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What Form of Gravitation Ensures Weakened Kepler's Third Law?

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1 Introduction

In this paper, we consider the two-body problem under gravitations of arbitrary functions. The *two-body problem* concerns the motion of a planet going around the Sun. From now on, we assume that the Sun is fixed and a planet goes around it. We denote by r the distance between the planet and the Sun, by m the mass of the planet, and by $mF(r)$ the magnitude of gravitation acting on the planet. We regard $F(r)$ as a function defined on the interval (R, ∞), where R is the radius of the Sun.

When the function $F(r)$ is proportional to the inverse square of r (Newton's Law of Universal Gravitation), the orbit draws an ellipse with the Sun at one focus (Kepler's 1st Law). However, if we change the function $F(r)$, then in almost all cases, the orbit seems to revolve and does not draw a closed curve. See Figure 1. We know two examples of $F(r)$ for which all bounded orbits are closed curves.

Proposition 1. (1) *If* $F(r) = \dfrac{k}{r^2}$ $(k > 0)$, *then every bounded orbit is an ellipse with the Sun at one focus.*

(2) *If* $F(r) = kr$ $(k > 0)$, *then every bounded orbit is an ellipse with the Sun at the elliptic center.*

Newton [2, Prop.10] solved the inverse problem of Proposition 1 (1). (Many readers may misunderstand. The man who solved Proposition 1 (1) is not Newton, but Johann Bernoulli. See [5], for example.) Surprisingly, Newton [2, Prop.9] also solved the inverse problem of Proposition 1 (2). Moreover, Newton [2, Prop.44] considered that what term makes the orbits revolve, and concluded that it must be an inverse cube one. By adding inverse cube terms to the functions of Proposition 1, we obtain the following result.

Proposition 2. (1) *If* $F(r) = \dfrac{k_1}{r^2} + \dfrac{k_2}{r^3}$ $(k_1 > 0,\ k_2 \geqq -Rk_1)$, *then every bounded orbit is a revolving ellipse around the Sun at one focus. In the case, the anomalistic period* $T = \dfrac{2\pi}{\sqrt{k_1}} a^{3/2}$, *where a is the mean distance.*

(2) *If* $F(r) = k_1 r + \dfrac{k_2}{r^3}$ $(k_1 > 0,\ k_2 \geqq -R^4 k_1)$, *then every bounded orbit is a revolving ellipse around the Sun at the elliptic center. In the case, the anomalistic period* $T = \dfrac{\pi}{\sqrt{k_1}}$.

The second half of Proposition 2 (1) asserts that Kepler's 3rd Law is true even if the gravitation contains an inverse cube term. We say that a curve is a *revolving ellipse* around a point O if it can be represented by a polar equation $r = \varphi(\alpha\theta)$, where $r = \varphi(\theta)$ is a polar equation of an ellipse with O as the origin. Every point of the curve is obtained by rotating a point of the ellipse.

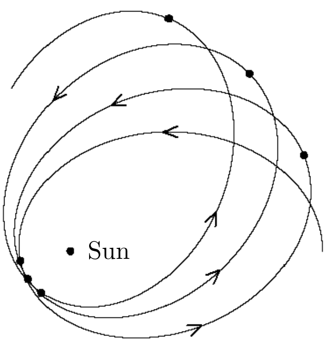

Figure 1. An orbit when $F(r) = r^{-2.1}$ with $r_1 = 1$, $r_2 = 5$. (The dots on the orbit indicate peri- and apoapsides.)

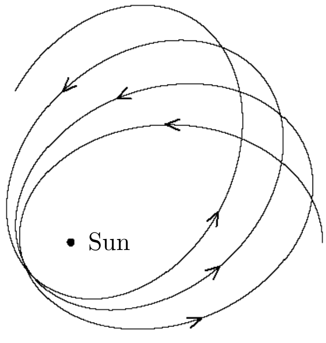

Figure 2. An orbit when $F(r) = 1/r^2 + 0.2/r^3$ with $r_1 = 1$, $r_2 = 5$. (A rotating ellipse around the Sun at one focus.)

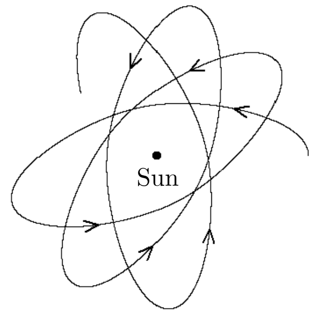

Figure 3. A orbit when $F(r) = r + 2/r^3$ with $r_1 = 1$, $r_2 = 3$. (A rotating ellipse around the Sun at the elliptic center.)

We call an orbit *bounded* if it neither collides with the Sun nor goes to infinity. Consider a bounded orbit. We denote respectively by r_1 and r_2 the minimum and the maximum of r. We call a point on the orbit attaining $r = r_1$ a *periapsis* and one attaining $r = r_2$ an *apoapsis.* The *anomalistic period* is the time which the planet requires to travel from a periapsis for the next periapsis. (The *sidereal period* is the time which the planet requires to make one revolution around the Sun. It seems to be a natural notion than the anomalistic one, however, it can not be defined mathematically because it may change as the starting point of one revolution changes.)

If we take $F(x)$ inappropriate, the planet has no bounded orbits except circular ones. The following result gives an *existence condition of bounded orbits.* In it, $U(r)$ denotes a primitive function of $F(r)$ and $L(r) := r^3F(r)$.

Proposition 3. ([3]) *For every two reals r_1, r_2 ($R < r_1 < r_2$), there is a bounded orbit with r_1, r_2 as the peri- and apoapsis distances respectively if and only if both $U(r)$ and $L(r)$ are monotone increasing on (R, ∞).*

The cases (1) and (2) of Proposition 1 have the property that all bounded orbits are closed curves. A question arises: Are there other examples of such gravitation? The following is the answer.

Theorem 4. ([1]) *Suppose that $F(r)$ is of class C^1 and satisfies the existence condition of bounded orbits. Then, all bounded orbits are closed curves if and only if $F(r)$ is one given in (1) or (2) of Proposition 1.*

The anomalistic period T is a function of variables r_1, r_2. Therefore, by a change of variables $a = \dfrac{r_1 + r_2}{2}$, $b = \dfrac{r_2 - r_1}{2}$, it can be considered as a function of variables a, b. The cases (1) and (2) of Proposition 2 have the property that T does not depend on b. (The author want to call this property *weakened Kepler's 3rd Law.*) A question arises: Are there other examples of such gravitation? The following is the answer.

Theorem 5. *Suppose that $F(r)$ is of class C^3 and satisfies the existence condition of bounded orbits. Then, the anomalistic period T does not depend on b if and only if $F(r)$ is one given in (1) or (2) of Proposition 2.*

The above theorem is a complete version of Theorem 2 of [3]. The author has proved Theorem 4 by using the same method as that of Theorem 5, however, he found that it was already done by M. J. Bertrand [1].

2 Preliminaries

We denote by (x, y) the coordinates of the planet with the Sun at the origin. We assume that the gravitation of the Sun attracts it, but no other forces acts it. Then by Newton's law of motion, we obtain the following differential equations:

$$m\frac{d^2x}{dt^2} = -mF(r)\cos\theta, \qquad m\frac{d^2y}{dt^2} = -mF(r)\sin\theta, \tag{2.1}$$

where (r, θ) is the polar coordinate of (x, y). We can arrange it to the following equations:

$$\frac{d}{dt}\Big(r^2\frac{d\theta}{dt}\Big) = 0, \qquad \frac{d^2r}{dt^2} = r\Big(\frac{d\theta}{dt}\Big)^2 - F(r). \tag{2.2}$$

By integrating them, we have that

$$r^2\frac{d\theta}{dt} = h, \qquad \Big(\frac{dr}{dt}\Big)^2 = -\frac{h^2}{r^2} - 2U(r) + C, \tag{2.3}$$

where h and C are integrating constants. The first equation stands for Kepler's 2nd Law. We denote by $g(r)$ the right-hand side of the second equation, that is,

$$g(r) = -\frac{h^2}{r^2} - 2U(r) + C. \tag{2.4}$$

From now on, consider a non-circular orbit. Since r_1, r_2 are the extrema of r, we have that

$$g(r_1) = -\frac{h^2}{r_1^2} - 2U(r_1) + C = 0, \quad g(r_2) = -\frac{h^2}{r_2^2} - 2U(r_2) + C = 0. \tag{2.5}$$

By solving it, we have that

$$h^2 = \frac{2r_1^2r_2^2}{r_2^2 - r_1^2}\big\{U(r_2) - U(r_1)\big\}, \quad C = \frac{2}{r_2^2 - r_1^2}\big\{r_2^2U(r_2) - r_1^2U(r_1)\big\}. \tag{2.6}$$

By putting it to (2.4), we obtain that

$$g(r) = -\frac{2r_1^2r_2^2}{r_2^2 - r_1^2}\{U(r_2) - U(r_1)\}\frac{1}{r^2} - 2U(r) + \frac{2}{r_2^2 - r_1^2}\{r_2^2U(r_2) - r_1^2U(r_1)\}. \tag{2.7}$$

We want to give a formula of the anomalistic period T. When the planet travels from a periapsis for the next apoapsis, $\frac{dr}{dt} = \sqrt{g(r)}$, and when from the apoapsis for the next periapsis, $\frac{dr}{dt} = -\sqrt{g(r)}$. So we obtain that

$$T = \int_{r_1}^{r_2} \frac{1}{\sqrt{g(r)}}\, dr + \int_{r_2}^{r_1} \frac{-1}{\sqrt{g(r)}}\, dr = 2\int_{r_1}^{r_2} \frac{1}{\sqrt{g(r)}}\, dr. \tag{2.8}$$

Hence T is a function of two variables r_1, r_2.

3 Proofs of Propositions 1 and 2

Proof of Proposition 1. (1) By eliminating t from (2.2), we have that

$$\left(\frac{dr}{d\theta}\right)^2 = -r^2 - \frac{2r^4}{h^2}U(r) + \frac{C}{h^2}r^4. \tag{3.1}$$

Put $U(r) = -\frac{k}{r}$ to it. By changing the variable $r = u^{-1}$, we have that

$$\left(\frac{du}{d\theta}\right)^2 = -\left(u^2 - 2\frac{k}{h^2}u - \frac{C}{h^2}\right). \tag{3.2}$$

Consider a non-circular bounded orbit. Since $u_1 := r_1^{-1}$ and $u_2 := r_2^{-1}$ are the extrema of u, we have that

$$\left(\frac{du}{d\theta}\right)^2 = -(u - u_1)(u - u_2). \tag{3.3}$$

By putting $u = \frac{u_1 + u_2}{2} + \frac{u_1 - u_2}{2}\cos\phi$, we can calculate

$$\theta = \pm\int \frac{1}{\sqrt{(u_1 - u)(u - u_2)}}\, du = \mp\phi + \theta_1, \tag{3.4}$$

where θ_1 is an integrating constant. Since $r = 1/u$, we obtain that

$$r = \left\{\frac{u_1 + u_2}{2} + \frac{u_1 - u_2}{2}\cos(\theta - \theta_1)\right\}^{-1} = \frac{l}{1 + e\cos(\theta - \theta_1)}, \tag{3.5}$$

where $l := \frac{2}{u_1 + u_2}$, $e := \frac{u_1 - u_2}{u_1 + u_2}$. Since $r_2 > r_1 > 0$, we have that $0 < e < 1$. Hence it is an ellipse with the origin at one focus.

(2) Put $U(r) = \frac{1}{2}kr^2$ to (3.1). By changing the variable $r = u^{-1/2}$ in (3.1), we have that

$$\left(\frac{du}{d\theta}\right)^2 = -4\left(u^2 - \frac{C}{h^2}u + \frac{k}{h^2}\right). \tag{3.6}$$

Since $u_1 := r_1^{-2}$ and $u_2 := r_2^{-2}$ are the extrema of u, we have that

$$\left(\frac{du}{d\theta}\right)^2 = -4(u - u_1)(u - u_2). \tag{3.7}$$

By the same calculation as (3.3), we obtain that

$$r = \left\{\frac{u_1 + u_2}{2} + \frac{u_1 - u_2}{2}\cos 2(\theta - \theta_1)\right\}^{-1/2} = \left\{u_1\cos^2(\theta - \theta_1) + u_2\sin^2(\theta - \theta_1)\right\}^{-1/2}. \tag{3.8}$$

It is an ellipse with the origin at its center.

Proof of Propostion 2. (1) Put $U(r) = -\dfrac{k_1}{r} - \dfrac{k_2}{2r^2}$ to (3.1). By changing the variable $r = u^{-1}$, we have that

$$\left(\frac{du}{d\theta}\right)^2 = -\alpha^2\left(u^2 - 2\frac{k_1}{\alpha^2 h^2}u - \frac{C}{\alpha^2 h^2}\right), \tag{3.9}$$

where $\alpha = \sqrt{1 - k_2/h^2}$. By the same calculation as (3.3), we obtain that

$$r = \frac{l}{1 + e\cos\alpha(\theta - \theta_1)}. \tag{3.10}$$

It is a revolving ellipse with the origin at one focus.

Put $U(r) = -\dfrac{k_1}{r} - \dfrac{k_2}{2r^2}$ to (2.7). Then we have that $g(r) = \dfrac{k_1(r_2 - r)(r - r_1)}{ar^2}$. By putting it to (2.8), we obtain that

$$T = 2\int_{r_1}^{r_2} \frac{dt}{dr}\,dr = 2\sqrt{\frac{a}{k_1}}\int_{r_1}^{r_2} \frac{r}{\sqrt{(r_2 - r)(r - r_1)}}\,dr = \frac{2\pi}{\sqrt{k_1}}a^{3/2}. \tag{3.11}$$

(2) Put $U(r) = \dfrac{1}{2}k_1 r^2 - \dfrac{k_2}{2r^2}$ to (3.1). By changing the variable $r = u^{-1/2}$, we obtain that

$$\left(\frac{du}{d\theta}\right)^2 = -4\alpha^2\left(u^2 - \frac{C}{\alpha^2 h^2}u + \frac{k_2}{\alpha^2 h^2}\right). \tag{3.12}$$

where $\alpha = \sqrt{1 - k_2/h^2}$. By the same calculation as (3.7), we obtain that

$$r = \left\{u_1\cos^2\alpha(\theta - \theta_1) + u_2\sin^2\alpha(\theta - \theta_1)\right\}^{-1/2}. \tag{3.13}$$

It is a revolving ellipse with the origin at its center.

Put $U(r) = \dfrac{k_1}{2}r^2 - \dfrac{k_2}{2r^2}$ to (2.7). Then we have that $g(r) = \dfrac{k_1(r_2^2 - r^2)(r^2 - r_1^2)}{r^2}$. By putting it to (2.8), we obtain that

$$T = 2\int_{r_1}^{r_2}\left(\frac{dt}{dr}\right)dr = \frac{2}{\sqrt{k_1}}\int_{r_1}^{r_2} \frac{r}{\sqrt{(r_2^2 - r^2)(r^2 - r_1^2)}}\,dr = \frac{\pi}{\sqrt{k_1}}. \tag{3.14}$$

4 Proof of Theorem 5

Proof of Theorem 5. By putting $U(r) = \dfrac{1}{r^2}\Phi(r)$, $r_1 = a + b$, $r_2 = a - b$, $r = a + \rho$, we have that

$$\begin{aligned} g(r) =& \frac{1}{r^2}\left\{ -\frac{2}{r_2^2 - r_1^2}\left(r_1^2\Phi(r_2) - r_2^2\Phi(r_1)\right) - 2\Phi(r) + \frac{2}{r_2^2 - r_1^2}\left(r^2\Phi(r_2) - r^2\Phi(r_1)\right)\right\} \\ =& \frac{b^2 - \rho^2}{b(a + \rho)^2}\left\{\frac{\Phi(a + b) - \Phi(a + \rho)}{b - \rho} - \frac{\Phi(a + \rho) - \Phi(a - b)}{b + \rho} - \frac{\Phi(a + b) - \Phi(a - b)}{2a}\right\}. \end{aligned} \tag{4.1}$$

For later convenience, we calculate under the assumption that $F(r)$ is of class C^n. Since $\Phi(r)$ is of class C^{n+1}, we can use Taylor expansion to (4.1) as follows:

$$g(r) = \frac{b^2 - \rho^2}{b(a + \rho)^2}\left\{\sum_{j=0}^{n+1}\frac{1}{j!}\Phi^{(j)}(a)\left(\frac{b^j - \rho^j}{b - \rho} - \frac{\rho^j - (-b)^j}{b + \rho}\right) - \sum_{j=0}^{n}\frac{1}{j!}\Phi^{(j)}(a)\frac{b^j - (-b)^j}{2a} + o(b^n)\right\} \tag{4.2}$$

$$= \frac{b^2 - \rho^2}{(1 + \rho/a)^2}\left\{\sum_{j=1}^{n-1}\alpha_j\sum_{i=0}^{[(j-1)/2]}\rho^{j-2i}b^{2i} + \sum_{j=0}^{[(n-1)/2]}\beta_{2j}b^{2j} + o(b^{n-1})\right\}, \tag{4.3}$$

where

$$\alpha_j = \frac{2}{(j + 2)!\,a^2}\Phi^{(j+2)}(a), \quad \beta_j = \alpha_j - \frac{1}{2a}\alpha_{j-1}. \tag{4.4}$$

We can represent these coefficients by derivatives of $L(r)$ as follows:

$$\beta_0 = \frac{1}{a^3}L'(a), \quad \alpha_j = \frac{2}{(j + 2)!\,a^2}\left(a^{-1}L''(a)\right)^{(j-1)}, \quad \beta_j = \alpha_j - \frac{1}{2a}\alpha_{j-1}, \tag{4.5}$$

where the superscript denotes the order of derivative with respect to a. Since $L(r)$ is monotone increasing, $L'(r) > 0$ on an open dense subset of (R, ∞). From now on, we consider the problem on this set. So we can assume that $\beta_0 > 0$.

Now since $F(r)$ is of class C^3, we can reduce (4.3) to

$$g(r) = \frac{b^2 - \rho^2}{(1 + \rho/a)^2} \big\{\beta_0 + \alpha_1\rho + \alpha_2\rho^2 + \beta_2 b^2 + o(b^2)\big\}. \tag{4.6}$$

Then we can calculate as follows:

$$\begin{aligned}
\frac{1}{\sqrt{g(r)}} =& \left\{ \frac{b^2-\rho^2}{(1+\rho/a)^2}\Big(\beta_0 + \alpha_1\rho + \alpha_2\rho^2 + \beta_2 b^2 + o(b^2)\Big)\right\}^{-1/2} \\
=& \left\{\frac{\beta_0(b^2-\rho^2)}{(1+\rho/a)^2}\right\}^{-1/2} \left\{1 + \frac{\alpha_1}{\beta_0}\rho + \frac{\alpha_2}{\beta_0}\rho^2 + \frac{\beta_2}{\beta_0}b^2 + o(b^2)\right\}^{-1/2} \\
=& \frac{(1+\rho/a)}{\sqrt{\beta_0(b^2-\rho^2)}}\left\{1 - \frac{\alpha_1}{2\beta_0}\rho - \frac{\alpha_2}{2\beta_0}\rho^2 - \frac{\beta_2}{2\beta_0}b^2 + \frac{3\alpha_1^2}{8\beta_0^2}\rho^2 + o(b^2)\right\} \\
=& \frac{1}{\sqrt{\beta_0(b^2-\rho^2)}}\left\{1 + \Big(\frac{1}{a} - \frac{\alpha_1}{2\beta_0}\Big)\rho + \Big(-\frac{\alpha_1}{2a\beta_0} - \frac{\alpha_2}{2\beta_0} + \frac{3\alpha_1^2}{8\beta_0^2}\Big)\rho^2 - \frac{\beta_2}{2\beta_0}b^2 + o(b^2)\right\}.
\end{aligned} \tag{4.7}$$

By putting it to (2.8), we obtain that

$$T = \frac{2\pi}{\sqrt{\beta_0}} + \frac{2\pi}{16\beta_0^{5/2}}\left\{-4\beta_0\Big(\frac{1}{a}\alpha_1 + \alpha_2 + 2\beta_2\Big) + 3\alpha_1^2\right\}b^2 + o(b^2). \tag{4.8}$$

Thus a necessary condition that T does not depend on b is that

$$-4\beta_0\Big(\frac{1}{a}\alpha_1 + \alpha_2 + 2\beta_2\Big) + 3\alpha_1^2 = 0. \tag{4.9}$$

By putting (4.6) to it, we have that

$$-L'(r)L'''(r) + \frac{1}{r}L'(r)L''(r) + \frac{1}{3}\big(L''(r)\big)^2 = 0. \tag{4.10}$$

It holds on an open dense subset of (R, ∞). Since $L(r)$ is of class C^3, it holds on the entire (R, ∞). By solving (4.10), we obtain that

$$L'(r) = (c_1 r^2 + c_2)^{3/2}, \tag{4.11}$$

where c_1, c_2 are integrating constants. When we solve (4.10), the signs $\pm$ appear in the right-hand side of (4.11), however, since $L(r)$ is monotone increasing, the sign must be plus.

Since $F(r) = r^{-3}L(r)$ is of class C^5, we can reduce (4.3) to

$$g(r) = \frac{b^2-\rho^2}{(1+\rho/a)^2}\big\{\beta_0 + \alpha_1\rho + \alpha_2\rho^2 + \beta_2 b^2 + \alpha_3\rho^3 + \alpha_3\rho b^2 + \alpha_4\rho^4 + \alpha_4\rho^2 b^2 + \beta_4 b^4 + o(b^4)\big\}. \tag{4.12}$$

By putting (4.11) to (4.5), we have that

$$\beta_0 = \frac{1}{a^3}(c_1a^2 + c_2)^{3/2}, \quad \alpha_j = \frac{6c_1}{(j+2)!\,a^2}\Big((c_1a^2+c_2)^{1/2}\Big)^{(j-1)}, \quad \beta_j = \alpha_j - \frac{1}{2a}\alpha_{j-1}. \tag{4.13}$$

By similar calculations as (4.7), (4.8), we obtain that

$$T = \frac{2\pi}{\sqrt{\beta_0}}\left\{1 - \frac{33c_1^2c_2^2}{320(c_1a^2+c_2)^4}b^4\right\} + o(b^4). \tag{4.14}$$

The author obtained the above result by using REDUCE Computer Algebra System [4]. Thus a necessary condition that T does not to depend on b is that $c_1 = 0$ or $c_2 = 0$.

When $c_1 = 0$, by integrating $L'(r) = c_2^{3/2}$, we have that $L(r) = k_1 r + k_2$, where $k_1 = c_2^{3/2}$ and k_2 is an integrating constant. Hence we obtain that $F(r) = \dfrac{k_1}{r^2} + \dfrac{k_2}{r^3}$. Since $U(r)$ is monotone increasing on (R, ∞), we have that $k_1 > 0$ and $k_2 \geqq -Rk_1$.

When $c_2 = 0$, by integrating $L'(r) = c_1^{3/2}r^3$, we have that $L(r) = k_1r^4 + k_2$, where $k_1 = c_1^{3/2}/4$ and k_2 is an integrating constant. Hence we obtain that $F(r) = k_1 r + \dfrac{k_2}{r^3}$. Since $U(r)$ is monotone increasing on (R, ∞), we have that $k_1 > 0$ and $k_2 \geqq -R^3k_1$.

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