle rotation $\mathbf{R}/\mathbf{Z} \ni x \mapsto x + \log_2(3/2) \in \mathbf{R}/\mathbf{Z}$, hence the continued fractional approximation $\log_2(3/2) \fallingdotseq [1,1,2,2] = 7/12$ induces a cyclic sequence of 12 tones. Historically, by connecting a couple of tetrachords, ancient Greek created a heptatonic scale, called the *diatonic scale*, which has been a basis for classical European music and is popularly used even today. As is well known in mathematical music community, the diatonic collection has significant features. Clough and Douthett[1] studied the diatonic collection from the viewpoint of maximal evenness: the 7 points are selected maximally even way from 12 points equally distributed on a circle.

Definition 2.1. *For $c, d, m \in \mathbf{Z}$ with $d \neq 0$, the* J-function *on* $\mathbf{Z}$ *is defined as*

$$J^m_{c,d}(k) = \left\lfloor \frac{ck+m}{d} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Since c is expressed as $c = Ad + b, 0 \le b \le d-1, A, b \in \mathbf{Z}$ and $\lfloor x+\alpha \rfloor - \lfloor x \rfloor \in \{0,1\}$ holds for any $x \in \mathbf{R}$ and $0 < \alpha < 1$, the J-function $J^m_{c,d}$ generates bi-infinite sequence $(J^m_{c,d}(k+1) - J^m_{c,d}(k) - A)_{k\in\mathbf{Z}} \in \{0,1\}^{\mathbf{Z}}$, called a mechanical word. It is shown that a periodic word with *Myhill's property* [1)], an embodiment of the maximal evenness, is a mechanical word and vice versa. (See details and proofs, e.g., [7][5].)

[1)] If the word is aperiodic, this property is known as a *balanced word* in combinatorics on words.

Definition 2.2 (Periodic Myhill's property)**.** *Let $\boldsymbol{v} = (v_k) \in \{0,1\}^{\mathbf{Z}}$ be a periodic bi-infinite sequence of $0, 1$ with period c. For a subword $\boldsymbol{w}$ in $\boldsymbol{v}$, $|\boldsymbol{w}|$ and $|\boldsymbol{w}|_1$ denotes the length of the word and the number of occurrences of the letter 1 in $\boldsymbol{w}$ respectively. We say $\boldsymbol{v}$ has* Myhill's property with period c *whenever $||\boldsymbol{x}|_1 - |\boldsymbol{y}|_1| = 1$ holds for any subwords $\boldsymbol{x}, \boldsymbol{y}$ in $\boldsymbol{v}$ with $|\boldsymbol{x}| = |\boldsymbol{y}| = k \not\equiv 0 \pmod c$. The set of periodic sequence having Myhill's property with period c is denoted by $\mathcal{M}_c$.*

Therefore the J-function for $0 < d \leq c$ induces a 'maximally even' embedding

$$J^m_{c,d} : \mathbf{Z}/d\mathbf{Z} \hookrightarrow \mathbf{Z}/c\mathbf{Z}.$$

We use notations $\mathcal{J}^m_{c,d} = (J^m_{c,d}(k))_{k=0,\ldots,d-1}$ and $|\mathcal{J}^m_{c,d}| = \{t \in \mathcal{J}^m_{c,d}\}$. If we adopt a semitone encoding for the chromatic scale as $C = 0, C^\sharp = 1, \ldots, B = 11$, the diatonic scale $CDEFGAB$ is expressed as a sequence $(0,2,4,5,7,9,11) = (J^5_{12,7}(k))_{k=0,\ldots,6} = \mathcal{J}^5_{12,7}$, hence the diatonic scale is embedded into the chromatic scale maximally even way. We note that the diatonic collection is selected successively in the process of stacking perfect fifth intervals. In [4] and [5], we have elucidated why the 'successive' 7 notes in Pythagorean tuning become maximally even, in terms of rotation dynamics on $\mathbf{R}/\mathbf{Z}$.

Definition 2.3. *For any subset $D \subset \mathbf{Z}/c\mathbf{Z}$, we associate a periodic sequence $\boldsymbol{v}(D) = (v_k) \in \{0,1\}^{\mathbf{Z}}$ of period c as*

$$\boldsymbol{v}(D)_k = v_k = 1 \text{ if and only if } k \equiv l \pmod c \text{ for some } l \in D.$$

A subset D is called a Myhill set *of period c whenever $\boldsymbol{v}(D) \in \mathcal{M}_c$.*

Theorem 2.4 (Dynamical characterization of Myhill set[5])**.** *Consider a subset $D \subset \mathbf{Z}/c\mathbf{Z}$ with the cardinality $\#|D| = d$ prime to c, and take a translation*

$$T : \mathbf{Z}/c\mathbf{Z} \ni x \mapsto x + d^{-1} \in \mathbf{Z}/c\mathbf{Z},$$

where d^{-1} is the multiplicative inverse of $d \in (\mathbf{Z}/c\mathbf{Z})^\times$. Then D is a Myhill set of the period c if and only if D is a collection of successive d images of some element $g \in \mathbf{Z}/c\mathbf{Z}$ by T, namely

$$D = \{g, T(g), T^2(g), \ldots, T^{d-1}(g)\} = \{g, g + d^{-1}, g + 2d^{-1}, \ldots, g + 1 - d^{-1}\}.$$

A J-function $J^m_{c,d}$ gives a Myhill set $|\mathcal{J}^m_{c,d}|$ describing a spatial feature how $\mathbf{Z}/d\mathbf{Z}$ is embedded in $\mathbf{Z}/c\mathbf{Z}$, while the theorem above states the Myhill set $|\mathcal{J}^m_{c,d}|$ has a temporal characterization in terms of a dynamics on $\mathbf{Z}/c\mathbf{Z}$. An explicit relation between $J^m_{c,d}$ and T is given as follows.

Proposition 2.5. *Given integers $c > d > 0$ prime to each other and m, take a pair (c^-, d^-) of integers satisfying*

$$c \cdot c^- + d \cdot d^- = 1. \tag{2.1}$$

Let T be a translation

$$T : \mathbf{Z}/c\mathbf{Z} \ni x \mapsto x + d^- \in \mathbf{Z}/c\mathbf{Z}.$$

Then we have

$$J^m_{c,d}(-(k+m+1)c^-) = T^k(g)$$

for $k = 0, \ldots, d-1$, where $g \equiv (m+1)d^- - 1 \pmod c$.

$$\begin{tikzcd}
\mathbf{Z}/d\mathbf{Z} \arrow[r, hook, "J^m_{c,d}"] \arrow[d, "\times(-c^-)"'] & \mathbf{Z}/c\mathbf{Z} \arrow[d, "T"] \\
\mathbf{Z}/d\mathbf{Z} \arrow[r, hook, "J^m_{c,d}"'] & \mathbf{Z}/c\mathbf{Z}
\end{tikzcd}$$

Proof. It comes from (2.1) that c^- is the multiplicative inverse of c in $\mathbf{Z}/d\mathbf{Z}$, and d^- is the multiplicative inverse of d in $\mathbf{Z}/c\mathbf{Z}$. Then we have

$$J^m_{c,d}(-(k+m+1)c^-) = \left\lfloor \frac{-(k+m+1)cc^- + m}{d} \right\rfloor = (k+m+1)d^- + \left\lfloor -\frac{k+1}{d} \right\rfloor = kd^- + (m+1)d^- - 1$$

for $k = 0, \ldots, d-1$, hence the assertion. $\square$

The following shows a relation among 'adjacent' scales (known as *related keys*) in terms of J-functions.

Proposition 2.6. *Given integers $c > d > 0$ prime to each other and m, take a pair (c^-, d^-) satisfying* (2.1)*. Then for $\nu \in \mathbf{Z}$ we have*

$$\mathcal{J}_{c,d}^{m+\nu}(-\nu c^-) = \mathcal{J}_{c,d}^m + \nu d^-, \tag{2.2}$$

where $\mathcal{J}_{c,d}^m(l)$ stands for the l-shift $(J_{c,d}^m(k+l))_{k=0,\dots,d-1}$ of $\mathcal{J}_{c,d}^m$.

Proof. Indeed, (2.1) drives

$$J_{c,d}^{m+\nu}(k - \nu c^-) = \left\lfloor \frac{c(k-\nu c^-) + m + \nu}{d} \right\rfloor = \left\lfloor \frac{ck + m + \nu d d^-}{d} \right\rfloor = J_{c,d}^m(k) + \nu d^-$$

holds for any $k \in \mathbf{Z}$. □

Let us apply above propositions to our case $c = 12, d = 7$. As $7 \cdot 7 + 12 \cdot (-4) = 1$, we put $c^- = -4$ and $d^- = 7$. The temporal translation T works as stacking $d^- = 7$ semitones in chromatic scale, corresponding to the adjacent $-c^- = 4$ (whole) tones move in diatonic scale, and both mean the diatonic scale is generated by stacking in perfect fifth. It is noticeable that by Proposition 2.6, changing the 'mode' m by $+1$ means a translation of the diatonic scale by $d^- = 7$ semitones, i.e., perfect fifth. In music theory, $\mathcal{J}_{12,7}^{m+1}$ and $\mathcal{J}_{12,7}^{m-1}$ are called the *dominant key*, and the *subdominant key* of $\mathcal{J}_{12,7}^m$ respectively. The relations between the mode m and diatonic scales are shown by Table 1. The sequence $D^\flat\ A^\flat\ E^\flat\ B^\flat\ F \cdots$ is also known as the *circle of fifth* in music theory.

m	0	1	2	3	4	5	6	7	8	9	10	11
Diatonic in major $\mathcal{J}_{12,7}^m$	$D^\flat$	$A^\flat$	$E^\flat$	$B^\flat$	F	C	G	D	A	E	B	$F^\sharp$

TABLE 1. Representation of major scales by J-functions.

2.2. **Douthett's 'beacon-filter' construction and second-order maximal evenness.** The discovery of the diatonic collection developed our new perception for music, the 'tonality', which was reinforced by progress of triadic harmony. Usually, a triad consists of three notes stacked in thirds. CEG, so-called the tonic chord in C major, consists of the frequencies with the $4:5:6$ ratio, and has been perceptually preferable for human beings because of its simple ratio. Cook and Fujisawa[2] proposed a mathematical model for perceptual stability/non-stability of triads. Contrast to such psychoacoustic understandings of triadic harmony, Douthett[3] proposed a new viewpoint for triads under the concept of maximal evenness. In fact, any triad stacked in thirds (in C major) are maximal evenly embedded into $\mathbf{Z}/7\mathbf{Z}$, so its realization in the chromatic scale is expressed[2)] as

$$\mathbf{Z}/3\mathbf{Z} \overset{J_{7,3}^m}{\hookrightarrow} \mathbf{Z}/7\mathbf{Z} \overset{J_{12,7}^5}{\hookrightarrow} \mathbf{Z}/12\mathbf{Z}. \tag{2.3}$$

Here we fix the second J-function to $J_{12,7}^5$ so as to obtain triads in C major scale, while we take the first $J_{7,3}^m$ with a continuous parameter m. Varying m continuously from 0 to 6, we obtain 7 triads staked in thirds, as $CEG, CEA, CFA, DFA, DFB, DGB, EGB$. In this sense, such triads are embedded in the chromatic scale 'second-order maximally even' way. Similarly, we obtain 7-th chords in the diatonic scale replacing

[2)] In [3], Douthett compared (2.3) to a beacon and filters. A beacon with three lamps evenly arranged on a circle stands for $\mathbf{Z}/3\mathbf{Z}$. $\mathbf{Z}/7\mathbf{Z}$ corresponds to the first filter, a circle of larger radius than that of beacon with equally distributed 7 holes, and concentric to it, and $\mathbf{Z}/12\mathbf{Z}$ corresponds to the second filter, a circle of larger radius than that of the first filter with equally distributed 12 holes, and concentric to the beacon. Then the J-functions controls the route of beams of the lamps how each beam travel through the holes. Varying m corresponds to the rotation of the beacon.

$J^m_{7,3}$ by $J^m_{7,4}$, as $CDFA \equiv DFAC, CEFA \equiv FACE, CEGA \equiv ACEG, CEGB, DEGB \equiv EGBD, DFGB \equiv GBDF, DFAB \equiv BDFA$, where $\equiv$ means the identification by octave equivalence. See Figure 1 and 2.

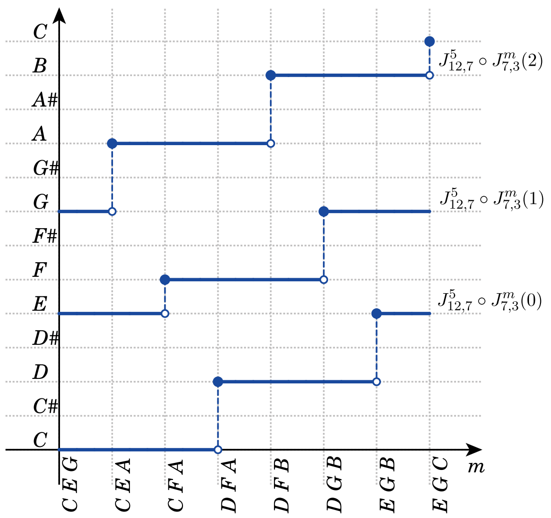

FIGURE 1. Triads as the second-order maximally even sets.

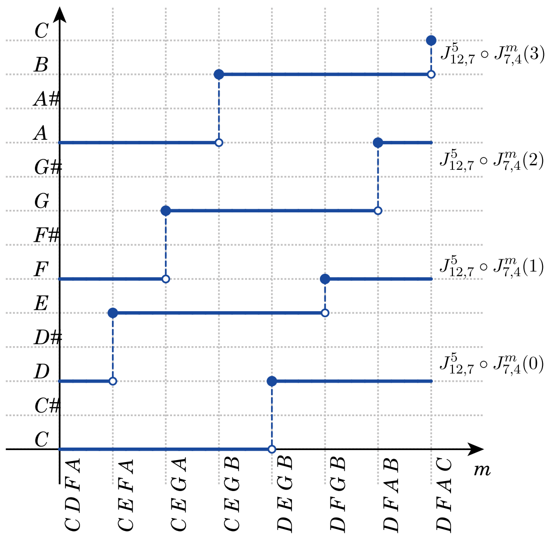

FIGURE 2. 7-th chords as the second-order maximally even sets.

In the sequence of triads $CEG, CEA, \dots$ or 7-th chords $CDFA, CEFA, \dots$, one sees that the difference of adjacent triads or 7-th chords occurs only one entry, like $CE\underline{G} \to CE\underline{A}, C\underline{E}A \to C\underline{F}A, \dots$, giving a smooth code progression in the diatonic collection. Indeed this fact is shown as follows.

Proposition 2.7. *Let c, d be non-zero integers prime to each other. Then for any $x \in \mathbf{R}$,*

$$\sum_{k \in \mathbf{Z}/d\mathbf{Z}} |J^{x+1}_{c,d}(k) - J^x_{c,d}(k)| = 1 \tag{2.4}$$

holds.

Proof. For $x \in \mathbf{R}$, one sees

$$\left\lfloor \frac{x}{d} \right\rfloor - \left\lfloor \frac{x-1}{d} \right\rfloor = 1$$

holds if and only if $\lfloor x \rfloor \equiv 0 \pmod d$. As c is prime to d, c is invertible in $\mathbf{Z}/d\mathbf{Z}$, and then there exists a unique $k \in \mathbf{Z}/d\mathbf{Z}$ fulfilling $ck + m \equiv 0 \pmod d$ for each $m \in \mathbf{Z}$, while for any $x \in \mathbf{R}$, $\lfloor x \rfloor$ is expressed by $\lfloor x \rfloor = ck + m, 0 \leq m \leq c-1$. Then any $x \in \mathbf{R}$ has a unique $k \in \mathbf{Z}/d\mathbf{Z}$ satisfying $|J^{x+1}_{c,d}(k) - J^x_{c,d}(k)| = 1$, hence the assertion. $\square$

3. J-functions and some characterizations of diatonic system

In this section we use the abbreviation $\mathcal{J}^{p,q}_{c,d,e} = J^p_{c,d}(\mathcal{J}^q_{d,e}) = (J^p_{c,d} \circ J^q_{d,e}(k))_{k=0,\dots,e-1}$.

3.1. **Triads, 7-th, augmented, diminished and half diminished chords.** We have already seen that triads or 7-th chords staked in thirds are obtained as second-order maximally even subsets in the chromatic scale $\mathbf{Z}/12\mathbf{Z}$, like $\mathcal{J}^{5,0}_{12,7,3} = CEG$ or $\mathcal{J}^{5,3}_{12,7,4} = CEGB$. To describe relations among chords, we require the followings.

Proposition 3.1. *For $c, d, l, m \in \mathbf{Z}$ with $d \neq 0$, we have a* translation

$$\mathcal{J}^{m+ld}_{c,d} = \mathcal{J}^m_{c,d} + l = (J^m_{c,d}(k) + l)_{k=0,\dots,d-1} \tag{3.1}$$

and a transposition

$$\mathcal{J}^{m+lc}_{c,d} = \mathcal{J}^{m}_{c,d}(l) = (J^{m}_{c,d}(k+l))_{k=0,\dots,d-1}. \tag{3.2}$$

Proof. (3.1) comes from

$$J^{m+ld}_{c,d}(k) = \left\lfloor \frac{ck+m+ld}{d} \right\rfloor = \left\lfloor \frac{ck+m}{d} \right\rfloor + l = J^{m}_{c,d}(k) + l,$$

and (3.2) from

$$J^{m+lc}_{c,d}(k) = \left\lfloor \frac{ck+m+lc}{d} \right\rfloor = \left\lfloor \frac{c(k+l)+m}{d} \right\rfloor = J^{m}_{c,d}(k+l).$$

□

For instance, we have translations of triads

$$\mathcal{J}^{5,0}_{12,7,3} = CEG \to \mathcal{J}^{5,3}_{12,7,3} = DFA \to \mathcal{J}^{5,6}_{12,7,3} = EGB \to \cdots,$$

and the transpositions

$$\mathcal{J}^{5,0}_{12,7,3} = CEG \to J^{5}_{12,7}(\mathcal{J}^{0}_{7,3}(1)) = \mathcal{J}^{5,7}_{12,7,3} = EGC \to J^{5}_{12,7}(\mathcal{J}^{0}_{7,3}(2)) = \mathcal{J}^{5,14}_{12,7,3} = GCE \to \cdots.$$

Thus, one sees correspondences among the J-representations above and the roman numeral analysis in music theory. In any diatonic scale $\mathcal{J}^{m}_{12,7}$ in mode m (the key of the scale), the triad $J^{m}_{12,7}\left(\mathcal{J}^{0}_{7,3}\right)$ is called the *tonic* denoted by roman numeral I, and its translations are denoted as (omitting $J^{m}_{12,7}$)

$$\mathcal{J}^{0}_{7,3} = \mathrm{I},\ \mathcal{J}^{3}_{7,3} = \mathrm{ii},\ \mathcal{J}^{6}_{7,3} = \mathrm{iii},\ \mathcal{J}^{9}_{7,3} = \mathrm{IV},\ \mathcal{J}^{12}_{7,3} = \mathrm{V},\ \mathcal{J}^{15}_{7,3} = \mathrm{vi},\ \mathcal{J}^{18}_{7,3} = \mathrm{vii}^{\circ}.$$

Also transpositions of any triad, say $\mathcal{J}^{0}_{7,3}$ are denoted as

$$\mathcal{J}^{0}_{7,3} = \mathrm{I},\ \mathcal{J}^{0}_{7,3}(1) = \mathcal{J}^{7}_{7,3} = \mathrm{I}^{6},\ \mathcal{J}^{0}_{7,3}(2) = \mathcal{J}^{14}_{7,3} = \mathrm{I}^{6}_{4}.$$

Contrast to triads or 7-th chords, diminished and augmented chords are described as maximally even sets in the chromatic scale, because they consist of evenly distributed four or three pitches. Up to transposition, there are three diminished chords

$$\mathcal{J}^{0}_{12,4} = CE^{\flat}F^{\sharp}A, \quad \mathcal{J}^{4}_{12,4} = C^{\sharp}EGB^{\flat}, \quad \mathcal{J}^{8}_{12,4} = DFA^{\flat}B,$$

and four augmented chords

$$\mathcal{J}^{0}_{12,3} = CEG^{\sharp}, \quad \mathcal{J}^{3}_{12,3} = C^{\sharp}FA, \quad \mathcal{J}^{6}_{12,3} = DF^{\sharp}A^{\sharp}, \quad \mathcal{J}^{9}_{12,3} = D^{\sharp}GB.$$

Note that we have already obtained the half diminished chord as a second-order maximally even set,

$$J^{5}_{12,7}(\mathcal{J}^{6}_{7,4}(-1)) = \mathcal{J}^{5,(6-7)}_{12,7,4} = \mathcal{J}^{5,-1}_{12,7,4} = BDFA.$$

One sees a triad stacked in thirds is a subset of a 7-th chord, like CEG in $CEGB$, however this kind of inclusion relation is also described by J-functions. Indeed, as $\mathcal{J}^{0}_{4,3} = (0,1,2)$, we see

$$\begin{aligned} CEG &= \mathcal{J}^{5,0}_{12,7,3} = (J^{5}_{12,7} \circ J^{0}_{7,3}(k))_{k=0,1,2} \\ &= (J^{5}_{12,7} \circ J^{3}_{7,4} \circ J^{0}_{4,3}(k))_{k=0,1,2} = (J^{5}_{12,7} \circ J^{3}_{7,4}(k))_{k=0,1,2} \sqsubset \mathcal{J}^{5,3}_{12,7,4} = CEGB, \end{aligned}$$

where $X \sqsubset Y$ means X is a subword of Y. This inclusion can be written as

$$CEG = \mathcal{J}^{5,0}_{12,7,3} = \mathcal{J}^{5,3,0}_{12,7,4,3} \sqsubset \mathcal{J}^{5,3}_{12,7,4} = CEGB.$$

Such a special inclusion occurs under some restricted conditions.

Lemma 3.2. *For* $0 \le m \le d$, *we have* $\mathcal{J}^{m}_{d+1,d} = (k)_{k=0,\dots,d;k\neq d-m}$.

Proof. The assertion comes from

$$\left\lfloor \frac{(d+1)k+m}{d} \right\rfloor = k + \left\lfloor \frac{k+m}{d} \right\rfloor = \begin{cases} k, & \text{for } k = 0,\dots,d-1-m, \\ k+1, & \text{for } k = d-m,\dots,d-1. \end{cases}$$

$\square$

Proposition 3.3. *For $c = 2d+1$ and $m \in \mathbf{Z}$, we have*

$$\mathcal{J}^m_{c,d}(-r) \sqsubset \mathcal{J}^l_{c,d+1}(-r),$$

where $l = c\left\lfloor \dfrac{m}{d} \right\rfloor + d - m - 1$ or $c\left\lfloor \dfrac{m}{d} \right\rfloor + d - m$, and $r \equiv m \pmod d$, $0 \le r \le d-1$.

Proof. By Lemma 3.2, we have $\mathcal{J}^0_{d+1,d} = (0,\dots,d-1)$. Putting $A = \lfloor m/d \rfloor$, thus $m = Ad + r$, it holds for $k = -r,\dots,d-1-r$ that

$$J^m_{c,d}(k) = \left\lfloor \frac{ck+m}{d} \right\rfloor = 2k + A + \left\lfloor \frac{k+r}{d} \right\rfloor = 2k + A,$$

and

$$\begin{aligned} J^l_{c,d+1}(k) &= \left\lfloor \frac{ck+l}{d+1} \right\rfloor = 2k + A + \left\lfloor \frac{-k+d-1-r}{d+1} \right\rfloor \text{ or } 2k + A + \left\lfloor \frac{-k+d-r}{d+1} \right\rfloor \\ &= 2k + A. \end{aligned}$$

Then applying Proposition 3.1, we have $J^{m-rc}_{c,d}(k) = J^{l-rc}_{c,d+1}(k)$ for $k = 0,\dots,d-1$. Therefore

$$\mathcal{J}^m_{c,d}(-r) = \left(J^{l-rc}_{c,d+1}(k)\right)_{k=0,\dots,d-1} \sqsubset \mathcal{J}^l_{c,d+1}(-r).$$

$\square$

Let us apply Proposition 3.3 for CEG, i.e., $c = 7, d = 3, m = 0$, which shows two possibilities $l = 2$ and 3, corresponding to $CEG = \mathcal{J}^{5,0}_{12,7,3} \sqsubset \mathcal{J}^{5,2}_{12,7,4} = CEGA$ and $CEG = \mathcal{J}^{5,0}_{12,7,3} \sqsubset \mathcal{J}^{5,3}_{12,7,4} = CEGB$ respectively. It comes from Lemma 3.2 that we have $CEB = \mathcal{J}^{5,3,1}_{12,7,4,3}$(omit G), $CGB = \mathcal{J}^{5,3,2}_{12,7,4,3}$(omit E) and $EGB = \mathcal{J}^{5,3,3}_{12,7,4,3}$(omit C), however only $CEG = \mathcal{J}^{5,0}_{12,7,3}$ and $EGB = \mathcal{J}^{5,6}_{12,7,3}$ are second-order maximally even, since associated mechanical words of CEB and CGB in the diatonic scale are $C-E-B-C = 241$ and $C-G-B-C = 421$ respectively, which are not Myhill. Thus hereafter, $X \sqsubset Y$ stands for the special inclusion whenever X is a maximally even subword of Y.

3.2. **Expansion of scales and chords over several octaves.** As 3 is prime to 7, 7 times repetition of stacking thirds in C major scale generate again the C major scale but rearranged; $CEGB\dot{D}\dot{F}\dot{A}$. Here $\dot{X}$ denotes the pitch an octave higher than original X. It is noticeable that this sequence is also obtained as $\mathcal{J}^5_{24,7}$, that is, as maximal evenly embedded 7 notes into two octaves. Generally we see the following.

Proposition 3.4. *Take natural numbers c, d, h, m where h is prime to d and $c > d$. Then it holds for any $k \in \mathbf{Z}/d\mathbf{Z}$ that*

$$\pi \circ J^m_{hc,d}(k) = J^m_{c,d}(hk),$$

where π denotes the natural projection, hence the natural projection of maximal evenly embedded d elements in $\mathbf{Z}/hc\mathbf{Z}$ is maximal evenly embedded in $\mathbf{Z}/c\mathbf{Z}$.

$$\begin{tikzcd} \mathbf{Z}/d\mathbf{Z} \arrow[r, hook, "J^m_{hc,d}"] \arrow[d, "\times h"', "\approx"] \arrow[dr, phantom, "\circlearrowright"] & \mathbf{Z}/hc\mathbf{Z} \arrow[d, "\pi"] \\ \mathbf{Z}/d\mathbf{Z} \arrow[r, hook, "J^m_{c,d}"'] & \mathbf{Z}/c\mathbf{Z} \end{tikzcd}$$

Proof. Indeed, multiplying by h is bijective as $h \in (\mathbf{Z}/d\mathbf{Z})^\times$, and for any $k \in \mathbf{Z}/d\mathbf{Z}$, we have

$$\pi \circ J^m_{hc,d}(k) = \pi\left(\left\lfloor \frac{hck+m}{d} \right\rfloor \pmod{hc}\right) = \left\lfloor \frac{c(hk)+m}{d} \right\rfloor \pmod c = J^m_{c,d}(hk).$$

$\square$

Taking $h = 2, 3, 4$ we obtain heptatonic scales $\mathcal{J}^5_{24,7} = CEGB\dot{D}\dot{F}\dot{A}$, $\mathcal{J}^5_{36,7} = CFB\dot{E}\dot{A}\ddot{D}\ddot{G}$ and $\mathcal{J}^5_{48,7} = CG\dot{D}\dot{A}\ddot{E}\ddot{B}\dddot{F}$ by stacking thirds, fourth and fifth, embedded in two, three and four octaves respectively. So we can take these scales for expansions of C major diatonic scale over several octaves.

The pentatonic scales are directory obtained by J-functions, e.g., $\mathcal{J}^0_{12,5} = CDEGA$, while they are also second-order maximally even, that is, $\mathcal{J}^0_{12,5} = \mathcal{J}^{5,0}_{12,7,5}$. We note $\mathcal{J}^0_{12,5}$ is a subset of three major scales, that is, C major $\mathcal{J}^5_{12,7} = CDEFGAB$, G major $\mathcal{J}^6_{12,7} = GABCDEF^\sharp$ and F major $\mathcal{J}^4_{12,7} = FGAB^\flat CDE$. This fact is explained by

$$\mathcal{J}^0_{12,5} = \mathcal{J}^{4,0}_{12,7,5} = \mathcal{J}^{5,0}_{12,7,5} = \mathcal{J}^{6,0}_{12,7,5}.$$

Table 2 shows the possible pentatonic scales $\mathcal{J}^m_{12,5}$ contained by C major scale. Also we note that expansions of pentatonic scales over two octaves drive stacked perfect fourth chords, like $BEA\dot{D}\dot{G}$.[3)]

$\mathcal{J}^m_{12,5}$	pentatonic	$\mathcal{J}^m_{24,5}$	expansion over 2 octaves	possible diatonic scale
$\mathcal{J}^{-1}_{12,5} = \mathcal{J}^{5,-1}_{12,7,5}$	$BDEGA \equiv GABDE$	$\mathcal{J}^{-1}_{24,5}$	$BEA\dot{D}\dot{G}$	C, G, D
$\mathcal{J}^0_{12,5} = \mathcal{J}^{5,0}_{12,7,5}$	$CDEGA$	$\mathcal{J}^0_{24,5}$	$CEA\dot{D}\dot{G} \equiv EAD\dot{G}\dot{C}$	C, G, F
$\mathcal{J}^1_{12,5} = \mathcal{J}^{5,1}_{12,7,5}$	$CDFGA \equiv FGACD$	$\mathcal{J}^1_{24,5}$	$CFA\dot{D}\dot{G} \equiv ADG\dot{C}\dot{F}$	$C, F, B^\flat$

TABLE 2. Pentatonic scales and possible scales. The expansions of them over two octaves form stacked fourth chords.

3.3. **Cadences and chord progressions.** A cadence indicates the central pitch of a passage strongly, and thus establish the tonality of a musical piece. The typical one is the dominant to tonic chord progression (authentic cadence[4)]), e.g., $\mathcal{J}^{5,-2}_{12,7,3} = BDG \to \mathcal{J}^{5,0}_{12,7,3} = CEG$ in C major, or if one emphasizes the roots of chords, $\mathcal{J}^{5,-9}_{12,7,3} = GBD \to \mathcal{J}^{5,0}_{12,7,3} = CEG$. Of course the reason why such chord progressions have been widely accepted may be explained by psychoacoustic, which is out of our purpose, although we describe the progressions in terms of J-functions. Actually, the relation

$$GBD = J^5_{12,7}\left(\mathcal{J}^0_{7,3} + 4\right) = J^5_{12,7}\left(\mathcal{J}^0_{7,3}\right) + 7 = CEG + 7,$$

shows that the essence of the authentic cadence (or fifth down chord progression) can be expressed as

$$J^m_{12,7}\left(\mathcal{J}^n_{7,3} + 4\right) \to J^m_{12,7}\left(\mathcal{J}^n_{7,3}\right). \tag{3.3}$$

The progression $GBDF \to CEG$ also creates a sense of resolution strongly, because $GBDF$ contains two leading tones B and F towards the tonic CEG. With the help of Proposition 3.3, this progression can be expressed as

$$GBDF = J^5_{12,7}\left(\mathcal{J}^5_{7,4}(2)\right) = J^5_{12,7}\left(\mathcal{J}^{19}_{7,4}\right) = J^5_{12,7}\left(\mathcal{J}^3_{7,4} + 4\right) \to J^5_{12,7}\left(\mathcal{J}^3_{7,4}\right) = CEGB \sqsupset J^5_{12,7}\left(\mathcal{J}^0_{7,3}\right) = CEG.$$

The diminished chords are useful for chord progressions, which work as a hub in chord progressions. The diminished chord $BDFA^\flat$ and 7-th chord $GBDF$ contain BDF commonly, of which elements B, D, F are

[3)]'So What' – one of the masterpiece of Miles Davis – starts from the chord progression $EAD\dot{G}\dot{B} \to DGC\dot{F}\dot{A}$, which are equivalent to stacked fourth chords $BEA\dot{D}\dot{G}$ and $ADG\dot{C}\dot{F}$ modulo two octaves.

[4)]It is said that Guillaume Dufay first introduced the authentic cadence in the festive motet *Nuper rosarum flores* in 1436. The epoch-making discovery of the dominant-tonic resolution had been influenced for the next almost 500 years, until the emergence of Erik Satie's *Gymnopédie*, which brought a quiet end to tonal music.

overtones of G modulo octave equivalence, thus $BDFA^\flat$ has a similar function as $GBDF$. Therefore the fifth down chord progression from the diminished chord can be adopted. This process is described as

$$\begin{aligned} BDFA^\flat &= \mathcal{J}_{12,4}^{-4} \sqsupset J_{12,4}^{-4}\left(\mathcal{J}_{4,3}^{0}\right) = BDF = J_{12,7}^{5}\left(\mathcal{J}_{7,4,3}^{19,3}\right) \sqsubset J_{12,7}^{5}\left(\mathcal{J}_{7,4}^{19}\right) = GBDF = J_{12,7}^{5}\left(\mathcal{J}_{7,4}^{3}+4\right) \\ &\to J_{12,7}^{5}\left(\mathcal{J}_{7,4}^{3}\right) = CEGB \sqsupset J_{12,7}^{5}\left(\mathcal{J}_{7,3}^{0}\right) = CEG. \end{aligned}$$

Note that BDF is also a maximally even subword of $BDFA^\flat$ obviously, thus we denote $BDF \sqsubset BDFA^\flat$. As the diminished $BDFA^\flat$ chord is evenly distributed in $\mathbf{Z}/12\mathbf{Z}$, the fifth down chord progressions can occur from $DFA^\flat, FA^\flat B$ and $A^\flat BD$, e.g.,

$$DFA^\flat B \sqsupset DFA^\flat \sqsubset B^\flat DFA^\flat = J_{12,7}^{2}\left(\mathcal{J}_{7,4}^{11}+4\right) \to J_{12,7}^{2}\left(\mathcal{J}_{7,4}^{11}\right) = E^\flat GB^\flat D.$$

One will see that diminished chords approximate 7-th chords. Particularly, the following 'root-invariant' progressions

$$\begin{aligned} CEGB &= \mathcal{C}\left(\mathcal{J}_{7,4}^{3}\right) = \mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{0}\right) \to CE^\flat F^\sharp A = \mathcal{J}_{12,4}^{0}, \\ GBDF &= \mathcal{C}\left(\mathcal{J}_{7,4}^{-9}\right) = \mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{2}(2)\right) \to GB^\flat C^\sharp E = \mathcal{J}_{12,4}^{4}(2), \\ DFAC &= \mathcal{C}\left(\mathcal{J}_{7,4}^{7}\right) = \mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{4}\right) \to DFA^\flat B = \mathcal{J}_{12,4}^{8} \end{aligned} \tag{3.4}$$

are often used, where $\overline{\mathcal{J}}_{c,d}^{m}$ denotes $(\lceil (ck+m)/d \rceil)_{k=0,\dots,d-1}$. These progressions are described as

$$\mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{\nu}\right) \to \mathcal{J}_{12,4}^{4n},$$

where $\nu = \lfloor 7n/3 \rfloor$. To obtain such 7-th to diminished progressions systematically, we adopt the following continuous approximation. The deformation of a diminished chord

$$\mathcal{J}_{12,4}^{4n} = \left(\left\lfloor \frac{12k+4n}{4} \right\rfloor\right)_{k=0,1,2,3} = (3k+n)_{k=0,1,2,3} = \left(\frac{12}{7}\left(\frac{7k+\frac{7n-m}{3}}{4}\right)+\frac{m}{7}\right)_{k=0,1,2,3}$$

implies that the term $\left(\frac{7k+(7n-m)/3}{4}\right)_{k=0,1,2,3}$ corresponds to a 7-th chord. Thus taking $\alpha(m,n) \in \mathbf{Z}$ for an approximation of $(7n-m)/3$, we get an approximate 7-th chord $\mathcal{J}_{12,7,4}^{m,\alpha(m,n)}$. For instance, if we put $\alpha(m,n) = \lceil (7n-m)/3 \rceil$, in the case of C major ($m=5$), we obtain approximations

$$\begin{aligned} ACEG &= \mathcal{C}\left(\mathcal{J}_{7,4}^{2}(-1)\right) = \mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{-1}(-1)\right) \sim ACE^\flat F^\sharp = \mathcal{J}_{12,4}^{0}(-1), \\ EGBD &= \mathcal{C}\left(\mathcal{J}_{7,4}^{4}(1)\right) = \mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{1}(1)\right) \sim EGB^\flat C^\sharp = \mathcal{J}_{12,4}^{4}(1), \\ BDFA &= \mathcal{C}\left(\mathcal{J}_{7,4}^{6}(-1)\right) = \mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{3}(-1)\right) \sim BDFA^\flat = \mathcal{J}_{12,4}^{8}(-1), \end{aligned} \tag{3.5}$$

which are unified as

$$\mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{\nu}\right) \sim \mathcal{J}_{12,4}^{4n}$$

with $\nu = \lceil (7n-5)/3 \rceil$. If we put $\alpha(m,n) = \lceil (7n-m)/4 \rceil + 1$, we again obtain (3.4) for C major $m=5$. If we put $\alpha(m,n) = \lfloor (7n-m)/3 \rfloor$, in the case of C major ($m=5$), we obtain approximations

$$\begin{aligned} FACE &= \mathcal{C}\left(\mathcal{J}_{7,4}^{1}(2)\right) = \mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{-2}(2)\right) \sim F^\sharp ACE^\flat = \mathcal{J}_{12,4}^{0}(2), \\ CEGB &= \mathcal{C}\left(\mathcal{J}_{7,4}^{3}\right) = \mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{0}\right) \sim C^\sharp EGB^\flat = \mathcal{J}_{12,4}^{4}, \\ BDFA &= \mathcal{C}\left(\mathcal{J}_{7,4}^{6}(-1)\right) = \mathcal{C}\left(\overline{\mathcal{J}}_{7,4}^{3}(-1)\right) \sim BDFA^\flat = \mathcal{J}_{12,4}^{8}(-1). \end{aligned} \tag{3.6}$$

By varying the mode m, one will obtain more approximations.

3.4. **J-function analysis for J. S. Bach's Prelude.** Here we apply the J-function representation to the famous work, *Prelude No.1 in C major, BWV 846 from 'the Well-tempered Clavier' by J. S. Bach.* In this work, each measure consists of 8 notes which form the similar passage, except the last three measures. The analysis for measures from 1 to 25 is shown in Table 3: the first 5 notes of the original passages at the second column, their standard forms as chords at the third, their J-representations at the fourth, and the chord progressions towards the next measure at the last. Here we adopt new notations for diatonic embeddings $J^4_{12,7}(\cdot) = \mathcal{F}(\cdot)\,, J^5_{12,7}(\cdot) = \mathcal{C}(\cdot)\,, J^6_{12,7}(\cdot) = \mathcal{G}(\cdot)$ and so on, taking account of Table 1.

One sees several kinds of chord progressions. A translation by whole tone $\mathcal{X} \to \mathcal{X} \pm 1$ occurs at measures $1 \to 2$ and $22 \to 23$. There are a lot of fifth down chord progressions $\mathcal{X} + 4 \to \mathcal{X}$. Smooth progressions $\mathcal{X}^* \to \mathcal{X}^{*\pm 1}$ occur at $4 \to 5$, $8 \to 9$ and $15 \to 16 \to 17$. Modulations (change of key) $\mathcal{P}(\mathcal{X}) = \mathcal{Q}(\mathcal{X})$ occur at 5 and 19. The progression from a triad or 7-th to a diminished $\mathcal{P}(\mathcal{X}) \to \mathcal{J}^*_{12,4}$ occur at $11 \to 12$, $13 \to 14$ and $21 \to 22$. Substitutions $\mathcal{X} \sqsubset \mathcal{Y}$, $\mathcal{X} \sqsupset \mathcal{Y}$ or $\mathcal{X} \sqsupset \mathcal{Y} \sqsubset \mathcal{Z}$ are seen almost everywhere. We note that at measures $12-13$, we may have to write

$$C^\sharp EGB^\flat \sqsupset C^\sharp EG \sqsubset AC^\sharp EG = \mathcal{D}\left(\mathcal{J}^7_{7,4} + 4\right) \to \mathcal{D}\left(\mathcal{J}^7_{7,4}\right) \to \mathcal{F}\left(\mathcal{J}^7_{7,4}\right) = \mathcal{C}\left(\mathcal{J}^7_{7,4}\right) = DFAC \sqsupset DFA,$$

where we substitute the F major $\mathcal{F}$ for the D minor[5)], that is, the progression

$$DF^\sharp AC = \mathcal{D}\left(\mathcal{J}^7_{7,4}\right) \to \mathcal{F}\left(\mathcal{J}^7_{7,4}\right) = DFAC$$

implies a major-minor transformation in D-key. At measure 23, we omit C which works as a passing tone.

We stop this analysis at measure 26 because there appears the suspended chord $GCDF$. As the original passage is $GD\dot{G}\dot{C}\dot{F}$, we can consider it as a maximal subword of $\mathcal{J}^1_{24,5} = ADG\dot{C}\dot{F}$ (thus $DGC\dot{F}$ is embedded into two octaves as a second-order maximally even set), however we yet have no reasonable description for the progression at measures $25 \to 26 \to 27$ under the maximal evenness ansatz.

3.5. **Concluding remarks.** We have been exploring tonal music from the angle of maximal evenness. Through the J-function analysis of Bach's Prelude, we see that the maximal evenness ansatz works well to describe the basic feature of tonal music, while several derivative musical phenomena are explained insufficiently. Our expression does not distinguish between major and minor, or more general Gregorian mode. We may need more flexible usage of maximal evenness ansatz, since music is a kind of language, indicated by the theory of Lerdahl and Jackendoff[6].

References

[1] J. Clough and J. Douthett, *Maximal Even Sets,* Journal of Music Theory **35**, pp. 93-173, 1991.

[2] N. Cook, and T. Fujisawa, *The psychophysics of harmony perception: Harmony is a three-tone phenomenon,* Empirical Musicology Review **1**(2), pp. 106-126, 2006.

[3] J. Douthett, *Filtered Point-Symmetry and Dynamical Voice-Leading,* in J. Douthett, M. Hyde and C. Smith(eds.), *Music Theory and Mathematics: Chords, Collections, and Transformations, Chapter 4,* University of Rochester Press, pp. 72-106, 2008.

[4] Y. Hashimoto, *Spatio-temporal symmetry on circle rotations and a notion on diatonic set theory,* Bull. of Aichi Univ. of Education, Natural Science **63**, pp. 1-9, 2014. http://hdl.handle.net/10424/5380.

[5] Y. Hashimoto, *A dynamical characterization of Myhill's property,* Bull. of Aichi Univ. of Education, Natural Science **64**, pp. 1-9, 2015. http://hdl.handle.net/10424/5978.

[6] F. Lerdahl and R. Jackendoff, *A Generative Theory of Tonal Music,* MIT Press, 1983.

[7] M. Lothaire, *Algebraic combinatorics on words,* Encyclopedia of mathematics and its applications **90**, Cambridge University Press, 2002.

[8] A. Popoff, M. Andreatta and A. Ehresmann, *A Categorical Generalization of Klumpenhouwer Networks,* Mathematics and Computation in Music **9110**, pp. 303-314, 2015.

[9] D. Tymoczko, *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice,* Oxford Studies in Music Theory, Oxford University Press, 2011.

[5)] In music theory, a diatonic scale starts from the sixth pitch is called the natural minor scale, e.g., for the C major scale $CDEFGAB$, $ABCDEFG$ is called the A minor scale.

meas.	original	standard f.	J-representation	progression
1	$CEG\dot{C}\dot{E}$	CEG	$\mathcal{C}(\mathcal{J}^0_{7,3})$	$\to \mathcal{C}(\mathcal{J}^0_{7,3}+1) \sqsubset \mathcal{C}(\mathcal{J}^0_{7,4}) = CDFA$
2	$CDA\dot{D}\dot{F}$	$DFAC$	$\mathcal{C}(\mathcal{J}^0_{7,4}(1))$	$= \mathcal{C}(\mathcal{J}^{-9}_{7,4}+4) \to \mathcal{C}(\mathcal{J}^{-9}_{7,4})$
3	$BDG\dot{D}\dot{F}$	$GBDF$	$\mathcal{C}(\mathcal{J}^{-9}_{7,4})$	$\equiv \mathcal{C}(\mathcal{J}^3_{7,4}+4) \to \mathcal{C}(\mathcal{J}^3_{7,4}) \sqsupset \mathcal{C}(\mathcal{J}^0_{7,3})$
4	$CEG\dot{C}\dot{E}$	CEG	$\mathcal{C}(\mathcal{J}^0_{7,3})$	$\to \mathcal{C}(\mathcal{J}^1_{7,3}) = CEA = \mathcal{C}(\mathcal{J}^{-6}_{7,3}(1))$
5	$CEA\dot{C}\dot{A}$	ACE	$\mathcal{C}(\mathcal{J}^{-6}_{7,3}) = \mathcal{G}(\mathcal{J}^{-6}_{7,3})$	$\equiv \mathcal{G}(\mathcal{J}^3_{7,3}+4) \to \mathcal{G}(\mathcal{J}^3_{7,3}) \sqsubset \mathcal{G}(\mathcal{J}^7_{7,4})$
6	$CDF^\sharp A\dot{D}$	$DF^\sharp AC$	$\mathcal{G}(\mathcal{J}^7_{7,4})$	$\equiv \mathcal{G}(\mathcal{J}^{-9}_{7,4}+4) \to \mathcal{G}(\mathcal{J}^{-9}_{7,4}) \sqsupset \mathcal{G}(\mathcal{J}^{12}_{7,3})$
7	$BDG\dot{D}\dot{G}$	GBD	$\mathcal{G}(\mathcal{J}^{12}_{7,3})$	$= \mathcal{G}(\mathcal{J}^0_{7,3}+4) \to \mathcal{G}(\mathcal{J}^0_{7,3}) \sqsubset \mathcal{G}(\mathcal{J}^3_{7,4})$
8	$BCEG\dot{C}$	$CEGB$	$\mathcal{G}(\mathcal{J}^3_{7,4})$	$\to \mathcal{G}(\mathcal{J}^2_{7,4}) = CEGA$
9	$ACEG\dot{C}$	$ACEG$	$\mathcal{G}(\mathcal{J}^2_{7,4}(-1))$	$\equiv \mathcal{G}(\mathcal{J}^7_{7,4}+4) \to \mathcal{G}(\mathcal{J}^7_{7,4})$
10	$DA\dot{D}\dot{F}^\sharp\dot{C}$	$DF^\sharp AC$	$\mathcal{G}(\mathcal{J}^7_{7,4})$	$= \mathcal{G}(\mathcal{J}^{-9}_{7,4}+4) \to \mathcal{G}(\mathcal{J}^{-9}_{7,4}) \sqsupset \mathcal{G}(\mathcal{J}^{12}_{7,3})$
11	$GBD\dot{G}\dot{B}$	GBD	$\mathcal{G}(\mathcal{J}^{12}_{7,3})$	$\sqsubset EGBD = \mathcal{G}(\overline{\mathcal{J}}^1_{7,4}(1)) \to \mathcal{J}^4_{12,4}(1) = EGB^\flat C^\sharp$
12	$GB^\flat E\dot{G}\dot{C}^\sharp$	$C^\sharp EGB^\flat$	$\mathcal{J}^4_{12,4}$	$\sqsupset C^\sharp EG = \mathcal{D}(\mathcal{J}^0_{7,3}) \sqsubset \mathcal{D}(\mathcal{J}^{-5}_{7,4}) = AC^\sharp EG$
				$\equiv \mathcal{D}(\mathcal{J}^7_{7,4}+4) \to \mathcal{C}(\mathcal{J}^7_{7,4}) \sqsupset \mathcal{C}(\mathcal{J}^3_{7,3})$
13	$FAD\dot{A}\dot{D}$	DFA	$\mathcal{C}(\mathcal{J}^3_{7,3})$	$\sqsubset BDFA = \mathcal{C}(\overline{\mathcal{J}}^3_{7,4}(-1)) \to \mathcal{J}^8_{12,4}(-1)$
14	$FA^\flat D\dot{F}\dot{B}$	$BDFA^\flat$	$\mathcal{J}^{-4}_{12,4}$	$\sqsupset BDF = \mathcal{C}(\mathcal{J}^{-3}_{7,3}) \sqsubset \mathcal{C}(\mathcal{J}^{-9}_{7,4}) = GBDF$
				$\equiv \mathcal{C}(\mathcal{J}^3_{7,4}+4) \to \mathcal{C}(\mathcal{J}^3_{7,4}) \sqsupset \mathcal{C}(\mathcal{J}^0_{7,3})$
15	$EGC\dot{G}\dot{C}$	CEG	$\mathcal{C}(\mathcal{J}^0_{7,3})$	$\to \mathcal{C}(\mathcal{J}^1_{7,3}) = \mathcal{C}(\mathcal{J}^{-6}_{7,3}(1)) = CEA,$
				$\mathcal{C}(\mathcal{J}^{-6}_{7,3}) \sqsubset \mathcal{C}(\mathcal{J}^1_{7,4}(-1)) = ACEF$
16	$EFA\dot{C}\dot{F}$	$FACE$	$\mathcal{C}(\mathcal{J}^1_{7,4}(2))$	$\to \mathcal{C}(\mathcal{J}^0_{7,4}(2))$
17	$DFA\dot{C}\dot{F}$	$DFAC$	$\mathcal{C}(\mathcal{J}^0_{7,4}(1))$	$= \mathcal{C}(\mathcal{J}^{-9}_{7,4}+4) \to \mathcal{C}(\mathcal{J}^{-9}_{7,4})$
18	$GD\dot{G}\dot{B}\dot{F}$	$GBDF$	$\mathcal{C}(\mathcal{J}^{-9}_{7,4})$	$\equiv \mathcal{C}(\mathcal{J}^3_{7,4}+4) \to \mathcal{C}(\mathcal{J}^3_{7,4}) \sqsupset \mathcal{C}(\mathcal{J}^0_{7,3})$
19	$CEG\dot{C}\dot{E}$	CEG	$\mathcal{C}(\mathcal{J}^0_{7,3}) = \mathcal{F}(\mathcal{J}^0_{7,3})$	$\sqsubset \mathcal{F}(\mathcal{J}^3_{7,4})$
20	$CGB^\flat\dot{C}\dot{E}$	$CEGB^\flat$	$\mathcal{F}(\mathcal{J}^3_{7,4})$	$\equiv \mathcal{F}(\mathcal{J}^1_{7,4}(2)+4) \to \mathcal{F}(\mathcal{J}^1_{7,4}(2))$
21	$F\dot{F}\dot{A}\dot{C}\dot{E}$	$FACE$	$\mathcal{F}(\mathcal{J}^1_{7,4}(2))$	$= \mathcal{F}(\overline{\mathcal{J}}^{-2}_{7,4}(2)) \to \mathcal{J}^0_{12,4}(2)$
22	$F^\sharp CA\dot{C}\dot{E}^\flat$	$F^\sharp ACE^\flat$	$\mathcal{J}^0_{12,4}(2)$	$\to \mathcal{J}^0_{12,4}(2) - 1 = \mathcal{J}^8_{12,4}(1) = FA^\flat BD$
23	$A^\flat F\dot{B}\dot{C}\dot{D}$	$A^\flat BDF$	$\mathcal{J}^8_{12,4}(2)$	$\sqsupset BDF = \mathcal{C}(\mathcal{J}^{-3}_{7,3}) \sqsubset \mathcal{C}(\mathcal{J}^5_{7,4}(2)) = GBDF$
24	$GF\dot{G}\dot{B}\dot{D}$	$GBDF$	$\mathcal{C}(\mathcal{J}^5_{7,4}(2))$	$= \mathcal{C}(\mathcal{J}^3_{7,4}+4) \to \mathcal{C}(\mathcal{J}^3_{7,4}) \sqsupset \mathcal{C}(\mathcal{J}^0_{7,3})$
25	$GC\dot{G}\dot{C}\dot{E}$	CEG	$\mathcal{C}(\mathcal{J}^0_{7,3})$	$\cdots$ to be continued $\cdots$

TABLE 3. J-function analysis for J. S. Bach's Prelude, BWV 846, measure 1–25. $\equiv$ means the octave equivalence.

[10] J. Timothy, *Foundations of Diatonic Theory: A Mathematically Based Approach to Music Fundamentals*, Key College Publishing, 2003.

(Received September 21, 2016)