Bulletin of Aichi Univ. of Education, 62 (Natural Sciences), pp. 7 − 16, March, 2013

The van der Corput Embedding of $ax + b$ and its Interval Exchange Map Approximation

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1. Introduction

In [4], we introduce interval preserving map approximation of a linear map $3x+1$, to attack the well-known and still unsolved $3x+1$ problem, which is firstly proposed by Lothar Collatz in 1930's:

Conjecture 1.1. *Consider a map $f : \mathbf{N} \to \mathbf{N}$ such that*

$$f(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd,} \\ n/2, & \text{if } n \text{ is even.} \end{cases}$$

Then for each natural number n, there exists a finite number t such that $f^t(n) = \underbrace{f \circ f \circ \cdots \circ f(n)}_{t-times} = 1.$

In the investigations[3][4], we use the *binary van der Corput embedding* β of natural numbers into unit interval $[0, 1]$, which gives a low-discrepancy sequence over $[0, 1]$, firstly introduced by van der Corput in 1935(cf.[7] pp 129), and construct a right continuous bijection F over $[0, 1)$ conjugate to $3x + 1$ by way of β. In this note, we investigate a right continuous bijection F over $[0, 1)$ conjugate to the linear function $ax+b$, its 'finite bit approximations' which bring dynamics of interval exchange maps on $[0, 1)$, and observe substitution dynamics and transducers associated with the approximations. As a result, we obtain a necessary and sufficient condition for the minimality of the dynamics $([0, 1), F)$ (Theorem 3.8).

2. The binary van der Corput embedding of natural numbers and a conjugacy of $ax + b$

We follow the notations in [4].

Definition 2.1. *Let $n = g_k \cdot 2^k + g_{k-1} \cdot 2^{k-1} + \cdots + g_0$ $(= (g_k g_{k-1} \cdots g_0)_2$ for short $)$ be a binary expansion of a natural number n. Then we define $\beta : \mathbf{N} \to [0, 1)$, given by*

$$\beta(n) = \frac{g_0}{2} + \frac{g_1}{2^2} + \cdots + \frac{g_k}{2^{k+1}} = (0.g_0 g_1 \cdots g_k)_2.$$

As $\beta : \mathbf{N} \to [0, 1)$ is one-to-one and β embeds natural numbers densely into $[0, 1)$, we call β the binary van der Corput embedding*.*

For a binary expression $n = (g_l g_{l-1} \cdots g_0)_2$, we put $l = \mathrm{ord}(n) := [\log_2 n]$ where $[x]$ stands for an integer not greater than x, and $n|^k$ denotes an *upper cut off* of n at k-th order;

$$n|^k = (g_{k-1} g_{k-2} \cdots g_0)_2 \equiv n \pmod{2^k}.$$

To make free from ambiguity, we always take the finite binary expression $\beta(n) = (0.g_0 g_1 \cdots g_l)_2$ of any natural number n. For a real number $x \in [0, 1)$ and $k \in \mathbf{N}$, $x|_k$ denotes a cut off of x at $(-k-1)$-th order in the binary expression;

$$x|_k = (0.e_1 e_2 \cdots e_k)_2 = \sum_{j=1}^{k} \frac{e_j}{2^j}, \quad \text{if } \ x = (0.e_1 e_2 \cdots)_2 = \sum_{j=1}^{\infty} \frac{e_j}{2^j}.$$

We use the notation $[\ x\ )_k$ as a segment $\left[x|_k, x|_k + \frac{1}{2^k}\right)$, hence a natural decomposition of segments

$$[\ (0.e_1 e_2 \cdots e_k)_2\ )_k = [\ (0.e_1 e_2 \cdots e_k 0)_2\ )_{k+1} \oplus [\ (0.e_1 e_2 \cdots e_k 1)_2\ )_{k+1}, \tag{2.1}$$

where $[a,b) \oplus [b,c)$ stands for a division of an interval $[a,c)$ at b. Then, the *k-th refinement* of the unit interval $[0,1)$ is given as

$$(2.2)\qquad [0,1) = \bigoplus_{j=0}^{2^k-1} \left[\ \beta(\nu_j^{(k)})\ \right)_k ,$$

where $\{\nu_j^{(k)} \mid j = 0,\dots,2^k-1\} = \{0,1,\dots,2^k-1\}$ with $\beta(\nu_i^{(k)}) < \beta(\nu_j^{(k)})$ if and only if $i < j$.

By definition, we easily to see the followings.

Lemma 2.2. *For $x, y \in [0,1)$ and $k,l,m,n \in \mathbf{N}$,*

(1) $x \in [\ x\ )_k$.
(2) $[\ x\ )_k \cap [\ y\ )_k \neq \emptyset$ *holds if and only if* $x|_k = y|_k$, *and hence* $[\ x\ )_k = [\ y\ )_k$.
(3) *if* $x \neq y$, *there exists* $l \in \mathbf{N}$ *such that* $[\ x\ )_l \cap [\ y\ )_l = \emptyset$.
(4) $[\ x\ )_{k+1} \subsetneq [\ x\ )_k$ *and* $\displaystyle\bigcap_{k=1}^{\infty} [\ x\ )_k = \{x\}$.
(5) $\beta(m)|_k = \beta(m|^k)$, *thus* $[\ \beta(m)\ )_k = [\ \beta(n)\ )_k$ *if and only if* $m|^k = n|^k$.
(6) *If* $l > ord(m)$, *then* $\beta(m + n \cdot 2^l) = \beta(m) + \dfrac{\beta(n)}{2^l}$.
(7) $(n|^k)\big|^l = n|^{\min\{k,l\}}$.

Let us consider a linear function $f(n) = an + b$ on natural numbers, where $a, b \in \mathbf{N}$ and $a > b$. We are to construct the conjugacy $F : [0,1) \to [0,1)$ of $f : \mathbf{N} \to \mathbf{N}$, that is, $F \circ \beta(n) = \beta \circ f(n)$ holds for any natural number n.

Lemma 2.3. *Suppose a is odd number. Then we have*

(1) $[\ \beta \circ f(m)\ )_k = [\ \beta \circ f(n)\ )_k$ *holds if and only if* $[\ \beta(m)\ )_k = [\ \beta(n)\ )_k$ *for any* $m, n \in \mathbf{N}$.
(2) $\{\beta \circ f(n) \mid n \in \mathbf{N}\}$ *is dense in* $[0,1)$.

Proof. (1) By Lemma 2.2 (5), $[\ \beta(m)\ )_k = [\ \beta(n)\ )_k$ means $m \equiv n \pmod{2^k}$, which equivalents to $am + b \equiv an + b \pmod{2^k}$ as a and 2^k are coprime, hence $[\ \beta(am+b)\ )_k = [\ \beta(an+b)\ )_k$.

(2) Given $x \in [0,1)$ and $k \in \mathbf{N}$, consider $m_k = \beta^{-1}(x|_k)$, and take $b' \in \mathbf{N}$ with $b' \equiv b - m_k \pmod{a}$. As a and 2^k is coprime, we have $2^{k\varphi(a)} = (2^k)^{\varphi(a)} \equiv 1 \pmod{a}$, hence $m_k + b' \cdot 2^{k\varphi(a)} \equiv m_k + b' \equiv b \pmod{a}$, where φ stands for Euler's totient function. Thus there exists $n \in \mathbf{N}$ with $m_k + b' \cdot 2^{k\varphi(a)} = an + b$, and hence

$$\beta(an+b) = \beta(m_k + b' \cdot 2^{k\varphi(a)}) = x|_k + \frac{\beta(b')}{2^{k\varphi(a)}} \in [\ x\ )_k .$$

As a result, for any $x \in [0,1)$ and $k \in \mathbf{N}$, there exists $n \in \mathbf{N}$ with $|x - \beta(an+b)| < 2^{-k}$. □

Now we suppose a is odd. Then the conjugacy F is defined on the dense set $\beta(\mathbf{N})$ by $F(\beta(n)) := \beta(f(n))$, and its extension to $[0,1)$ is given as follows. For any $x \in [0,1)$, consider a sequence $n_k = \beta^{-1}(x|_k)$, $k = 1,2,\dots$. Obviously we see $n_{k+1} \equiv n_k \pmod{2^k}$, hence $an_{k+1} + b \equiv an_k + b \pmod{2^k}$. It follows from Lemma 2.2 (4) that

$$[\ \beta(an_{k+1}+b)\ )_{k+1} \subsetneq [\ \beta(an_{k+1}+b)\ )_k = [\ \beta(an_k+b)\ )_k ,$$

holds, and then $\displaystyle\bigcap_{k=1}^{\infty} [\ \beta(an_k+b)\ )_k$ consists of a unique point, denoted by $\displaystyle\lim_{k\to\infty} \beta(an_k+b)$. Thus we define the conjugacy $F : [0,1) \to [0,1)$ of the linear function f:

$$(2.3)\qquad F(x) = \lim_{k\to\infty} \beta\left(a\beta^{-1}(x|_k) + b\right).$$

Lemma 2.4. *For $x, y \in [0,1)$, we have*

(1) $[\ F(x|_k)\ )_k = [\ F(x)\ )_k$, *hence* $F(x)|_k = F(x|_k)\big|_k$.
(2) $F([\ x\ )_k) \subset [\ F(x)\ )_k$, *hence* F *is right continuous.*
(3) $[\ F(x)\ )_k = [\ F(y)\ )_k$ *if and only if* $[\ x\ )_k = [\ y\ )_k$, *hence* F *is injective.*

Proof. (1) By definition, we have

$$\{F(x)\} = \bigcap_{k=1}^{\infty} \left[\ \beta(a\beta^{-1}(x|_k) + b)\ \right)_k \subset \left[\ \beta(a\beta^{-1}(x|_k) + b)\ \right)_k = [\ F(x|_k)\ )_k\,,$$

that is, $F(x) \in [\ F(x|_k)\ )_k$, while $F(x) \in [\ F(x)\ )_k$ by Lemma 2.2 (1). Thus we see $[\ F(x|_k)\ )_k \cap [\ F(x)\ )_k \neq \emptyset$, which means $[\ F(x|_k)\ )_k = [\ F(x)\ )_k$ by Lemma 2.2 (2).

(2) $y \in [\ x\ )_k$ means $y|_k = x|_k$. It follows from (1) that

$$F(y) \in [\ F(y)\ )_k = [\ F(y|_k)\ )_k = [\ F(x|_k)\ )_k = [\ F(x)\ )_k\,.$$

For any $\varepsilon > 0$, take $k \in \mathbf{N}$ with $2^{-k} < \varepsilon$. Then for any y with $x \leq y \leq x + 2^{-(k+1)}$, we see $y \in [\ x\ )_k$ and

$$F(y) \in F([\ x\ )_k) \subset [\ F(x)\ )_k\,,$$

that is, $|F(y) - F(x)| < 2^{-k} < \varepsilon$, showing the right continuity of F.

(3) Put $m_k = \beta^{-1}(x|_k)$ and $n_k = \beta^{-1}(y|_k)$, then we see $[\ F(x)\ )_k = [\ F(x|_k)\ )_k = [\ \beta \circ f(m_k)\ )_k$ and $[\ F(y)\ )_k = [\ F(y|_k)\ )_k = [\ \beta \circ f(n_k)\ )_k$ by (1). It follows from Lemma 2.3 (1) that $[\ F(x)\ )_k = [\ F(y)\ )_k$ if and only if $[\ x\ )_k = [\ \beta(m_k)\ )_k = [\ \beta(n_k)\ )_k = [\ y\ )_k$. Suppose that $x \neq y$, then $[\ x\ )_k \cap [\ y\ )_k = \emptyset$ holds for some k. Then we have $[\ F(x)\ )_k \cap [\ F(y)\ )_k = \emptyset$, hence $F(x) \neq F(y)$. □

Lemma 2.4 (1) is an extension of a natural property

$$f(n|^k)\big|^k = f(n)|^k \tag{2.4}$$

of the linear function $f(n) = an + b$ with $a, b \in \mathbf{N}$, while (2)(3) implies that F brings segment-wise exchange over the refinement (2.2) of the unit interval in any order k. Indeed we have the following.

Proposition 2.5. *For any* $k \in \mathbf{N}$ *and* $x \in [0, 1)$, *the conjugacy* F *gives a right continuous bijection*

$$F : [\ x\ )_k \to [\ F(x)\ )_k\,.$$

Particularly, F *preserves the Lebesgue measure* μ *on* $[0, 1)$.

Proof. All we have to show is surjectivity of F. For any $y \in [0, 1)$ and $l \in \mathbf{N}$, there exists $n_l \in \mathbf{N}$ with $F \circ \beta(n_l) = \beta \circ f(n_l) \in [\ y\ )_l$ by Lemma 2.3 (2). By Lemma 2.4, we see

$$F([\ \beta(n_l)\ )_l) \subset [\ F(\beta(n_l))\ )_l = [\ y\ )_l\,.$$

Let x be the unique accumulation point $\{x\} = \bigcap_{l}^{\infty} [\ \beta(n_l)\ )_l$. Then $F(x) \in [\ F(\beta(n_l))\ )_l \subset [\ y\ )_l$ holds for any l, meaning that

$$F(x) \in \bigcap_{l}^{\infty} [\ y\ )_l = \{y\},$$

hence $y = F(x)$ and F is a surjection $[0, 1) \to [0, 1)$. Combining Lemma 2.4 (2),(3) and surjectivity of F, we see $F([\ x\ )_k) = [\ F(x)\ )_k$ for any $k \in \mathbf{N}$. Note that $\mu([\ x\ )_k) = \mu([\ F(x)\ )_k) = 2^{-k}$. It is shown that the σ-field generated by the segments $[\ x\ )_k\,, k \in \mathbf{N}, x \in [0, 1)$ coincides with the Lebesgue measurable sets. Hence the assertion. □

Note that for any natural number n, $[\ \beta(n)\ )_k = [\beta(n), \beta(n)+2^{-k})$ if $k > \mathrm{ord}(n)$, thus $w \in [\ \beta(n)\ )_k$ means $w \geq \beta(n)$. Therefore we state the right continuity of F only. Indeed, for $f(n) = n+1$, we see

$$\lim_{\substack{w \to (0.11)_2 \\ w < (0.11)_2}} F(w) = (0.1)_2 \neq (0.001)_2 = F((0.11)_2).$$

3. Segment-wise linear approximation of the conjugacy F

In view of Proposition 2.5, we construct a sequence of segment-wise linear functions F_k's approximating the conjugacy F.

Definition 3.1. *For each $k \in \mathbf{N}$, we define the k-th approximant F_k as*

$$F_k(x) = x - x|_k + F(x)\big|_k.$$

If $x \in [\ \beta(n)\ )_k$, we see

$$F_k(x) = x - \beta(n)\big|_k + F(\beta(n))\big|_k,$$

thus $F_k([\ \beta(n)\ )_k) = [\ F \circ \beta(n)\ )_k$, compatible with Proposition 2.5. Moreover, the fact $F(x), F_k(x) \in [\ F \circ \beta(n)\ )_k$ for any $x \in [\ \beta(n)\ )_k$ means

$$|F(x) - F_k(x)| < 2^{-k}$$

for any $x \in [\ \beta(n)\ )_k$, and then for any $x \in [0,1)$. Therefore the sequence F_k, $k = 1, 2, \ldots$ approximates F uniformly on $[0,1)$, so F_k simulates the behavior of F that exchange the segments $[\ \beta(n)\ )_k$'s. We also note that (2.3) is expressed as $F(x) = \lim_{k\to\infty} F_k(x)$.

In the following subsections, we investigate the approximants F_k's to extract dynamical characteristics of the map $f(n) = an + b$.

3.1. **Behavior of carries and exchange of segments.** For each $k \in \mathbf{N}$, we define an integer valued function

$$\tau_k(n) = \left[\frac{f(n|^k)}{2^k}\right] = \left[\frac{an|^k + b}{2^k}\right],$$

then we see

$$f(n|^k) = \tau_k(n) \cdot 2^k + f(n)\big|^k. \tag{3.1}$$

Namely, the function τ_k describes the amount of the carry from the lower k bits in the calculation $n \mapsto an+b$.

Theorem 3.2. *Suppose that a is odd and $0 \leq b < a$. Then, for $n, k \in \mathbf{N}$, we have*

(1) $\tau_k(n) \in \{0, 1, \ldots, a-1\}$.

(2) *For binary expressions $n = (g_p \cdots g_0)_2$ and $f(n) = (h_q \cdots h_0)_2$, we have*

$$h_k = \big(g_k + \tau_k(n)\big)\big|^1,\ \ k = 1, \ldots, q$$

and

$$\tau_{k+1}(n) = \left[\frac{ag_k + \tau_k(n)}{2}\right],$$

where $g_k = 0$ for $k > p$.

(3) *Let $(0.g_0g_1\cdots g_p)_2$ be the binary expression of $\beta(n)$. Then, the image of the segment*

$$[\ \beta(n)\ )_k = [\ (0.g_0\cdots g_{k-1})_2\ )_k = [\ (0.g_0\cdots g_{k-1}0)_2\ )_{k+1} \oplus [\ (0.g_0\cdots g_{k-1}1)_2\ )_{k+1}$$

by F is given as

$$F\Big([\ \beta(n)\ )_k\Big) = \begin{cases} [\ F((0.g_0\cdots g_{k-1}0)_2)\ )_{k+1} \oplus [\ F((0.g_0\cdots g_{k-1}1)_2)\ )_{k+1}, & \text{if } \tau_k(n) \text{ is even,} \\ [\ F((0.g_0\cdots g_{k-1}1)_2)\ )_{k+1} \oplus [\ F((0.g_0\cdots g_{k-1}0)_2)\ )_{k+1}, & \text{if } \tau_k(n) \text{ is odd.} \end{cases}$$

Proof. (1) As $n|^k \le 2^k - 1$ and $a > b$, we have

$$0 \le \left[\frac{an|^k + b}{2^k}\right] \le \left[\frac{a \cdot 2^k + b - a}{2^k}\right] = \left[a - \frac{a-b}{2^k}\right] \le a - 1.$$

(2) By definition, we see $n|^{k+1} = (g_k g_{k-1} \cdots g_0)_2 = g_k \cdot 2^k + n|^k$. We also see $f(n)\big|^{k+1} = h_k \cdot 2^k + f(n)\big|^k$, while, by (3.1), we have

$$\begin{aligned} f(n)\big|^{k+1} &= f(n|^{k+1})\big|^{k+1} = \Big(a(g_k \cdot 2^k + n|^k) + b\Big)\Big|^{k+1} = \Big(ag_k \cdot 2^k + f(n|^k)\Big)\Big|^{k+1} \\ &= \Big((ag_k + \tau_k(n)) \cdot 2^k + f(n)\big|^k\Big)\Big|^{k+1} = \big(ag_k + \tau_k(n)\big)\big|^1 \cdot 2^k + f(n)\big|^k. \end{aligned} \tag{3.2}$$

As a is odd, we obtain the equation $h_k = \big(ag_k + \tau_k(n)\big)\big|^1 = \big(g_k + \tau_k(n)\big)\big|^1$.

Again by (3.1) and $f(n)\big|^{k+1} = h_k \cdot 2^k + f(n)\big|^k$, we see

$$f(n|^{k+1}) = \tau_{k+1}(n) \cdot 2^{k+1} + f(n)\big|^{k+1} = (2\tau_{k+1}(n) + h_k)2^k + f(n)\big|^k$$

while, as $n|^{k+1} = g_k \cdot 2^k + n|^k$, we have

$$f(g_k \cdot 2^k + n|^k) = ag_k \cdot 2^k + f(n|^k) = (ag_k + \tau_k(n)) \cdot 2^k + f(n)\big|^k,$$

hence $2\tau_{k+1}(n) + h_k = ag_k + \tau_k(n)$. Since $0 \le h_k/2 < 1$ and $\tau_{k+1}(n)$ is an integer, we come to

$$\tau_{k+1}(n) = \left[\tau_{k+1}(n) + \frac{h_k}{2}\right] = \left[\frac{ag_k + \tau_k(n)}{2}\right].$$

(3) It comes from (3.2) that

$$\beta\left(f(n)\right)\big|_{k+1} = \beta\left(f(n)\right)\big|_k + \frac{1}{2^{k+1}}\beta(g_k + \tau_k(n))\big|_1.$$

Then we have $(g_k + \tau_k(n))|^1 = g_k$ whenever $\tau_k(n)$ is even, and $(g_k + \tau_k(n))|^1 = 1 - g_k$ whenever $\tau_k(n)$ is odd. Thus we see

$$\Big(F \circ \beta(n|^k + 2^k)\Big)\Big|_{k+1} = \begin{cases} \Big(F \circ \beta(n)\Big)\Big|_k + \dfrac{1}{2^{k+1}}, & \text{if } \tau_k(n) \text{ is even,} \\ \Big(F \circ \beta(n)\Big)\Big|_k - \dfrac{1}{2^{k+1}}, & \text{if } \tau_k(n) \text{ is odd,} \end{cases}$$

hence the assertion. □

3.2. **Substitution dynamics induced by the conjugacy F.**

Definition 3.3. *For each* approximation order k, *we label the segment* $[\ \beta(n)\ )_k$ *as*

$$L_k(n) := \tau_0(n)\tau_1(n)\cdots\tau_k(n).$$

We say the segment $[\ \beta(n)\ )_k$ *is* painted in color $\tau_k(n)$.

By Theorem 3.2 (1), the label $L_k(n)$ is a string over the alphabet $A = \{0, 1, \ldots, a-1\}$ with length $k+1$. Note that $\tau_0(n) = b$ by definition, thus the original unit interval $[0,1)$ is labeled as b. As the approximation order k increases, each segment is divided into two segments:

$$[\ \beta(n)\ )_k = \Big[\ \beta(n|^k)\ \Big)_k \rightsquigarrow \Big[\ \beta(n|^k)\ \Big)_{k+1} \oplus \Big[\ \beta(n|^k + 2^k)\ \Big)_{k+1}. \tag{3.3}$$

Accordingly, the refinements $\big[\ \beta(n|^k)\ \big)_{k+1}$ and $\big[\ \beta(n|^k + 2^k)\ \big)_{k+1}$ are labeled respectively as

$$\begin{aligned} L_{k+1}(n|^k) &= \tau_0(n)\cdots\tau_k(n)\left[\frac{\tau_k(n)}{2}\right] \\ L_{k+1}(n|^k + 2^k) &= \tau_0(n)\cdots\tau_k(n)\left[\frac{a + \tau_k(n)}{2}\right] \end{aligned} \tag{3.4}$$

by Theorem 3.2 (2).

Consider a free monoid $A^* = \bigcup_{k=0}^{\infty} A^k$ over A, with the concatenation $\oplus$ as a multiplication operation, and the empty string ε as a unit, where $A^k = \{x_1 \oplus x_2 \oplus \cdots \oplus x_k \mid x_i \in A\}$ and $A^0 = \{\varepsilon\}$. In view of (3.3) and (3.4), we define a substitution $\zeta : A^* \to A^*$ as

$$\zeta : x \mapsto \left[\frac{x}{2}\right] \oplus \left[\frac{a+x}{2}\right],$$

which simulates the way to repaint the segments that the refinement (3.3) causes:

$$\tau_k(n) \mapsto \tau_{k+1}(n|^k) \oplus \tau_{k+1}(n|^k + 2^k) = \left[\frac{\tau_k(n)}{2}\right] \oplus \left[\frac{a+\tau_k(n)}{2}\right].$$

Consequently, the refinement of the unit interval

$$[0,1) \rightsquigarrow [\ (0.0)_2\ )_1 \oplus [\ (0.1)_2\ )_1 \rightsquigarrow [\ (0.00)_2\ )_2 \oplus [\ (0.01)_2\ )_2 \oplus [\ (0.10)_2\ )_2 \oplus [\ (0.11)_2\ )_2 \rightsquigarrow \cdots$$

induces an orbit of ζ with initial state b:

$$b \mapsto \zeta(b) = \left[\frac{b}{2}\right] \oplus \left[\frac{a+b}{2}\right] \mapsto \zeta^2(b) = \left[\frac{[b/2]}{2}\right] \oplus \left[\frac{a+[b/2]}{2}\right] \oplus \left[\frac{a+[(a+b)/2]}{2}\right] \oplus \left[\frac{a+[(a+b)/2]}{2}\right] \mapsto \cdots.$$

Summing up, we have the following.

Proposition 3.4. *The k-th refinement* (2.2) *of the unit interval is painted in the pattern $\zeta^k(b)$.*

3.3. **A finite state transducer which represents $ax+b$.** Theorem 3.2 (2) is also expressed in terms of a finite state automaton. Let $T_{a,b} = (A, I, O, \{b\}, \{0\}, \delta)$ be a deterministic finite state transducer, with states A, an input alphabet $I = \{0,1\}$, an output alphabet $O = \{0,1\}$, an initial state b, a final state 0 and a transition function $\delta : A \times I \to A \times O$. According to Theorem 3.2 (2), we define the transition rule as

$$\delta(x,g) = \begin{cases} \left(\left[\dfrac{ag+x}{2}\right], g\right), & \text{if } x \text{ is even,} \\ \left(\left[\dfrac{ag+x}{2}\right], 1-g\right), & \text{if } x \text{ is odd.} \end{cases} \tag{3.5}$$

Corollary 3.5. *Given the binary expressions $(g_p \cdots g_0)_2$ of a natural number n, define a sequence in $A \times O$,*

$$(x_{i+1}, c_{i+1}) = \delta(x_i, g_i),\ \ i = 0, \ldots, q-1,$$

where $x_0 = b$, $q = p + \mathrm{ord}(a) + 1$ and $g_i = 0$ if $i > p$. Then we have

$$f(n) = (c_q c_{q-1} \cdots c_0)_2,$$

and the label of the segment $[\ \beta(n)\ )_k$ is given as

$$L_k(n) = x_0 x_1 \cdots x_k$$

for each $k = 0, 1, \ldots$.

Proof. Let $(h_q \cdots h_0)_2$ be the binary expression of $f(n)$. By definition, we see $x_{i+1} = [(ax_i + g_i)/2]$. Recall that $\tau_0(n) = b$ for any $n \in \mathbf{N}$. It comes from Theorem 3.2 (2) that

$$\tau_i(n) = x_i \ \text{ and } \ h_i = (g_i + x_i)|^1 = \begin{cases} g_i, & \text{if } x_i \text{ is even,} \\ 1 - g_i, & \text{if } x_i \text{ is odd} \end{cases}$$

hold, hence $c_i = h_i$. □

3.4. **Minimality of the dynamical system** $([0,1),F)$**.** As we have seen, for each approximation order k, the approximant F_k exchanges the segments $\left[\,\beta(\nu_j^{(k)})\,\right)_k$'s in the refinement (2.2), and hence induces a permutation π_k over $\{0,1,\ldots,2^k-1\}$, satisfying

$$F_k(\left[\,\beta(\nu_j^{(k)})\,\right)_k) = \left[\,\beta(\nu_{\pi_k(j)}^{(k)})\,\right)_k,$$

or, equivalently,

$$\nu_{\pi_k(j)}^{(k)} = \left(a\nu_j^{(k)}+b\right)\Big|^k.$$

Note that, as the approximation order k increases, we see

$$\nu_{2j}^{(k+1)} = \nu_j^{(k)} \quad\text{and}\quad \nu_{2j+1}^{(k+1)} = \nu_j^{(k)} + 2^k \tag{3.6}$$

by definition of the refinement (2.2). Combining Theorem 3.2 (3) and (3.6), we have

$$\nu_{\pi_{k+1}(2j)}^{(k+1)} = \nu_{\pi_k(j)}^{(k)} = \nu_{2\pi_k(j)}^{(k+1)} \quad\text{and}\quad \nu_{\pi_{k+1}(2j+1)}^{(k+1)} = \nu_{\pi_k(j)}^{(k)} + 2^k = \nu_{2\pi_k(j)+1}^{(k+1)}$$

whenever $\tau_k(\nu_j^{(k)})$ is even, and

$$\nu_{\pi_{k+1}(2j)}^{(k+1)} = \nu_{\pi_k(j)}^{(k)} + 2^k = \nu_{2\pi_k(j)+1}^{(k+1)} \quad\text{and}\quad \nu_{\pi_{k+1}(2j+1)}^{(k+1)} = \nu_{\pi_k(j)}^{(k)} = \nu_{2\pi_k(j)}^{(k+1)}$$

whenever $\tau_k(\nu_j^{(k)})$ is odd. As a result, we obtain relations between the permutation π_k and π_{k+1}.

Lemma 3.6. *If the color $\tau_k(\nu_j^{(k)})$ of the segment $\left[\,\beta(\nu_j^{(k)})\,\right)_k$ is even, we have*

$$\pi_{k+1}(2j) = 2\pi_k(j) \quad\textit{and}\quad \pi_{k+1}(2j+1) = 2\pi_k(j)+1,$$

and if $\tau_k(\nu_j^{(k)})$ is odd,

$$\pi_{k+1}(2j) = 2\pi_k(j)+1 \quad\textit{and}\quad \pi_{k+1}(2j+1) = 2\pi_k(j).$$

Let $\pi_k = \pi_{k,1}\pi_{k,2}\cdots\pi_{k,m}$ be a cycle decomposition, determined uniquely up to the order of $\pi_{k,i}$'s. For each cyclic permutation $\pi_{k,i} = (p_1p_2\cdots p_l)$, consider the maximum subset $Q\subset\{p_1,\ldots,p_l\}$ such that $\tau_k(\nu_q^{(k)})$ is odd if and only if $q\in Q$. For a cycle $\pi_{k,i} = (p_1p_2\cdots p_l)$, let $\pi'_{k+1,i}$ be the permutation defined as Lemma 3.6, that is,

$$\pi'_{k+1,i}(2p_j+\eta) = 2\pi_{k,i}(p_j) + \left(\tau_k(\nu_{p_j}^{(k)})+\eta\right)\Big|^1 = 2p_{j+1} + \left(\tau_k(\nu_{p_j}^{(k)})+\eta\right)\Big|^1, \quad \eta\in\{0,1\},$$

where $l+1$ is understood as 1.

Proposition 3.7. *$\pi'_{k+1,i}$ is a cyclic permutation whenever $\#Q$ is odd, and $\pi'_{k+1,i}$ consists of two cyclic permutations whenever $\#Q$ is even.*

Proof. Put $P_0 = \{2p_1,2p_2,\ldots,2p_l\}$ and $P_1 = \{2p_1+1,2p_2+1,\ldots,2p_l+1\}$. Consider an orbit $x_1 = 2p_1, x_2 = \pi'_{k+1,i}(x_1), x_3 = \pi'_{k+1,i}(x_2),\ldots$, and a map $\phi(x_j) = \eta$ if $x_j\in P_\eta$. Notice that $x_{sl+j}\in\{2p_j,2p_j+1\}$ for $j=1,\ldots,l$ and $s\in\mathbf{N}$ by definition. Then we see $\phi(x_{sl+j+1}) = \phi(x_{sl+j})$ whenever $p_j\notin Q$, and $\phi(x_{sl+j+1}) = 1-\phi(x_{sl+j})$ whenever $p_j\in Q$, hence

$$x_{sl+j+1} = \pi'_{k+1,i}(x_{sl+j}) = 2p_{j+1}+\phi(x_{sl+j+1})$$

for each $j=1,\ldots,l$ and $s\in\mathbf{N}$.

Suppose that $\#Q$ is even. It is seen that $\phi(x_{l+1}) = \phi(x_1) = 0$, hence

$$x_{l+1} = 2p_1+\phi(x_{l+1}) = 2p_1 = x_1$$

namely, the cycle $(x_1x_2,\cdots x_l)$ is a factor of $\pi'_{k+1,i}$. Considering another orbit $y_1 = 2p_1+1, y_2 = \pi'_{k+1,i}(y_1),\ldots$, a similar argument shows the cycle $(y_1y_2\cdots y_l)$ is also a factor of $\pi'_{k+1,i}$. Consequently, we obtain a decomposition $\pi'_{k+1,i} = (x_1x_2\cdots x_l)(y_1y_2\cdots y_l)$.

Suppose that $\#Q$ is odd, then we see $\phi(x_{l+1}) = 1 - \phi(x_1) = 1$, hence

$$x_{l+1} = 2p_1 + \phi(x_{l+1}) = 2p_1 + 1 = y_1,$$

showing that $\pi'_{k+1,i}$ is a cycle of the length $2l$:

$$\pi'_{k+1,i} = (x_1 x_2 \cdots x_l y_1 y_2 \cdots y_l).$$

□

We are to find a condition that the permutation π_k induced by the approximant F_k becomes a cycle in any approximation order k. For a string $\boldsymbol{w} = w_1 \oplus w_2 \oplus \cdots \oplus w_l \in A^*$, we put $S(\boldsymbol{w}) = \sum_{i=1}^{l} w_i$. Note that the parity of $S(\boldsymbol{w})$ equals that of the number of entries w_i's which are odd.

In view of Proposition 3.7, the original unit interval $[0,1)$ must be painted an odd color at least, namely, $b = \tau_0([0,1))$ is odd. Then, $S(\zeta(b))$ also should be odd. We have assumed that a is odd. Putting $a = 2m+1$, we have

$$S(\zeta(b)) = \left[\frac{b}{2}\right] + \left[\frac{a+b}{2}\right] = m + 1 + 2\left[\frac{b}{2}\right]$$

as $[(b+1)/2] = [b/2] + 1$, hence m should be even; $a \equiv 1 \pmod 4$. Conversely, we have the following.

Theorem 3.8. *The permutation π_k associated with the approximant F_k is a cyclic permutation in any approximation order k, hence the dynamical system $([0,1), F)$ is minimal, if and only if $a \equiv 1 \pmod 4$ and b is odd.*

Proof. The necessity for b being odd and $a \equiv 1 \pmod 4$ is shown above. Suppose $a = 4m+1$ and b being odd. It comes from Theorem 3.2 (2) that $S(\zeta(x)) = 2m + 2[x/2]$ if x is even, and $S(\zeta(x)) = 2m+1+2[x/2]$ if x is odd. Then, $S(\zeta(\boldsymbol{w}))$ is odd if and only if $S(\boldsymbol{w})$ is odd for any $\boldsymbol{w} \in A^*$. As a result, we see that $S(\zeta^k(b))$ is odd for any k, hence π_k is a cyclic permutation by Proposition 3.7, which means for any segment $[\,x\,)_k$, the orbit $[\,x\,)_k, F_k([\,x\,)_k), F_k^2([\,x\,)_k), \ldots, F_k^{2^k-1}([\,x\,)_k)$ covers $[0,1)$:

$$\coprod_{i=0}^{2^k-1} F_k^i([\,x\,)_k) = [0,1). \tag{3.7}$$

Take any $x \in [0,1)$. (3.7) shows that, for any $y \in [0,1)$, there exists $0 \le t \le 2^k - 1$ with $y \in F^t([\,x\,)_k)$, while $F^t(x) \in \left[\,F^t(x)\,\right)_k = F^t([\,x\,)_k)$. Thus we have $|y - F^t(x)| < 2^{-k}$. Consequently, the orbit $\{F^s(x) \mid s \in \mathbf{N}\}$ is dense in $[0,1)$ for any $x \in [0,1)$, namely the dynamical system $([0,1), F)$ is minimal. □

This result is contrast to Keane condition[6][16]: no left endpoint of any segment is mapped to another ones, which brings the minimality of the dynamics of interval exchange maps. In our case, the left endpoint $x|_k$ of any segment $[\,x\,)_k$ is always mapped to another ones by F_k.

Example 3.4.1 (The case $3x+1$)**.** This is the original case of Collatz. The substitution ζ and the transducer $T_{3,1}$ are illustrated in FIGURE 1 and 2 respectively. As $3 \not\equiv 1 \pmod 4$, the dynamics $([0,1), F)$ is not minimal. Indeed, TABLE 1 shows that permutations π_k's associated with F_k's are decomposed in two cycles. A typical orbit of F_5, the approximation order $k = 5$, is illustrated in FIGURE 5

k	π_k
0	(0)
1	(0)(1)
2	(0 2)(1 3)
3	(0 4 1 5)(2 7 3 6)
4	(0 8 2 11 1 9 3 10)(4 14 6 12 5 15 7 13)

TABLE 1. Permutations π_k associated with $3x + 1$

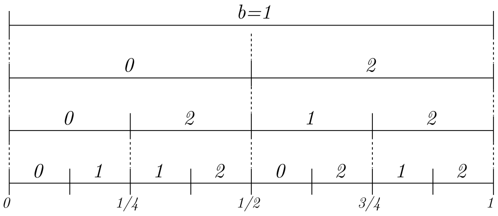

FIGURE 1. The color pattern of refinement induced by substitution ζ

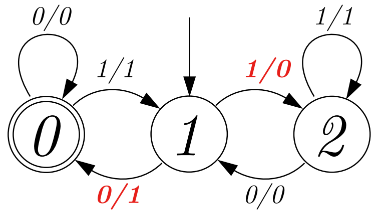

FIGURE 2. Transitive diagram of $T_{3,1}$

Example 3.4.2 (The case $5x + 1$)**.** The substitution ζ and the transducer $T_{5,1}$ are illustrated in FIGURE 3 and 4 respectively. As $5 \equiv 1 \pmod 4$, the conjugacy F of $f(x) = 5x+1$ induces a minimal dynamical system on $[0, 1)$. Actually, TABLE 2 shows that permutations π_k's associated with F_k's are always cyclic. A typical orbit of F_5, the approximation order $k = 5$, is illustrated in FIGURE 6.

k	π_k
0	(0)
1	(0 1)
2	(0 2 1 3)
3	(0 4 3 7 1 5 2 6)
4	(0 8 6 15 3 11 4 13 1 9 7 14 2 10 5 12)

TABLE 2. Permutations π_k associated with $5x + 1$

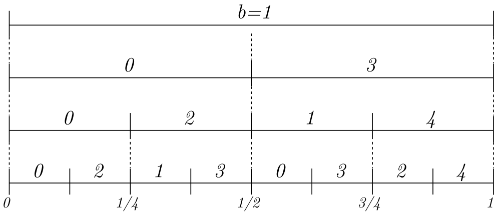

FIGURE 3. The color pattern of refinement induced by substitution ζ

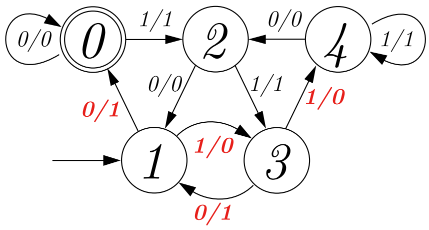

FIGURE 4. Transitive diagram of $T_{5,1}$

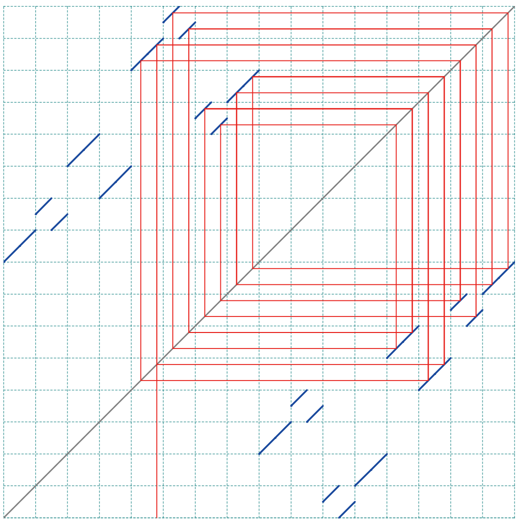

FIGURE 5. Approximant F_5 for $3x + 1$ and a typical orbit

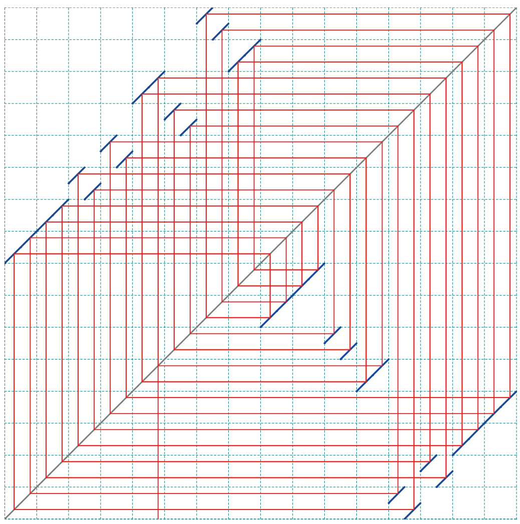

FIGURE 6. Approximant F_5 for $5x + 1$ and a typical orbit

4. **Remarks**

Notice that F preserves the Lebesgue measure μ on $[0,1)$ by Proposition 2.5, therefore, it will be a worthwhile question whether the minimality implies the ergodicity in our case. However, I have no statement about the ergodicity of F at this moment.

In [3] and [4], we have discussed the van der Corput embedding of the Collatz procedure

$$G(x) = \begin{cases} 2x, & x \in [0,1/2), \\ F(x) = \lim_{k\to\infty} \beta(3\beta^{-1}(x|_k)+1), & x \in [1/2,1). \end{cases}$$

Then, to solve the original Collatz conjecture 1.1, we are to prove the statement: 'for each $n \in \mathbf{N}$, there exists a finite number t with $G^t(\beta(n)) = 1/2$.' Note that G does not preserve intervals, and so the Lebesgue measure. The finite bit approximation

$$G_k(x) = \begin{cases} x + x|_k, & \text{for } x \in [0,1/2), \\ F_k(x), & \text{for } x \in [1/2,1) \end{cases}$$

leads us to a mild version of the Collatz conjecture, a finite combinatorial problem proposed in [4], of which research is in progress.

Problem 4.1 ($3x+1$ problem on G_k)**.** *Show that for any $x \in [0,1)$, there exists $t \in \mathbf{N}$ such that*

$$G_k^t(x) \in [\ 0\ )_k \cup [\ \beta(1)\ )_k = \Big[\ 0,\ 1/2^k\ \Big) \cup \Big[\ 1/2,\ 1/2+1/2^k\ \Big).$$

References

[1] K. Dajani and C. Kraaikamp, *Ergodic theory of numbers*, Carus Mathematical Monographs **29**, Mathematical Association of America, 2002.
[2] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd ed. Wiley, 2003.
[3] Y. Hashimoto, *A fractal set associated with the Collatz problem*, Bull. of Aichi Univ. of Education, Natural Science **56**, pp 1-6, 2007.
[4] Y. Hashimoto, *Interval preserving map approximation of* $3x+1$ *problem*, Bull. of Aichi Univ. of Education, Natural Science **61**, pp 5-14, 2012.
[5] J. Hopcroft, R. Motwani and J. Ullman, *Introduction to Automata Theory, Languages and Computations*, 2nd Ed. Addison-Wesley, 2001.
[6] M. Keane, *Interval Exchange Transformations*, Math. Z. **141**, pp 25-31, 1975.
[7] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Dover Publications, 2005.
[8] J. Lagarias, *The Ultimate Challenge: The* $3x+1$ *Problem*, AMS, 2011.
[9] M. Lothaire, *Algebraic combinatorics on words*, Encyclopedia of mathematics and its applications **90**, Cambridge University Press, 2002.
[10] F. Oliveira and F. L. C. da Rocha, *Minimal non-ergodic* C^1*-diffeomorphisms of the circle*, Ergod. Th. & Dynam. Sys. **21** no. 6, pp 1843-1854, 2001.
[11] M. Queffélec, *Substitution Dynamical Systems - Spectral Analysis*, Lect. Notes in Math. **1294**, 2nd Ed., Springer, 2010.
[12] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8**, pp 477-493, 1957.
[13] M. Shirvani and T. D. Rogers, *On Ergodic One-Dimensional Cellular Automata*, Commun. Math. Phys. **136**, pp 599-605, 1991.
[14] G. Wirsching, *The Dynamical System Generated by the* $3x+1$ *Function*, Lect. Notes in Math. **1681**, Springer, 1998.
[15] H. Xie, *Grammatical complexity and one-dimensional dynamical systems*, Directions in chaos **6**, World Scientific, 1997.
[16] J. C. Yoccoz, *Continued Fraction Algorithms for Interval Exchange Maps: an Introduction*, in '*Frontiers in Number Theory, Physics, and Geometry* **I**', pp 403-438, 2005.

(Received September 13, 2012)