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Radial Bargmann representation for the Fock space of type B

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Let $\nu_{\alpha,q}$ be the probability and orthogonality measure for the q-Meixner-Pollaczek orthogonal polynomials, which has appeared in the work of Bożejko, Ejsmont, and Hasebe [J. Funct. Anal. **269**, 1769–1795 (2015)] as the distribution of the (α, q)-Gaussian process (the Gaussian process of type B) over the (α, q)-Fock space (the Fock space of type B). The main purpose of this paper is to find the radial Bargmann representation of $\nu_{\alpha,q}$. Our main results cover not only the representation of q-Gaussian distribution by van Leeuwen and Maassen [J. Math. Phys. **36**, 4743–4756 (1995)] but also of q^2-Gaussian and symmetric free Meixner distributions on $\mathbb{R}$. In addition, non-trivial commutation relations satisfied by (α, q)-operators are presented. © *2016 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4939748]

I. INTRODUCTION

Bożejko-Ejsmont-Hasebe[14] considered a deformation of the (algebraic) full Fock space with two parameters α and q, namely, the (α, q)-Fock space (or the Fock space of type B) $\mathcal{F}_{\alpha,q}(H)$ over a complex Hilbert space H. The deformation with $\alpha = 0$ is equivalent to the q-deformation by Bożejko-Speicher[16] and Bożejko-Kümmerer-Speicher,[15] and the corresponding q-Bargmann-Fock space has been constructed by van Leeuwen-Maassen.[29]

For the construction of $\mathcal{F}_{\alpha,q}(H)$, their starting point is to replace the Coxeter group of type A, that is, symmetric group S_n for the q-Fock space by the Coxeter group of type B, $\Sigma_n := \mathbb{Z}_2^n \rtimes S_n$ in (A1) of the Appendix. This replacement provides us a more general symmetrization operator on $H^{\otimes n}$ than that of Ref. 16 to define the (α, q)-inner product $\langle \cdot, \cdot \rangle_{\alpha,q}$ in (A3). One can define annihilation $B^-_{\alpha,q}(f)$ and creation $B^+_{\alpha,q}(f)$ operators acting on $\mathcal{F}_{\alpha,q}(H)$ and the (α, q)-Gaussian process (the Gaussian process of type B) $G_{\alpha,q}(f)$ for $f \in H$ as the sum of them, $G_{\alpha,q}(f) := B^-_{\alpha,q}(f) + B^+_{\alpha,q}(f)$. It is one of their main interests to find a probability distribution $\mu_{\alpha,q,f}$ on $\mathbb{R}$ of $G_{\alpha,q}(f)$, $\|f\|_H = 1$, with respect to the vacuum state $\langle \Omega, \cdot\, \Omega \rangle_{\alpha,q}$. $\mathcal{F}_{\alpha,q}(H)$ equipped with $\langle \cdot, \cdot \rangle_{\alpha,q}$, $B^-_{\alpha,q}(f)$, and $B^+_{\alpha,q}(f)$ is a typical example of interacting Fock spaces in the sense of Accardi-Bożejko.[1] It suggests that the theory of orthogonal polynomials plays intrinsic roles in all previous works mentioned above. In fact, the measure $\mu_{\alpha,q,f}$ given in Ref. 14 [Theorem 3.3] is derived essentially from the orthogonality measure $\nu_{\alpha,q}$ associated with the q-Meixner-Pollaczek orthogonal polynomials $\{P_n^{(\alpha,q)}(x)\}$ for $\alpha, q \in (-1, 1)$ given by the recurrence relation,

$$\begin{cases} P_0^{(\alpha,q)}(x) = 1,\ P_1^{(\alpha,q)}(x) = x, \\ xP_n^{(\alpha,q)}(x) = P_{n+1}^{(\alpha,q)}(x) + (1 + \alpha q^{n-1})[n]_q P_{n-1}^{(\alpha,q)}(x), \quad n \geq 1, \end{cases}$$

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where $[n]_q = 1 + q + \cdots + q^{n-1}$ is the q number. However, the Bargmann representation (measure on $\mathbb{C}$) of $\nu_{\alpha,q}$ has not been obtained yet except the case of $\alpha = 0$ for $0 \le q < 1$,[29] for $q = 1$,[10,9] for $q = 0$[12] and t-deformed cases of these,[8,24] and for $q > 1$.[23]

Therefore, the main purpose of this paper is to find the radial Bargmann representation of the probability measure $\nu_{\alpha,q}$ on $\mathbb{R}$. Our results cover the radial Bargmann representations of q-Gaussian, symmetric free Meixner (Kesten), and q^2-Gaussian distributions on $\mathbb{R}$.

The organization of this paper will be as follows. In Section II, we shall explain how the (α, q)-Fock space is related to the notion of one-mode interacting Fock spaces and Bargmann representation. In Section III, the radial Bargmann representation of $\nu_{\alpha,q}$ is constructed explicitly in Theorem 3.11. In Section IV, commutation relations satisfied by one-mode (α, q)-annihilation and creation operators will be treated. In the Appendix, we shall give a minimum reference on the Coxeter group of type B extracted from Ref. 14.

II. KEY IDEAS AND OUR PURPOSE

Let us point out some of the keys to calculate the distribution of $G_{\alpha,q}(f)$ in Ref. 14. It is shown that a linear map, Φ: Span$\{f^{\otimes n} \mid f \in H, n \ge 0\} \to L^2(\mathbb{R}, \mu_{\alpha,q,f})$ given by $\Phi(f^{\otimes n}) = P_n^{(\alpha\langle f,\overline{f}\rangle_H,q)}(x)$, is an isometry and a relation under $\|f\|_H = 1$,

$$\begin{aligned} G_{\alpha,q}(f) f^{\otimes n} &= (B^+_{\alpha,q}(f) + B^-_{\alpha,q}(f))(f^{\otimes n}) \\ &= f^{\otimes(n+1)} + (1 + \alpha\langle f, \overline{f}\rangle_H) q^{n-1}[n]_q f^{\otimes(n-1)}, \end{aligned}$$

is satisfied where $\overline{f}$ denotes a self-adjoint involution of $f \in H$ in (A2). This corresponds to the three terms recursion relation satisfied by $P_n^{(\alpha\langle f,\overline{f}\rangle_H,q)}(x)$ through Φ. Then, it is proved that $\mu_{\alpha,q,f} = \nu_{\alpha\langle f,\overline{f}\rangle_H,q}$ (see $\nu_{\alpha,q}$ in (3.3)) in the sense of

$$\langle \Omega, G_{\alpha,q}(f)^n \Omega\rangle_{\alpha,q} = \int x^n \mu_{\alpha,q,f}(dx), \tag{2.1}$$

where Ω denotes the vacuum vector. Therefore, in order to get the Bargmann representation of $\nu_{\alpha\langle f,\overline{f}\rangle_H,q}$, it is enough to consider that of $\nu_{\alpha,q}$ in the sense of Definition 2.2 given later.

Since the structure mentioned above can be reduced to the one-mode analogue of (α, q)-Fock spaces, let us recall fundamental relationships between one-mode interacting Bargmann-Fock spaces and the theory of orthogonal polynomials of one variable.

Definition 2.1. Let $\{\omega_n\}_{n=0}^\infty$ with $\omega_0 := 1$ be an infinite sequence of positive real numbers and $\{\alpha_n\}_{n=0}^\infty$ be of real numbers. A one-mode interacting Bargmann-Fock space $\mathcal{B}$ is defined as $\bigoplus_{n=0}^\infty \mathbb{C}\Phi_n$ equipped with $\Phi_n := z^n/[\omega_n]!$, $[\omega_n]! := \prod_{k=0}^n \omega_k$, the inner product $\langle \Phi_m, \Phi_n\rangle_{\mathcal{B}} = \delta_{m,n}$ for all $m, n \in \mathbb{N} \cup \{0\}$, operators of creation a^+, annihilation a^-, and conservation a° defined by

$$\begin{cases} a^+\Phi_n := \sqrt{\omega_{n+1}}\Phi_{n+1}, & n \ge 0, \\ a^-\Phi_0 = 0,\ a^-\Phi_n := \sqrt{\omega_n}\Phi_{n-1}, & n \ge 1, \\ a^\circ\Phi_n := \alpha_n\Phi_n, & n \ge 0. \end{cases} \tag{2.2}$$

Let $(\{\omega_n\}_{n=0}^\infty, \{\alpha_n\}_{n=0}^\infty)$ be a pair of sequences in Definition 2.1 and define a sequence of monic polynomials $\{P_n(x)\}$ recurrently by

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x - \alpha_0, \\ xP_n(x) = P_{n+1}(x) + \omega_n P_{n-1} + \alpha_n P_n(x), \quad n \ge 1. \end{cases} \tag{2.3}$$

Then, there exists a probability measure μ on $\mathbb{R}$ with finite moments of all orders such that $\{P_n(x)\}$ is the orthogonal polynomials with $\langle P_m(x), P_n(x)\rangle_{L^2(\mathbb{R},\mu)} = \delta_{m,n}[\omega_n]!$ for all $m, n \in \mathbb{N} \cup \{0\}$. (See Refs. 19 and 21, for example.)

It is easy to see that a linear map

$$U : \mathcal{B} = \bigoplus_{n=0}^{\infty} \mathbb{C}\Phi_n \to L^2(\mathbb{R}, \mu)$$

defined by $U\left(\sqrt{[\omega_n]!}\Phi_n\right) = P_n(x)$ is an isometry and in addition, $a^+ + a^- + a^\circ = U^* X U$ is satisfied due to (2.2) and (2.3), where X represents the multiplication operator by x in $L^2(\mathbb{R}, \mu)$. This intertwining relation provides a notion of the quantum decomposition of a classical random variable X and

$$\langle \Phi_0, (a^+ + a^- + a^\circ)^n \Phi_0 \rangle_{\mathcal{B}} = \int x^n \mu(dx). \tag{2.4}$$

Therefore, if $\omega_n = (1 + \alpha q^{n-1})[n]_q$, $\alpha_n = 0$, the equality in (2.4) is a one-mode analogue of (2.1).

Now, it is interesting to consider the following moment problem to realize the inner product by the integral.

Problem 1. For a given $\{\omega_n\}$ of μ, find a probability measure γ_μ satisfying the equality

$$\int_{\mathbb{C}} \overline{z}^m z^n \gamma_\mu(d^2 z) = \delta_{m,n}[\omega_n]! \tag{2.5}$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

Definition 2.2. A measure γ_μ satisfying equality (2.5) is called a Bargmann representation (measure on $\mathbb{C}$) of a measure μ on $\mathbb{R}$.

It was proved in Ref. 28 (see also Refs. 8 and 24) that if a measure μ admits any Bargmann representation, then it also admits a radial (rotation invariant) Bargmann representation

$$\gamma_\mu(d^2 z) = \frac{1}{2\pi} \lambda_{[0,2\pi)}(d\theta)\rho_\mu(dr),\ z = re^{i\theta},\ r \geq 0,\ \theta \in [0, 2\pi),$$

where $\lambda_{[0,2\pi)}$ is the Lebesgue measure on $[0, 2\pi)$. It says that the angular part takes care of orthogonality of (2.5). Therefore, Problem 1 can be transformed into Problem 2.

Problem 2. Find a positive radial measure ρ_μ satisfying

$$\int_0^\infty r^{2n} \rho_\mu(dr) = [\omega_n]!$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

Main purpose. We shall consider Problem 2 associated with $\omega_n = (1 + \alpha q^{n-1})[n]_q$, $\alpha_n = 0$ of $\nu_{\alpha.q}$ in Section III. Furthermore, commutation relations satisfied by a^+, a^- acting on $\mathcal{B}$ associated with $\omega_n = (1 + \alpha q^{n-1})[n]_q$ will be presented in Section IV.

Remark 2.3. (1) One can notice that γ_μ and ρ_μ are determined only by $[\omega_n]!$. Therefore, it is enough in general for the Bargmann representation in the above sense to consider the symmetric measure μ with $\alpha_n = 0$ for all n, which implies that a° is a zero operator.

(2) If μ is symmetric, then $\alpha_n = 0$ for all n is implied. The converse statement is true if μ is determined by its moments.

III. (α, q)-BARGMANN REPRESENTATION

A. q-Meixner-Pollaczek polynomials

Let us recall standard notations from q-calculus, which can be found in Refs. 20 and 22, for example. The q-shifted factorials are defined by

$$(a; q)_0 := 1, \quad (a; q)_k := \prod_{\ell=1}^{k} (1 - aq^{\ell-1}),\ k = 1, 2, \ldots, \infty,$$

and the product of q-shifted factorials is defined by

$$(a_1, a_2; q)_k := (a_1; q)_k (a_2; q)_k, \quad k = 1, 2, \ldots, \infty.$$

Remark 3.1. The q-shifted factorials are a natural extension of the Pochhammer symbol $(\cdot)_n$ because one can see that $\lim_{q\to 1}[k]_q = k$ implies

$$\lim_{q\to 1} \frac{(q^k; q)_n}{(1-q)^n} = (k)_n, \tag{3.1}$$

where $(k)_0 := 1$, $(k)_n := k(k+1)\cdots(k+n-1)$, $n \geq 1$.

As we have mentioned, $\{P_n^{(\alpha,q)}(x)\}$ for $\alpha, q \in (-1,1)$ is the q-Meixner-Pollaczek polynomials satisfying the recurrence relation,

$$\begin{cases} P_0^{(\alpha,q)}(x) = 1, \; P_1^{(\alpha,q)}(x) = x, \\ xP_n^{(\alpha,q)}(x) = P_{n+1}^{(\alpha,q)}(x) + (1+\alpha q^{n-1})[n]_q P_{n-1}^{(\alpha,q)}(x), \quad n \geq 1. \end{cases} \tag{3.2}$$

It is known in Ref. 22 [14.9.2] and Ref. 14 [page 1781] that the orthogonality measure $\nu_{\alpha.q}$ for such polynomials has the density of the form

$$\frac{(q,\gamma^2;q)_\infty}{2\pi} \sqrt{\frac{1-q}{4-(1-q)x^2}} \left(\frac{g(x,1;q)g(x,-1;q)g(x,\sqrt{q};q)g(x,-\sqrt{q};q)}{g(x,i\gamma;q)g(x,-i\gamma;q)} \right), \tag{3.3}$$

supported on the interval $(-2/\sqrt{1-q}, 2/\sqrt{1-q})$, where

$$g(x,b;q) = \prod_{k=0}^{\infty} (1 - 4bx(1-q)^{-1/2} q^k + b^2 q^{2k})$$

and

$$\gamma = \begin{cases} \sqrt{-\alpha}, & \alpha < 0, \\ i\sqrt{\alpha}, & \alpha \geq 0. \end{cases}$$

Example 3.2. (1) If $\alpha = 0$, then q-Meixner-Pollaczek polynomials get back to the q-Hermite polynomials $H_n^{(q)}(x)$ whose orthogonality measure is the standard q-Gaussian measure on $(-2/\sqrt{1-q}, 2/\sqrt{1-q})$ given by

$$\nu_q(dx) := \frac{\sqrt{1-q}}{\pi} \sin\theta \prod_{n=1}^{\infty} (1-q^n)|1-q^n e^{2i\theta}|^2 dx,$$

where $x\sqrt{1-q} = 2\cos\theta$, $\theta \in [0,\pi]$. Furthermore, one can get the standard Gaussian law as $q \to 1$, the Bernoulli law as $q \to -1$, and the standard Wigner's semi-circle law if $q = 0$. See Refs. 15 and 16.

(2) The measure $\nu_{\alpha,0}$ is the symmetric free Meixner law.[2,13,26]

(3) The measure $\nu_{q,q}$ is the q^2-Gaussian law scaled by $\sqrt{1+q}$.

(4) If $\alpha = -q^{2\beta}$ as suggested in Remark 3.1, then the measure $\nu_{-q^{2\beta},q}$ under a certain scaling converges to the classical symmetric Meixner law as $q \uparrow 1$,

$$\frac{2^{2\beta}}{2\pi\Gamma(2\beta)} |\Gamma(\beta+ix)|^2 dx, \quad x \in \mathbb{R}. \tag{3.4}$$

See also Ref. 22 [14.9.15].

B. Problem

For $\alpha, q \in (-1,1)$, we would like to know when there exists a radial measure $\rho_{\nu_{\alpha,q}}$ satisfying

$$\int_0^\infty r^{2k} \rho_{\nu_{\alpha,q}}(dr) = (-\alpha; q)_k [k]_q!, \qquad k \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

Here, $[k]_q!$ denotes the q-factorials defined by

$$[0]_q! := 1, \quad [k]_q! := \prod_{\ell=1}^{k} [\ell]_q = \frac{(q;q)_k}{(1-q)^k}, \; k \geq 1.$$

It is easy to get the inequality for $\alpha, q \in (-1,1)$,

$$\left|(-\alpha;q)_k [k]_q!\right| \leq \left(\frac{4}{1-|q|}\right)^k, \; k \in \mathbb{N} \cup \{0\}. \tag{3.6}$$

Due to Carleman criterion for the moment problem, this inequality implies that a radial measure $\rho_{\nu_{\alpha,q}}$ is determined uniquely by the sequence $\{(-\alpha;q)_k [k]_q!\}$.

We shall follow the procedure below to construct $\rho_{\nu_{\alpha,q}}$ in (3.5).

(1) Recall the radial part of the q-Gaussian measure on $\mathbb{C}$ (q-Bargmann measure), $\rho_{\nu_q} = \rho_{\nu_{0,q}}$, obtained in Ref. 29,

$$\int_0^\infty r^{2k} \, \rho_{\nu_q}(dr) = [k]_q!. \tag{3.7}$$

(2) Find a radial (possibly signed) measure $\rho_{\alpha,q}$ having the moment $(-\alpha;q)_k$.

(3) Compute the multiplicative (Mellin) convolution $\rho_{\nu_q} \circledast \rho_{\alpha,q}$ to get $\rho_{\nu_{\alpha,q}}$.

Remark 3.3. It is known[29] that a radial measure ρ_{ν_q} in (3.7) does not exist for $q < 0$. However, one can see that the positivity assumption on q can be relaxed for $\rho_{\nu_{\alpha,q}}$ if $\alpha = q$. It will be discussed right after the proof of Proposition 3.6 and in Proposition 3.7.

C. Construction of (α, q)-radial measures

Lemma 3.4. Suppose that $\alpha \in (-1,1)$ and $q \in [0,1)$. Let

$$\rho_{\alpha,q} := (-\alpha;q)_\infty \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{(q;q)_n} \delta_{q^{n/2}},$$

which is a signed measure. Then, we have

$$\int_0^\infty r^{2k} \, \rho_{\alpha,q}(dr) = (-\alpha;q)_k, \qquad k \in \mathbb{N} \cup \{0\}.$$

In particular, if taking $\alpha = -q$, then one can see $\rho_{\nu_q} = D_{(1-q)^{-1/2}}(\rho_{-q,q})$, namely,

$$\int_0^\infty r^{2k} \, D_{(1-q)^{-1/2}}(\rho_{-q,q})(dr) = \frac{(q;q)_k}{(1-q)^k} = [k]_q!,$$

where $D_t(\lambda)$ is the push-forward of λ by the map $x \mapsto tx$ for a measure λ on $\mathbb{R}$.

Proof. First, we have

$$\int_0^\infty r^{2k} \, \rho_{\alpha,q}(dr) = (-\alpha;q)_\infty \sum_{n=0}^{\infty} \frac{(-\alpha q^k)^n}{(q;q)_n}.$$

Since Euler's formula (see Ref. 20 [1.3.15]),

$$\frac{1}{(a;q)_\infty} = \sum_{n=0}^{\infty} \frac{a^n}{(q;q)_n}, \tag{3.8}$$

is known, we replace a by $-\alpha q^k$ in (3.8) to obtain

$$\begin{aligned} \int_0^\infty r^{2k} \, \rho_{\alpha,q}(dr) &= \frac{(-\alpha;q)_\infty}{(-\alpha q^k;q)_\infty} \\ &= (-\alpha;q)_k. \end{aligned}$$

The proof is complete. □

Remark 3.5. (1) The last equality in proof is due to the formula

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

See Ref. 20 [1.2.30], for example.

(2) Euler's formula is considered as the q-analogue of exponential function e^a due to

$$\lim_{q\to 1} \frac{1}{((1-q)a;q)_n} = e^a.$$

Let

$$\begin{bmatrix} n \\ \ell \end{bmatrix}_q := \frac{[n]_q!}{[\ell]_q![n-\ell]_q!} = \frac{(q;q)_n}{(q;q)_\ell (q;q)_{n-\ell}}$$

be the q-binomial coefficients and $h_n(z \mid q)$ be the Rogers-Szegö polynomials[20,27] defined by

$$h_n(z \mid q) = \sum_{\ell=0}^{n} \begin{bmatrix} n \\ \ell \end{bmatrix}_q z^\ell.$$

Proposition 3.6. Suppose that $\alpha \in (-1,1)$ and $q \in [0,1)$. Let

$$\rho_{\nu_{\alpha,q}} := \begin{cases} (-\alpha,q;q)_\infty \displaystyle\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} \mid q)\delta_{(1-q)^{-1/2}q^{n/2}}, & q > 0, \\ -\alpha\delta_0 + (1+\alpha)\delta_1, & q = 0, \end{cases} \tag{3.9}$$

which is a signed measure in general. Then, we have

$$\int_0^\infty r^{2k}\, \rho_{\nu_{\alpha,q}}(dr) = \frac{(-\alpha,q;q)_k}{(1-q)^k} = (-\alpha;q)_k [k]_q!, \qquad k \in \mathbb{N} \cup \{0\}. \tag{3.10}$$

Proof. First of all, it is easy to show (3.10) for the case $q = 0$. Therefore, let us assume $q > 0$. One can compute the multiplicative (Mellin) convolution $\circledast$ to get $\rho_{\nu_{\alpha,q}}$ as follows:

$$\begin{aligned} \rho_{\nu_{\alpha,q}} &= \rho_{\alpha,q} \circledast D_{(1-q)^{-1/2}}(\rho_{-q,q}) \\ &= (-\alpha,q;q)_\infty \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} \frac{(-\alpha)^\ell q^{n-\ell}}{(q;q)_\ell (q;q)_{n-\ell}} \right) \delta_{(1-q)^{-1/2}q^{n/2}} \\ &= (-\alpha,q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} \mid q)\delta_{(1-q)^{-1/2}q^{n/2}}. \end{aligned}$$

On the other hand, by Lemma 3.4, we have

$$\int_0^\infty r^{2k}\, D_{(1-q)^{-1/2}}(\rho_{-q,q})(dr) = \frac{(q;q)_k}{(1-q)^k} = [k]_q!.$$

Therefore, one can get

$$\begin{aligned} \int_0^\infty r^{2k}\, \rho_{\nu_{\alpha,q}}(dr) &= \int_0^\infty r^{2k}\, \rho_{\alpha,q}(dr) \int_0^\infty r^{2k}\, D_{(1-q)^{-1/2}}(\rho_{-q,q})(dr) \\ &= (-\alpha;q)_k [k]_q!, \qquad k \in \mathbb{N} \cup \{0\}. \end{aligned}$$

□

In Proposition 3.6, we have obtained $\rho_{\nu_{\alpha,q}}$ for $\alpha \in (-1,1)$ and $q \in (0,1)$. Due to the term

$$\delta_{(1-q)^{-1/2}q^{n/2}} \text{ in } \rho_{\nu_{\alpha,q}},$$

it seems impossible for $q \in (-1,0)$ to define $\rho_{\nu_{\alpha,q}}$. However, if $-1 < \alpha = q < 0$, then $\nu_{q,q}$ coincides with a scaled q^2-Gaussian measure, and hence, the Bargmann measure exists.

Proposition 3.7. Suppose $-1 < \alpha = q < 0$. *We define*

$$\begin{aligned}\rho_{\nu_{q,q}} &:= D_{(1+q)^{1/2}}(\rho_{\nu_{q^2}}) \\ &= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n}\delta_{(1-q)^{-1/2}(-q)^n}.\end{aligned} \tag{3.11}$$

Then, one can see

$$\int_0^\infty r^{2k}\,\rho_{\nu_{q,q}}(dr) = (1+q)^k[k]_{q^2}! = (-q;q)_k[k]_q!.$$

Proof. One can see by direct computations

$$\begin{aligned}(-q;q)_k[k]_q! &= \left\{\prod_{\ell=1}^{k}(1-(-q)q^{\ell-1})\right\}\left\{\prod_{\ell=1}^{k}\frac{1-q^\ell}{1-q}\right\} \\ &= (1+q)^k\prod_{\ell=1}^{k}\frac{1-q^{2\ell}}{1-q^2} \\ &= (1+q)^k[k]_{q^2}!.\end{aligned}$$

Thus, $\rho_{\nu_{q,q}}$ can be defined as the radial measure for q^2-Gaussian measure on $\mathbb{C}$ scaled by $(1+q)^{1/2}$, namely, $\rho_{\nu_{q,q}} = D_{(1+q)^{1/2}}(\rho_{\nu_{q^2}})$. □

Remark 3.8. If we use the fact that $h_n(-1 \mid q) = 0$ for odd $n \geq 1$ (see proof of Lemma 3.9), we can extend definition (3.9) to the case $-1 < \alpha = q < 0$. This will give an alternative way to define $\rho_{\nu_{q,q}}$ for $-1 < q < 0$, but both definitions give the same measure.

We need some properties of the Rogers-Szegö polynomials to know when the measure $\rho_{\nu_{\alpha,q}}$ becomes positive.

Lemma 3.9 (Ref. 25). Suppose that $q \in (-1,1)$.

(1) *If* $n \geq 0$ *is odd, then* $h_n(x \mid q) \geq 0$ *if and only if* $x \geq -1$. *Moreover, the point* $x = -1$ *is the unique zero of* $h_n(x \mid q)$ *on* $\mathbb{R}$.
(2) *If* $n \geq 0$ *is even, then* $h_n(x \mid q) > 0$ *for all* $x \in \mathbb{R}$.

Proof. It is known that all the zeros of $h_n(z \mid q)$ lie on the unit circle $|z| = 1$. See Ref. 25 or Ref. 27 [Theorem 1.6.11]. Note that the result was obtained for $q \in [0,1)$, but the proof can be extended to $q \in (-1,1)$ without any modifications.

By definition, one can see

$$\begin{bmatrix} n \\ \ell \end{bmatrix}_q = \frac{(1-q^{n-\ell+1})(1-q^{n-\ell+2})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^\ell)} > 0,$$

and hence, $h_n(1 \mid q) > 0$ for all $n \geq 0$. Thus, $h_n(x \mid q) \neq 0$ for $x \in \mathbb{R} \setminus \{-1\}$. It then suffices to show that $h_n(-1 \mid q) > 0$ for all even $n \geq 0$ and $h_n(-1 \mid q) = 0$ for all odd $n \geq 1$. The recurrence relation for the Rogers-Szegö polynomials is known to be

$$h_{n+1}(z \mid q) = (z+1)h_n(z \mid q) - (1-q^n)zh_{n-1}(z \mid q), \qquad n \geq 1. \tag{3.12}$$

See Ref. 27 [1.6.76] (note that formula (1.6.76) has an error of a sign). It is easy to see that $h_0(-1 \mid q) = 1 > 0, h_1(-1 \mid q) = 0$, so by induction and (3.12), one can show $h_n(-1 \mid q) > 0$ for all even $n \geq 0$ and $h_n(-1 \mid q) = 0$ for all odd $n \geq 1$. □

We need the following lemma in proof of Theorem 3.11 for the non-existence part of a radial Bargmann measure.

Lemma 3.10. Let μ *be a signed measure on* $\mathbb{R}$ *with compact support and let* ν *be a non-negative measure on* $\mathbb{R}$. *If* μ *and* ν *have the same finite moments of all orders, then* $\mu = \nu$.

Proof. We denote by m_n the moments of μ (and ν by assumption). Since μ is compactly supported, say on $[-R,R]$,

$$|m_n| = \left|\int_{[-R,R]} x^n\,\mu(dx)\right| \le \|\mu\| R^n, \qquad n \in \mathbb{N}\cup\{0\},$$

where $\|\mu\|$ denotes the total variation of μ. Therefore, ν is also supported on $[-R,R]$. By Weierstrass' approximation, we have

$$\int_{[-R,R]} f(x)\,\mu(dx) = \int_{[-R,R]} f(x)\,\nu(dx) \tag{3.13}$$

for all $f \in C([-R,R])$. This implies that $\mu = \nu$ since, if we use the Hahn decomposition $\mu = \mu_+ - \mu_-$, then (3.13) implies

$$\int_{[-R,R]} f(x)\,\mu_+(dx) = \int_{[-R,R]} f(x)\,(\nu+\mu_-)(dx),$$

and hence, $\mu_+ = \nu + \mu_-$ as non-negative finite measures. □

In summary, the complete answer to the unique existence of a radial Bargmann representation of $\nu_{\alpha,q}$ is stated as follows.

Theorem 3.11. *Suppose that $\alpha, q \in (-1,1)$. The probability measure $\nu_{\alpha,q}$ has a radial Bargmann representation if and only if either (i) $q \ge 0$ and $\alpha \le q$ or (ii) $\alpha = q \ne 0$.*

In fact, the radial measure is given uniquely by

$$\rho_{\nu_{\alpha,q}} = \begin{cases} -\alpha\delta_0 + (1+\alpha)\delta_1 & (\alpha \le q = 0),\\ (-\alpha,q;q)_\infty \displaystyle\sum_{n=0}^\infty \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} \mid q)\delta_{(1-q)^{-1/2}q^{n/2}} & (q>0,\ \alpha<q),\\ (q^2;q^2)_\infty \displaystyle\sum_{n=0}^\infty \frac{q^{2n}}{(q^2;q^2)_n}\delta_{(1-q)^{-1/2}|q|^n} & (\alpha = q \ne 0). \end{cases}$$

Proof. 1. Existence and uniqueness. If $q \in [0,1)$ and $\alpha \le q$, then by Proposition 3.6 and Lemma 3.9, the signed measure $\rho_{\nu_{\alpha,q}}$ is in fact a non-negative measure and becomes the radial part of a Bargmann measure. The case $\alpha = q < 0$ was discussed in Proposition 3.7. Due to Carleman criterion for the moment problem, the inequality given in (3.6) guarantees the uniqueness of $\rho_{\nu_{\alpha,q}}$ for these cases.

2. Non-existence. (1) If $q \in (0,1)$ and $\alpha > q$, then $\rho_{\nu_{\alpha,q}}$ is not a non-negative measure and is really a signed measure since $h_n(-\alpha/q \mid q) < 0$ for odd integers $n \ge 0$ and $q > 0$ from Lemma 3.9. By Lemma 3.10, if a radial Bargmann measure exists, then it must be equal to the signed measure $\nu_{\alpha,q}$. This is a contradiction. Thus, a radial Bargmann measure does not exist.

(2) If $q = 0$ and $\alpha > q = 0$, then by (3.9), $\nu_{\alpha,0}$ is really a signed measure, and hence, by the same argument as above, a radial Bargmann measure does not exist.

(3) Let

$$\beta_k(\alpha,q) := (-\alpha;q)_k [k]_q!, \qquad k \ge 0, \alpha, q \in (-1,1).$$

Given $q < 0$ and $\alpha \ne q$, suppose that there exists a radial part of a Bargmann measure, ρ. Let ρ^2 be the push-forward of ρ by the map $x \mapsto x^2$. Then,

$$\beta_k(\alpha,q) = \int_0^\infty x^k\,\rho^2(dx) = \int_0^\infty x^{2k}\,\rho(dx). \tag{3.14}$$

By the way, by Proposition 3.6, it holds that $\beta_k(\alpha,q') = \int_0^\infty x^{2k}\,\rho_{\nu_{\alpha,q'}}(dx)$ for any $q' > 0$, that is,

$$\beta_k(\alpha,q') = (-\alpha,q';q')_\infty \sum_{n=0}^\infty \frac{(q')^n}{(q';q')_n} h_n(-\alpha(q')^{-1} \mid q') \frac{(q')^{kn}}{(1-q')^k}, \qquad q' > 0, \tag{3.15}$$

which is true even for $q' = q$ by analytic continuation.

Now let us consider the signed measure

$$\mu := (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} \mid q)\delta_{(1-q)^{-1}q^n}, \qquad \alpha \neq q < 0,$$

supported on the points $\frac{q^n}{1-q}$ for $n = 0, 1, 2, 3, \ldots$. Then, by (3.15) for $q' = q$ and by (3.14),

$$\int_{\mathbb{R}} x^k \, \mu(dx) = \beta_k(\alpha, q) = \int_0^\infty x^k \, \rho^2(dx), \qquad k \in \mathbb{N} \cup \{0\}.$$

By Lemma 3.10, the signed measure μ and the probability measure ρ^2 should be equal. However, the support of μ is not contained in $[0, \infty)$, and hence, μ cannot be equal to ρ^2. This is a contradiction. □

Example 3.12. (1) The radial measure $\rho_{\nu_{0,q}}$ for $q \in [0, 1)$ is of the q-Bargmann.[29]

(2) The radial measure $\rho_{\nu_{q,q}}$ for $q \in (-1, 1)$ is of the q^2-Bargmann.

(3) $\lim_{q\uparrow 1} \rho_{\nu_{\alpha,q}}$ is of the classical Bargmann.[10,9]

(4) Consider $\alpha = -q^{2\beta}, \beta > 0$. This choice of α is suggested by (3.1) in Remark 3.1. In fact, one can see

$$\lim_{q\uparrow 1} \frac{(1 - q^{2\beta+n-1})[n]_q}{4(1-q)} = \frac{1}{4}(n + 2\beta - 1)n.$$

This limit sequence is the Jacobi sequence of the symmetric Meixner distribution in (3.4), so that $\rho_{\nu_{-q^{2\beta},q}}$ under suitable scaling converges weakly as $q \uparrow 1$ to the radial measure with the density,

$$\frac{2\pi r}{\Gamma(2\beta)} \int_0^\infty h(r, t/4) e^{-t} t^{2\beta-1} dt,$$

where

$$h(r,t) = \frac{1}{\pi t} \exp\left(-\frac{r^2}{t}\right), \; r \in \mathbb{R}, \; t > 0.$$

This is an integral representation of the radial density for the Bessel kernel measure, which can be also represented by the modified Bessel function.[5,7]

(5) $\rho_{\nu_{\alpha,0}}$ for $\alpha \in (-1, 0]$ is the radial measure for the symmetric free Meixner distribution. See Remark 3.13.

Remark 3.13. Let μ_t be a t-deformed probability measure of a probability measure μ on $\mathbb{R}$ defined through the Cauchy transform G_μ of μ,

$$\frac{1}{G_{\mu_t}(z)} := \frac{t}{G_\mu(z)} + (1-t)z, \quad t \geq 0,$$

examined by Bożejko-Wysoczański.[17,18] Krystek-Wojakowski[24] discussed the radial Bargmann representation of a t-deformed probability measure μ_t, t-Bargmann representation for short, and obtained necessary and sufficient condition for the admissibility of the representation. The t-Bargmann representation of the Kesten measure κ_t has the form

$$\rho_{\kappa_t} = \left(1 - \frac{1}{t}\right)\delta_0 + \frac{1}{t}\delta_{\sqrt{t}}, \quad t \geq 1.$$

In Ref. 8, the t-Bargmann representation of a symmetric free Meixner law $\varphi_{s,t}$, with two positive parameters s, t is treated and is admitted if and only if $t \geq 1$. In fact, one can see $\rho_{\varphi_{s,t}} = D_s(\rho_{\kappa_t})$ and hence,

$$\rho_{\nu_{(1-t)/t,0}} = \rho_{\varphi_{1/\sqrt{t},t}} = D_{1/\sqrt{t}}(\rho_{\kappa_t}), \quad t \geq 1.$$

Therefore, case (5) in Example 3.12 can be viewed as a t-Bargmann representation, too.

Furthermore, let us state the t-deformed version of Theorem 3.11 for $q \neq 0$ without proof.

Proposition 3.14. The t-deformed version of $\rho_{\nu_{\alpha,q}}$ *for* $q \neq 0$ *is given by*

$$\left(1 - \frac{1}{t}\right)\delta_0 + \frac{1}{t}\rho_{\nu_{\alpha,q}}, \quad t \geq 1.$$

Remark 3.15. The t-Bargmann representation of ν_q is treated in Ref. 24 for $q = 1$ and Ref. 8 for $0 \leq q < 1$.

Before closing this section, let us give a short remark about relations with the free infinite divisibility. Many of the particular examples have so far suggested that the free infinite divisibility of a probability measure implies the existence of a radial Bargmann representation. The converse is not true in general because the Askey-Wimp-Kerov distribution $\mu_{9/10}$ for instance, discussed in Ref. 11, is not freely infinitely divisible, but it has a Bargmann representation with a gamma distribution as its radial measure. However, not many counterexamples have been found.

Therefore, we conjecture that the free infinite divisibility of our (α, q)-Gaussian distribution is equivalent to the existence of its radial Bargmann measure.

Conjecture. Suppose that $\alpha, q \in (-1, 1)$. The probability measure $\nu_{\alpha,q}$ is freely infinitely divisible if and only if $\alpha = q$ or $\alpha < q \geq 0$.

This conjecture is guaranteed to be true in the restricted subfamilies $\{\nu_{\alpha,0} \mid \alpha \in (-1,1)\}$ (Ref. 26 [Theorem 3.2]), $\{\nu_{0,q} \mid -1 < q < 1\}$ (Refs. 3 and 4 [Example 3.11] for the free infinite divisibility), and $\{\nu_{q,q} \mid q \in (-1,1)\}$ (all measures in this family are freely infinitely divisible since they are q^2-Gaussians).

IV. COMMUTATION RELATIONS AMONG ONE-MODE (α, q)-OPERATORS

Definition 4.1. Suppose that $\alpha, q \in (-1, 1)$ and f is analytic on $\mathbb{C}$.

(1) Let Z be the multiplication operator defined by

$$(Zf)(z) := zf(z).$$

(2) Let D_q be the Jackson derivative given by

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

(3) The α-deformed Jackson derivative is given as

$$D_{\alpha,q} := \begin{cases} D_q + \alpha q^{2N} D_{1/q}, & q \neq 0, \\ D_0 + \alpha \dfrac{d}{dz}\Big|_0, & q = 0, \end{cases}$$

where N is the number operator. For $q \neq 0$, we can also write

$$D_{\alpha,q} = D_q + \frac{\alpha}{q^2} D_{1/q} q^{2N}.$$

Remark 4.2. It is easy to check that the α-deformed Jackson derivative is equivalently defined as

$$(D_{\alpha,q} f)(z) = (D_q f)(z) + \alpha (D_{1/q} f)(q^2 z), \quad q \neq 0.$$

For example, if $f(z) = z^n$, $(D_{\alpha,q} f)(z) = (1 + \alpha q^{n-1})[n]_q z^{n-1}$ holds. In fact, the α-deformed Jackson derivative is an analogue of the operator in Ref. 14 [Theorem2.5].

Then, one can realize one-mode analogue of (α,q)-operators on an appropriate domain of the one-mode interacting Bargmann-Fock space $\mathcal{B}$, with $\omega_n = (1+\alpha q^{n-1})[n]_q$ and $\alpha_n = 0$ by

$$a^+ := Z,\ a^- := D_{\alpha,q}, \text{ and } \Phi_n := \frac{z^n}{\sqrt{[\omega_n]!}}.$$

In fact, it is easy to check that

$$\begin{cases} a^+\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1}, \\ a^-\Phi_n = \sqrt{\omega_n}\Phi_{n-1} \end{cases}$$

hold and the q-commutation relation, one-mode analogue of (A4),

$$\begin{aligned} [a^-,a^+]_q\Phi_n &:= (a^-a^+ - qa^+a^-)\Phi_n \\ &= (I+\alpha q^{2N})\Phi_n, \end{aligned}$$

is satisfied. Let us put $M_{\alpha,q} = I + \alpha q^{2N}$ and then one can get the expression

$$M_{\alpha,q} = (1+\alpha)I - \alpha(1-q^2)ZD_{q^2},$$

due to $(ZD_{q^2})\Phi_n = [n]_{q^2}\Phi_n$.

Therefore, one can obtain the following.

Theorem 4.3. *Suppose $\alpha \in (-1,1)$ and $q \in (-1,1)$. Then, the following are satisfied.*

(1) $[a^-,a^+]_q = M_{\alpha,q}$, $[a^-,M_{\alpha,q}]_{q^2} = (1-q^2)a^-$, $[M_{\alpha,q},a^+]_{q^2} = (1-q^2)a^+$.

(2) $M_{\alpha,q} = (1+\alpha)I - \alpha(1-q^2)ZD_{q^2}$.

(3) *In particular, if $\alpha = q$, then one can obtain a more refined relation, $[a^-,a^+]_{q^2} = (1+q)I$.*

Example 4.4. (1) $\alpha = 0$ implies $[a^-,a^+]_q = I$. Hence, $M_{0,q} = I$ commutes with both a^+ and a^-,

$$[a^-,M_{0,q}]_1 = [M_{0,q},a^+]_1 = 0.$$

Therefore, the case $\alpha \neq 0$ provides non-trivial commutation relations.

(2) If $\alpha = -q^{2\beta}$ for $\beta > 0$, then the limiting case of the scaled operator is obtained as

$$\lim_{q\uparrow 1}\frac{M_{-q^{2\beta},q}}{1-q^2} = \lim_{q\uparrow 1}\frac{I - q^{2\beta}q^{2N}}{1-q^2} = N + \beta.$$

Moreover, let us consider the scaled operators,

$$A^\pm := \lim_{q\uparrow 1}\frac{a^\pm}{\sqrt{1-q^2}}.$$

Then, one can get

$$[A^-,A^+]_1 = N + \beta$$

and hence,

$$[A^-,N]_1 = A^-,\ [N,A^+]_1 = A^+.$$

It should be noted that these are the commutation relations for the classical Meixner-Pollaczek polynomials with respect to the symmetric Meixner distribution in (3.4). See Ref. 6.

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APPENDIX: ON A COXETER GROUP OF TYPE B

Let Σ_n be the set of bijections σ of the $2n$ points $\{\pm 1, \pm 2, \ldots, \pm n\}$ with $\sigma(-k) = -\sigma(k)$. Equipped with the composition operation as a product, Σ_n becomes what is called a Coxeter group of type B. It is generated by $\pi_0 := (1, -1)$ and $\pi_i := (i, i+1)$, $1 \le i \le n-1$, which satisfy the generalized braid relations

$$\begin{cases} \pi_i^2 = e, & 0 \le i \le n-1, \\ (\pi_0\pi_1)^4 = (\pi_i\pi_{i+1})^3 = e, & 1 \le i \le n-1, \\ (\pi_i\pi_j)^2 = e, & |i-j| \ge 2,\ 0 \le i,j \le n-1. \end{cases} \tag{A1}$$

An element $\sigma \in \Sigma_n$ expresses an irreducible form

$$\sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \le i_1, \ldots, i_k \le n-1,$$

and in this case,

$$\ell_1(\sigma) := \text{the number of } \pi_0 \text{ in } \sigma,$$

$$\ell_2(\sigma) := \text{the number of } \pi_i,\ 1 \le i \le n-1, \text{ in } \sigma$$

are well defined. Let H be a separable Hilbert space. For a given self-adjoint involution $f \mapsto \overline{f}$ for $f \in H$, an action of Σ_n on $H^{\otimes n}$ is defined by

$$\begin{cases} \pi_0(f_1 \otimes \cdots \otimes f_n) = \overline{f_1} \otimes f_2 \otimes \cdots \otimes f_n, & n \ge 1, \\ \pi_i(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n, & n \ge 2,\ 1 \le i \le n-1. \end{cases} \tag{A2}$$

The (α, q)-inner product on the full Fock space $\mathcal{F}(H)$ is defined by

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha,q} := \delta_{m,n} \sum_{\sigma \in \Sigma_n} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \prod_{j=1}^{n} \langle f_j, g_{\sigma(j)} \rangle_H,\ \alpha, q \in (-1,1), \tag{A3}$$

with conventions $0^0 = 1$ and $g_{-k} = \overline{g_k}$, $k = 1, 2, \ldots, n$. For example, if one may define the involution as $\overline{f} := -f$, then $g_{-k} = -g_k$. Equipped with this inner product, the full Fock space $\mathcal{F}(H)$ is denoted by $\mathcal{F}_{\alpha,q}(H)$ to emphasize on the dependence of the inner product on α, q.

The (α, q)-creation operator $B^+_{\alpha,q}(f)$ is the usual left creation operator on the full Fock space, and the (α, q)-annihilation operator $B^-_{\alpha,q}(f)$ is its adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{\alpha,q}$. They satisfy the commutation relation

$$B^-_{\alpha,q}(f) B^+_{\alpha,q}(g) - q B^+_{\alpha,q}(g) B^-_{\alpha,q}(f) = \langle f, g \rangle_H I + \alpha \langle \overline{f}, g \rangle_H q^{2N}, \quad f, g \in H. \tag{A4}$$

The readers can consult Ref. 14 for details.

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