What Form of Gravitation Ensures Kepler's Third Law?

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1 Introduction

According to Newton's law of universal gravitation, the Sun attracts planets with a force proportional to the inverse square of the distance between them. By using the law, we can prove *Kepler's laws of planetary motion*:

   1st law: Every planet moves along an elliptic orbit with the Sun at one focus.
   2nd law: The line segment from a planet to the Sun sweeps equal areas in each equal times.
   3rd law: The orbital period of planets is proportional to the 3/2 power of the mean distance.

Surprisingly, Newton himself tried to modify his law of universal gravitation. He proved that: if a planetary orbit is a "revolving ellipse", then the gravitation consists of inverse square and inverse cube terms (Newton's theorem of revolving orbits). His motivation was how to understand the apsidal precession of the Moon's orbit around the Earth. See [1], [2].

Consider a planet moving along a non-circular orbit around the Sun. We denote by r the distance of the planet from the Sun, and by r_1, r_2 the minimum and the maximum of r. We call a point on the orbit attaining $r = r_1$ a *periapsis*, and one attaining $r = r_2$ an *apoapsis*. The *anomalistic period* is the time which the planet requires to travel from a periapsis for the next periapsis. The *sidereal period* is the time which the planet requires to make one revolution around the Sun. In the case of Kepler motion, the anomalistic and the sidereal periods are equal. In this paper, we interpret the "orbital period" of Kepler's 3rd law as the anomalistic period. We put $a = \dfrac{r_1 + r_2}{2}, \; b = \dfrac{r_2 - r_1}{2}$. Then the anomalistic period T is a function of two variables a, b. We call a the *mean distance* of the planet from the Sun.
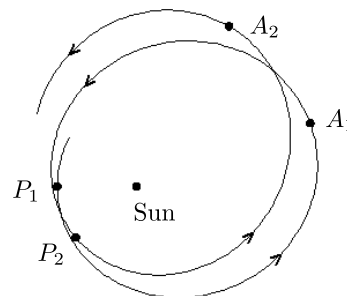
From now on, we assume that the Sun is fixed, and that the planet goes around it. We denote by m the mass of the planet, and by $mF(r)$ the gravitation from the Sun. We assume that $F(r)$ is defined outside the radius R of the Sun, that is, $r > R$.

Figure. The apsidal precession of an orbit. The position of periapsis P_1 moves to the next position P_2.

Theorem 1. $F(r) = \dfrac{k_1}{r^2} + \dfrac{k_2}{r^3}$, *where* $k_1 > 0$, $k_2 \geqq -Rk_1$, *then the anomalistic period* $T = \dfrac{2\pi}{\sqrt{k_1}} a^{3/2}$.

When $F(r)$ is the function given above, the orbit comes to be a "revolving ellipse". Theorem 1 asserts that even such a gravitation ensures Kepler's 3rd law. A question arises: does other form of gravitation ensure Kepler's 3rd law? We obtain a negative answer.

Theorem 2. *Suppose that* $F(r)$ *is of class* C^3, *and satisfies two conditions:* (i) *a primitive function* $U(r)$ *of* $F(r)$ *and* $L(r) := r^3 F(r)$ *are both monotone increasing on an interval* (R, ∞), (ii) $F(r)$ *is bounded on* (R, ∞). *If the anomalistic period* T *does not depend on* b, *then* $F(r) = \dfrac{k_1}{r^2} + \dfrac{k_2}{r^3}$, *where* $k_1 > 0$, $k_2 \geqq -Rk_1$.

The condition (i) is a natural assumption as the following proposition shows.

Proposition 3. *For any two reals* r_1, r_2 $(R < r_1 < r_2)$, *there is an orbit for which the minimum and the maximum of* r *are respectively equal to* r_1, r_2 *if and only if the condition* (i) *of Theorem 2 holds. Moreover, if* (i) *holds, then the anomalistic period* T *is a function of two variables* r_1, r_2 $(R < r_1 < r_2)$.

After the author had proved Theorem 2, he searched preceding studies and found the same result as Theorem 1 in [3] and [4]. (However, the author presumes that neither of them is the first article which states Theorem 1.) About Theorem 2, he found no articles which state a similar result.

2   Preliminaries

We denote by (x, y) the coordinates of the planet with the Sun at the origin. We assume that the gravitation of the Sun attracts it, but no other force act it. Then by Newton's law of motion, we obtain the following differential equations:

$$m\frac{d^2x}{dt^2} = -mF(r)\cos\theta, \qquad m\frac{d^2y}{dt^2} = -mF(r)\sin\theta, \tag{2.1}$$

where (r, θ) is the polar coordinate of (x, y). We can arrange (2.1) into the following equation:

$$r^2\frac{d\theta}{dt} = h, \qquad \frac{d^2r}{dt^2} = \frac{h^2}{r^3} - F(r), \tag{2.2}$$

where h is an integrating constant. The first equation of (2.2) stands for Kepler's second law. By integrating the second equation of (2.2), we obtain that

$$\left(\frac{dr}{dt}\right)^2 = -\frac{h^2}{r^2} - 2U(r) + C, \tag{2.3}$$

where C is an integrating constant. We denote by $g(r)$ the right-hand side of (2.3).

From now on, consider a non-circular orbit. Since (2.3) vanishes at $r = r_1, r_2$, we obtain that

$$g(r_1) = -\frac{h^2}{r_1^2} - 2U(r_1) + C = 0, \quad g(r_2) = -\frac{h^2}{r_2^2} - 2U(r_2) + C = 0. \tag{2.4}$$

By solving (2.4), we have that

$$h^2 = \frac{2r_1^2r_2^2}{r_2^2 - r_1^2}\{U(r_2) - U(r_1)\}, \quad C = \frac{2}{r_2^2 - r_1^2}\{r_2^2U(r_2) - r_1^2U(r_1)\}. \tag{2.5}$$

By putting (2.5) into $g(r)$, we have that

$$g(r) = -\frac{2r_1^2r_2^2}{r_2^2 - r_1^2}\{U(r_2) - U(r_1)\}\frac{1}{r^2} - 2U(r) + \frac{2}{r_2^2 - r_1^2}\{r_2^2U(r_2) - r_1^2U(r_1)\}. \tag{2.6}$$

By putting $U(r) = \frac{1}{r^2}\Phi(r)$, $r_1 = a + b$, $r_2 = a - b$, $r = a + \rho$, we have that

$$\begin{aligned}
g(r) &= \frac{1}{r^2}\left\{-\frac{2}{r_2^2 - r_1^2}\left(r_1^2\Phi(r_2) - r_2^2\Phi(r_1)\right) - 2\Phi(r) + \frac{2}{r_2^2 - r_1^2}\left\{r^2\Phi(r_2) - r^2\Phi(r_1)\right\}\right\} \\
&= \frac{b^2 - \rho^2}{b(a+\rho)^2}\left\{\frac{\Phi(a+b) - \Phi(a+\rho)}{b - \rho} - \frac{\Phi(a+\rho) - \Phi(a-b)}{b + \rho} - \frac{\Phi(a+b) - \Phi(a-b)}{2a}\right\}.
\end{aligned} \tag{2.7}$$

3   Proof of Proposition 3 and Theorem 1

Proof of Proposition 3. By introducing a new variable $w = \frac{dr}{dt}$, we transform the second equation of (2.2) into a system of differential equations:

$$\frac{dr}{dt} = w, \qquad \frac{dw}{dt} = \frac{h^2}{r^3} - F(r). \tag{3.1}$$

We can solve (3.1) as $w^2 = g(r)$. Notice that $g(r_1) = g(r_2) = 0$. The equation $w^2 = g(r)$ on rw-plane represents a closed curve passing through $(r_1, 0)$ and $(r_2, 0)$ if and only if $g(r) > 0$ on the interval (r_1, r_2).

Sufficiency. Take two reals r_1, r_2 $(R < r_1 < r_2)$. Since $U(r)$ is monotone increasing, the right-hand side of the 1st equation of (2.5) is positive. Note that $r^3g'(r) = h^2 - L(r)$. Since $L(r)$ is monotone increasing, $r^3g'(r)$ is monotone decreasing. Since $g(r_1) = g(r_2) = 0$, if $g'(r)$ has a constant sign on the interval (r_1, r_2), we obtain that $g(r) = 0$ on (r_1, r_2), a contradiction. So $g'(r)$ changes its sign positive to negative at a point in (r_1, r_2). Therefore $g(r) > 0$ on (r_1, r_2). Hence $w^2 = g(r)$ is an orbit whose periapsis and apoapsis distances are r_1 and r_2 respectively.

Necessity. Assume that $U(r)$ is not monotone increasing. Then there are two points $r_1 < r_2$ satisfying $U(r_1) > U(r_2)$ or there is an interval $[r_1, r_2]$ on which $U(r)$ is constant. In the first case, the right-hand side of the first equation of (2.5) is negative, a contradiction. In the second case, by putting it into (2.6), we have that $g(r) = 0$ on $[r_1, r_2]$, a contradiction. Hence $U(r)$ is monotone increasing.

Assume that $L(r)$ is not monotone increasing. Then there are two points $r_1 < r_2$ satisfying $L(r_1) > L(r_2)$ or there is an interval $[r_1, r_2]$ on which $L(r)$ is constant. In the first case, we have that $r_1^3 g'(r_1) < r_2^3 g'(r_2)$. Since $g(r) > 0$ on (r_1, r_2), we have that $g'(r_1) \geqq 0$ and $g'(r_2) \leqq 0$, a contradiction. In the second case, $r^3 g'(r)$ is constant on $[r_1, r_2]$. Since $g'(r_1) \geqq 0$ and $g'(r_2) \leqq 0$, we obtain that $r^3 g'(r) = 0$ on $[r_1, r_2]$. Thus we have that $g(r) = 0$ on $[r_1, r_2]$, a contradiction. Hence $L(r)$ is monotone increasing.

Finally, we show that the anomalistic period T is a function of two variables r_1, r_2. When the planet travels from apoapsis for periapsis, $\dfrac{dr}{dt} = \sqrt{g(r)}$, and when the planet travels from periapsis for apoapsis, $\dfrac{dr}{dt} = -\sqrt{g(r)}$. So we obtain that

$$T = \int_{r_1}^{r_2} \frac{1}{\sqrt{g(r)}} \, dr + \int_{r_2}^{r_1} \frac{-1}{\sqrt{g(r)}} \, dr = 2 \int_{r_1}^{r_2} \frac{1}{\sqrt{g(r)}} \, dr. \tag{3.2}$$

Hence we finish the proof. □

Proof of Theorem 1. By putting $\Phi(r) = -k_1 r - k_2$ into (2.7), we have that

$$g(r) = \frac{b^2 - \rho^2}{b(a+\rho)^2} \left\{ -\frac{k_1 b - k_1 \rho}{b - \rho} + \frac{k_1 b + k_1 \rho}{b + \rho} + \frac{2k_1 b}{2a} \right\} = \frac{k_1(b^2 - \rho^2)}{a(a+\rho)^2}. \tag{3.3}$$

By putting it into (3.2), we obtain that

$$T = 2\sqrt{\frac{a}{k_1}} \int_{-b}^{b} \frac{a + \rho}{\sqrt{b^2 - \rho^2}} \, d\rho = \frac{2\pi}{\sqrt{k_1}} a^{3/2}. \tag{3.4}$$

Hence we finish the proof. □

4   Proof of Theorem 2

Proof of Theorem 2. Since $\Phi(r)$ is of class C^4, we can use Taylor expansion to (2.7) as follows:

$$g(r) = \frac{b^2 - \rho^2}{b(a+\rho)^2} \left\{ \sum_{j=0}^{4} \frac{1}{j!} \Phi^{(j)}(a) \left(\frac{b^j - \rho^j}{b - \rho} - \frac{\rho^j - (-b)^j}{b + \rho} \right) - \sum_{j=0}^{3} \frac{1}{j!} \Phi^{(j)}(a) \frac{b^j - (-b)^j}{2a} + o(b^3) \right\}$$

$$= \frac{b^2 - \rho^2}{(1 + \rho/a)^2} \left\{ \beta_0 + \alpha_1 \rho + \alpha_2 \rho^2 + \beta_2 b^2 + o(b^2) \right\}, \tag{4.1}$$

where

$$\beta_0 = \frac{1}{a^2} \Phi''(a) - \frac{1}{a^3} \Phi'(a), \quad \alpha_1 = \frac{1}{3a^2} \Phi'''(a), \quad \alpha_2 = \frac{1}{6a^2} \Phi^{(4)}(a), \quad \beta_2 = \alpha_2 - \frac{1}{2a} \alpha_1. \tag{4.2}$$

We can calculate them as follow:

$$\beta_0 = \frac{1}{a^3} L'(a), \quad \alpha_1 = \frac{1}{3a^3} L''(a), \quad \alpha_2 = \frac{1}{12a^3} L'''(a) - \frac{1}{12a} L''(a). \tag{4.3}$$

Since $L(r)$ is monotone increasing, $L'(r) > 0$ on an open dense subset of (R, ∞). From now on, we calculate on this set. By putting (4.1) into (3.2), we can calculate as follows:

$$\frac{1}{\sqrt{g(r)}} = \left\{ \frac{b^2 - \rho^2}{(1 + \rho/a)^2} \left(\beta_0 + \alpha_1 \rho + \alpha_2 \rho^2 + \beta_2 b^2 + o(b^2) \right) \right\}^{-1/2}$$

$$= \left\{ \frac{\beta_0(b^2 - \rho^2)}{(1 + \rho/a)^2} \right\}^{-1/2} \left\{ 1 + \frac{\alpha_1}{\beta_0} \rho + \frac{\alpha_2}{\beta_0} \rho^2 + \frac{\beta_2}{\beta_0} b^2 + o(b^2) \right\}^{-1/2}$$

$$= \frac{(1 + \rho/a)}{\sqrt{\beta_0(b^2 - \rho^2)}} \left\{ 1 - \frac{\alpha_1}{2\beta_0} \rho - \frac{\alpha_2}{2\beta_0} \rho^2 - \frac{\beta_2}{2\beta_0} b^2 + \frac{3\alpha_1^2}{8\beta_0^2} \rho^2 + o(b^2) \right\}$$

$$= \frac{1}{\sqrt{\beta_0(b^2 - \rho^2)}} \left\{ 1 + \left(\frac{1}{a} - \frac{\alpha_1}{2\beta_0} \right) \rho + \left(-\frac{\alpha_1}{2a\beta_0} - \frac{\alpha_2}{2\beta_0} + \frac{3\alpha_1^2}{8\beta_0^2} \right) \rho^2 - \frac{\beta_2}{2\beta_0} b^2 + o(b^2) \right\}. \tag{4.4}$$

By putting (4.4) into (3.2), we obtain that

$$T = \frac{2\pi}{\sqrt{\beta_0}} + \frac{2\pi}{16 \beta_0^{5/2}} \left\{ -4\beta_0 \left(\frac{1}{a} \alpha_1 + \alpha_2 + 2\beta_2 \right) + 3\alpha_1^2 \right\} b^2 + o(b^2). \tag{4.5}$$

Therefore, if T does not contain b, then we obtain that

$$-4\beta_0\left(\frac{1}{a}\alpha_1 + \alpha_2 + 2\beta_2\right) + 3\alpha_1^2 = 0, \qquad \text{that is,} \ -12\beta_0\alpha_2 + 3\alpha_1^2 = 0. \tag{4.6}$$

By putting (4.3) into (4.6), we have that

$$-L'(r)L'''(r) + \frac{1}{a}L'(r)L''(r) + \frac{1}{3}\left(L''(r)\right)^2 = 0. \tag{4.7}$$

The equation (4.7) holds on an open dense subset of (R, ∞). Since $L(r)$ is of class C^3, the equation (4.7) holds on the entire (R, ∞). By solving (4.7), we have that

$$\left(L'(r)\right)^{2/3} = c_1 r^2 + c_2, \tag{4.8}$$

where c_1, c_2 are integrating constants.

When $c_1 \neq 0$, we have that $F(r) = \frac{1}{4}c_1^{3/2}r + o(r^{-1})$ as $r \to \infty$. It contradicts to the condition (ii). When $c_1 = 0$, we have that $F(r) = \frac{k_1}{r^2} + \frac{k_2}{r^3}$, where $k_1 = \pm c_2^{3/2}$ and k_2 is an integrating constant.

Finally, we prove that $k_1 > 0$ and $k_2 \geqq -Rk_1$. When we assume $k_1 = 0$, we have that $L(r) = k_2$, contradicting to (i). When we assume $k_1 < 0$, then $U'(r) < 0$ for a large r, contradicting to (i). When we assume $k_2 < -Rk_1$, then $U'(r) < 0$ for a small $r > R$, contradicting to (i). $\qquad\square$

5 Newton's Version of Kepler's Third Law

The truth is that *Kepler's 3rd law is not correct*. Because: when we derived Kepler's 3rd law by Newton's law of universal gravitation, we assumed that the planet goes around the Sun. However, the truth is that both the planet and the Sun go around the common center of mass. It was already pointed out by Newton in Principia.

In this section, we assume that both the planet of mass m and the Sun of mass M go around the common center of mass. We can modify Theorem 1 as follows.

Theorem 1r. *If $F(r) = \dfrac{k_1}{r^2} + \dfrac{k_2}{r^3}$, where $k_1 > 0$, $k_2 \geqq -Rk_1$, then the anomalistic period $T = \dfrac{2\pi}{\sqrt{k_1(1+\varepsilon)}}a^{3/2}$, where $\varepsilon = m/M$.*

Proof. We denote by (x, y) the coordinates of the planet with the common center of mass at the origin. Then the coordinates of the Sun is $(-\varepsilon x, -\varepsilon y)$. So we can modify (2.1) as follows:

$$m\frac{d^2 x}{dt^2} = -mF\big((1+\varepsilon)r\big)\cos\theta, \qquad m\frac{d^2 y}{dt^2} = -mF\big((1+\varepsilon)r\big)\sin\theta. \tag{5.1}$$

By putting $k_1' = k_1/(1+\varepsilon)^2$, $k_2' = k_2/(1+\varepsilon)^3$, we can transform (5.1) into the same form as (2.1). Therefore by Theorem 1, we have that $T = \dfrac{2\pi}{\sqrt{k_1'}}(a')^{3/2}$, where a' is the mean distance of the planet and *the origin*. Since we have $a' = a/(1+\varepsilon)$, we obtain that $T = \dfrac{2\pi}{\sqrt{k_1(1+\varepsilon)}}a^{3/2}$. $\qquad\square$

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