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Interval Preserving Map Approximation of 3*x* + 1 Problem

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1. Introduction

The well-known and still unsolved $3x+1$ problem was firstly proposed by Lothar Collatz in 1930's, who had great interest in representation of integer functions by directed graphs.

Conjecture 1.1. *Consider a map* $f:\mathbf{N}\to\mathbf{N}$ *such that*

$$f(n)=\begin{cases}3n+1, & \text{if } n \text{ is odd,}\\ n/2, & \text{if } n \text{ is even.}\end{cases}$$

Then for each natural number n*, there exists a finite number* t *such that* $f^t(n)=\underbrace{f\circ f\circ\cdots\circ f(n)}_{t-times}=1.$

In [2], by introducing the *reverse binary embedding of natural numbers* into $[0,1)$, we obtained a graph $\mathfrak{K}$ of the Collatz function f, which is a Cantor set generated by an iterated function system. Our interest is in the dynamics on the Cantor set $\mathfrak{K}$ presented by the iteration of the Collatz procedure f. In this note, we advance the analysis of the dynamics. We introduce interval preserving maps on $[0,1)$, approximating the dynamics of the Collatz procedure, and consider a $3x+1$ problem of 'finite bit' version.

2. Reverse binary embedding of natural numbers and a conjugacy of Collatz procedure

Definition 2.1. *Let* $n=a_k\cdot 2^k+a_{k-1}\cdot 2^{k-1}+\cdots+a_0$ $(=(a_ka_{k-1}\cdots a_0)_2$ *for short* $)$ *be a binary expansion of a natural number* n*. The* reverse binary embedding β *of* n *is given by*

$$\beta(n)=\frac{a_0}{2}+\frac{a_1}{2^2}+\cdots+\frac{a_k}{2^{k+1}}=(0.a_0a_1\cdots a_k)_2.$$

By definition, $\beta:\mathbf{N}\to[0,1)$ is one-to-one and $\mathbf{N}$ is densely embedded into $[0,1)$. Moreover, even and odd numbers are embedded in $[0,1/2)$ and $[1/2,1)$ respectively. We note that the binary expression $x=(0.b_1b_2\cdots)_2$ of a real number $x\in[0,1)$ has ambiguity. If there exists $k\in\mathbf{N}$ such that $b_l=1$ for all $l\geq k$, then $x=(0.b_1b_2\cdots b_{k-1})_2+2^{-k+1}$, hence x has another binary expression. Thus we always assume that any binary expression $x=(0.b_1b_2\cdots)_2$ has infinitely many 0's in the sequence b_j's, including the case that x has a finite expression $x=(0.b_1b_2\cdots b_k)_2$. For a real number $x\in[0,1)$ and $k\in\mathbf{N}$, $x|_k$ denotes a cut off of x at $(-k-1)$-th order in the binary expression;

$$x|_k=(0.b_1b_2\cdots b_k)_2=\sum_{j=1}^{k}\frac{b_j}{2^j},\quad \text{if}\quad x=(0.b_1b_2\cdots)_2=\sum_{j=1}^{\infty}\frac{b_j}{2^j}.$$

We use a notation $[\ x\ )_k$ as a segment $\left[x|_k, x|_k+\frac{1}{2^k}\right)$, then we have a natural decomposition of segments

$$[\ (0.b_1b_2\cdots b_k)_2\ )_k=[\ (0.b_1b_2\cdots b_k0)_2\ )_{k+1}\oplus[\ (0.b_1b_2\cdots b_k1)_2\ )_{k+1}\,, \tag{2.1}$$

where $[a,b)\oplus[b,c)$ stands for a division of an interval $[a,c)$ at b. For a natural number n, we put $\mathrm{ord}(n)=[\log_2 n]$, where $[x]$ stands for an integer not greater than x, and for a binary expression $n=(a_la_{l-1}\cdots a_0)_2$, $n|^k$ denotes an *upper cut off* of n at k-th order;

$$n|^k=(a_{k-1}a_{k-2}\cdots a_0)_2\equiv n\mod 2^k.$$

Apparently we have

Lemma 2.2. *For* $x,y\in[0,1)$ *and* $m,n,k\in\mathbf{N}$*,*

(1) $x \in [\,x\,)_k$.
(2) $[\,x\,)_k \cap [\,y\,)_k \neq \emptyset$ *holds if and only if* $x|_k = y|_k$, *and hence* $[\,x\,)_k = [\,y\,)_k$.
(3) *if* $x \neq y$, *there exists* $l \in \mathbf{N}$ *such that* $[\,x\,)_l \cap [\,y\,)_l = \emptyset$.
(4) $[\,x\,)_{k+1} \subsetneq [\,x\,)_k$ *and* $\displaystyle\bigcap_{k=1}^{\infty} [\,x\,)_k = \{x\}$.
(5) $\beta(m)|_k = \beta(m|^k)$.
(6) *Thus* $[\,\beta(m)\,)_k = [\,\beta(n)\,)_k$ *if and only if* $m|^k = n|^k$.

To observe the dynamics of the Collatz procedure, we consider a map $F : [0,1) \to [0,1)$ conjugate to f, that is, $F \circ \beta(n) = \beta \circ f(n)$ holds for any natural number n. The embedding β brings not only a well-defined map $F : \beta(\mathbf{N}) \to \beta(\mathbf{N})$, but also an extension of F on $[0,1)$ as follows.

Since $\beta(2n) = \beta(n)/2$, we have $\beta(n) = \beta \circ f(2n) = F \circ \beta(2n) = F(\beta(n)/2)$, hence $F(x) = 2x$ for $x \in [0,1/2) \cap \beta(\mathbf{N})$. Thus we extend F on $[0,1/2)$ by $F(x) = 2x$. In order to extend F on $[1/2,1)$, we need the followings.

Lemma 2.3. *For any natural number* m, n, $[\,\beta(3m+1)\,)_k = [\,\beta(3n+1)\,)_k$ *holds if and only if* $[\,\beta(m)\,)_k = [\,\beta(n)\,)_k$.

Proof. By Lemma 2.2, $[\,\beta(m)\,)_k = [\,\beta(n)\,)_k$ means $m \equiv n \mod 2^k$. Then $3m+1 \equiv 3n+1 \mod 2^k$, and hence $[\,\beta(3m+1)\,)_k = [\,\beta(3n+1)\,)_k$. The converse also holds since 3 and 2^k are prime to each other. □

Consider a sequence $n_k = \beta^{-1}(x|_k)$, $k = 1, 2, \ldots$, then we see $\beta(n_k) = x|_k = (x|_{k+1})|_k = \beta(n_{k+1})|_k$. By Lemma 2.2 (5), we see $n_{k+1} \equiv n_k \mod 2^k$, hence $3n_{k+1} + 1 \equiv 3n_k + 1 \mod 2^k$. (2.1) shows

$$[\,\beta(3n_{k+1}+1)\,)_{k+1} \subsetneq [\,\beta(3n_{k+1}+1)\,)_k = [\,\beta(3n_k+1)\,)_k\,,$$

and then $\displaystyle\bigcap_{k=1}^{\infty} [\,\beta(3n_k+1)\,)_k$ consists of a unique point, expressed as $\displaystyle\lim_{k\to\infty} \beta(3n_k+1)$. Thus we define a conjugacy $F : [0,1) \to [0,1)$ of the Collatz procedure f;

$$F(x) = \begin{cases} 2x, & x \in [0,1/2), \\ \displaystyle\lim_{k\to\infty} \beta(3\beta^{-1}(x|_k)+1), & x \in [1/2,1). \end{cases} \tag{2.2}$$

We see that for any odd number n, $\beta(n)|_k = \beta(n) \in [1/2,1)$ by taking $k > \mathrm{ord}(n)$, therefore $F(\beta(n)) = \beta(3n+1) = \beta(f(n))$. We also see that $[\,x\,)_k = [\,\beta(n_k)\,)_k$ and $[\,F(x)\,)_k = [\,\beta(3n_k+1)\,)_k$ as $F(x) \in [\,\beta(3n_k+1)\,)_k$ by definition.

Lemma 2.4. $\{\beta(3n+1) \mid n \in \mathbf{N}\}$ *is dense in* $[0,1)$.

Proof. As $2 \equiv (-1) \mod 3$, we have $2^{2k} \equiv 1 \mod 3$ and $2^{2k+1} \equiv (-1) \mod 3$. Given $x \in [0,1)$ and $k \in \mathbf{N}$, consider $m_k = \beta^{-1}(x|_k)$. Suppose $m_k \equiv 0 \mod 3$, then there exists $n \in \mathbf{N}$ with $m_k + 2^{2k} = 3n+1$, hence

$$\beta(3n+1) = \beta(m_k + 2^{2k}) = x|_k + \frac{1}{2^{2k+1}} \in [\,x\,)_k\,.$$

Similarly when $m_k \equiv 1 \mod 3$ (resp. $m_k \equiv (-1) \mod 3$), we find $n \in \mathbf{N}$ with $m_k = 3n+1$ (resp. $m_k + 2^{2k+1} = 3n+1$), showing that $\beta(3n+1) \in [\,x\,)_k$. As a result, for any $x \in [0,1)$ and $k \in \mathbf{N}$, there exists $n \in \mathbf{N}$ with $|x - \beta(3n+1)| < 2^{-k}$. □

The following proposition implies that F is approximated by interval exchange maps.

Proposition 2.5. *For any odd number* n *and any natural number* k, *the conjugacy* F *is a right continuous bijection*

$$F : [\,\beta(n)\,)_k \to [\,\beta(3n+1)\,)_k\,.$$

Proof. Consider $x, y \in [\ \beta(n)\ )_k$ with $x \neq y$, then we find a natural number $l > k$ with $[\ x\ )_l \cap [\ y\ )_l = \emptyset$. Applying Lemma 2.2 and 2.3 for $x|_l$ and $y|_l$, we see $\left[\ \beta(3\beta^{-1}(x|_l)+1)\ \right)_l \cap \left[\ \beta(3\beta^{-1}(y|_l)+1)\ \right)_l = \emptyset$, showing $F(x) \neq F(y)$, that is, F is injective on $[\ \beta(n)\ )_k$. Lemma 2.3 and 2.4 means that $[\ \beta(n)\ )_k \cap \beta(\mathbf{N})$ is embedded in $[\ \beta(3n+1)\ )_k$ densely by F. Then for any $y \in [\ \beta(3n+1)\ )_k$ and natural number $l > k$, we choose $m_l \in \mathbf{N}$ such that $\beta(m_l) \in [\ \beta(n)\ )_k$ and $\beta(3m_l+1) \in [\ y\ )_l$. As $\beta(3m_{l+1}+1) \in [\ y\ )_{l+1} \subset [\ y\ )_l$, we see $\beta(3m_{l+1}+1)|_l = \beta(3m_l+1)|_l$, hence $\beta(m_{l+1})|_l = \beta(m_l)|_l$ by Lemma 2.3. Thus $[\ \beta(m_{l+1})\ )_{l+1} \subsetneq [\ \beta(m_l)\ )_l$. Let x be a unique accumulation point $\{x\} = \bigcap_{l>k}^{\infty} [\ \beta(m_l)\ )_l$, then $x|_l = \beta(m_l)|_l = \beta(m_l|^l)$. We see $y|_l = \beta(3m_l+1)|_l = \beta(3m_l|^l+1)|_l$ as $3m_l + 1 \equiv 3m_l|^l + 1 \mod 2^l$, hence

$$\beta(3\beta^{-1}(x|_l)+1) = \beta(3m_l|^l+1) \in [\ y\ )_l\,.$$

Taking $l \to \infty$, we have $F(x) = y$, showing that F is a surjection $[\ \beta(n)\ )_k \to [\ \beta(3n+1)\ )_k$.

For any $x \in [\ \beta(n)\ )_k$ and $l > k$, we see $F(x) \in \left[\ \beta(3\beta^{-1}(x|_l)+1)\ \right)_l$ by the definition of F. We also have $F(w) \in \left[\ \beta(3\beta^{-1}(w|_l)+1)\ \right)_l = \left[\ \beta(3\beta^{-1}(x|_l)+1)\ \right)_l$ whenever $w \in [\ x\ )_l$, that is, $|F(w) - F(x)| < 2^{-l}$ if $w \in [\ x\ )_l$. Therefore F is right continuous. $\square$

Note that for any natural number n, $[\ \beta(n)\ )_k = [\beta(n), \beta(n) + 2^{-k})$ if $k > \mathrm{ord}(n)$, thus $w \in [\ \beta(n)\ )_k$ means $w \geq \beta(n)$. Therefore we state the right continuity only in Proposition 2.5. Indeed, for example, we see

$$\lim_{\substack{w \to (0.11)_2 \\ w < (0.11)_2}} F(w) = (0.001)_2 \neq (0.0101)_2 = F((0.11)_2).$$

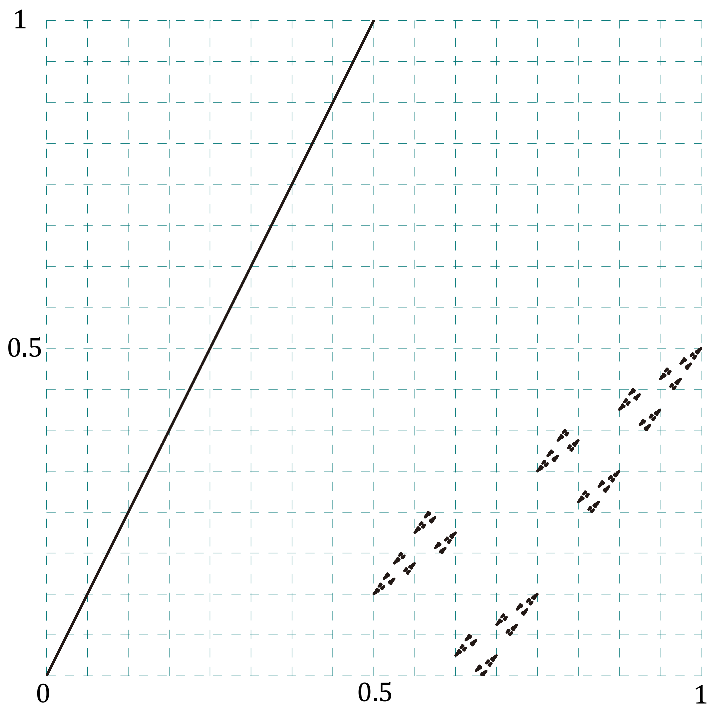

FIGURE 1. The graph of F conjugate to the Collatz procedure

3. Piecewise linear approximant of the conjugacy F

In view of Proposition 2.5, we construct a sequence of piecewise linear maps F_k approximating the conjugacy F: for each $k \in \mathbf{N}$, we define the k-th approximant F_k as

$$F_k(x) = \begin{cases} 2x, & \text{for } x \in [0, 1/2), \\ x - x|_k + F(x|_k)|_k, & \text{for } x \in [1/2, 1), \end{cases}$$
$$= \begin{cases} 2x, & \text{for } x \in [0, 1/2), \\ x - x|_k + \beta(3\beta^{-1}(x|_k) + 1)|_k, & \text{for } x \in [1/2, 1). \end{cases}$$

By definition, F_k is right continuous. Since $w|_k = \beta(n)|_k$ if $w \in [\ \beta(n)\ )_k$ and $\beta(3n|^k + 1)|_k = \beta(3n+1)|_k$, we see

$$F(w) = w - \beta(n)|_k + \beta(3n+1)|_k, \quad \text{for } w \in [\ \beta(n)\ )_k .$$

Thus $F_k([\ \beta(n)\ )_k) = [\ \beta(3n+1)\ )_k$ for any odd number n, compatible with Proposition 2.5. As a result, we see $F(x), F_k(x) \in [\ \beta(3n+1)\ )_k$ for any $x \in [\ \beta(n)\ )_k$, showing that

$$|F(x) - F_k(x)| < 2^{-k}$$

holds for any $x \in [1/2, 1)$. Therefore the sequence F_k, $k = 1, 2, \dots$ approximates F uniformly on $[1/2, 1)$.

In the following subsections, we investigate the approximants F_k's to extract dynamical characteristics of the map $3x + 1$.

3.1. Behavior of carries from lower to upper bits.

For $n, k \in \mathbf{N}$, we define an integer valued function

$$\tau_k(n) = \left[\frac{3n|^k + 1}{2^k} \right].$$

The function τ_k describes the number of bits carried in the calculation $3n + 1$ at k-th bit, as follows.

Proposition 3.1. *Given an odd number n and take $k \in \mathbf{N}$, then we have*

(1) $\tau_k(n) \in \{0, 1, 2\}$.

(2) *For binary expressions $n = (a_l \cdots a_0)_2$ and $3n + 1 = (c_l \cdots c_0)_2$ with $l \geq \operatorname{ord}(n) + 2$, we have*

$$c_k = \big(a_k + \tau_k(n)\big)\big|^1 \quad \text{for } k \leq l.$$

(3)
$$\tau_{k+1}(n|^k) = \begin{cases} 0, & \text{if } \tau_k(n) = 0, 1, \\ 1, & \text{if } \tau_k(n) = 2, \end{cases} \qquad \tau_{k+1}(n|^k + 2^k) = \begin{cases} 1, & \text{if } \tau_k(n) = 0, \\ 2, & \text{if } \tau_k(n) = 1, 2. \end{cases}$$

(4) *If $\tau_k(n) = 0$ or 2,*

$$\left[\ F(\beta(n|^k))\ \right)_k = \left[\ F(\beta(n|^k))\ \right)_{k+1} \oplus \left[\ F(\beta(n|^k + 2^k))\ \right)_{k+1}$$

holds, and if $\tau_k(n) = 1$,

$$\left[\ F(\beta(n|^k))\ \right)_k = \left[\ F(\beta(n|^k + 2^k))\ \right)_{k+1} \oplus \left[\ F(\beta(n|^k))\ \right)_{k+1}$$

holds.

Proof. (1) As $n|^k \leq 2^k - 1$, we have

$$0 \leq \left[\frac{3n|^k + 1}{2^k} \right] \leq \left[\frac{3 \cdot 2^k - 2}{2^k} \right] = \left[3 - \frac{1}{2^{k-1}} \right] \leq 2.$$

(2) By definition, we see $n|^{k+1} = (a_k a_{k-1} \cdots a_0)_2 = a_k \cdot 2^k + n|^k$ and $3n|^k + 1 = \tau_k(n) \cdot 2^k + (3n+1)|^k$. We also see $(3n+1)|^{k+1} = c_k \cdot 2^k + (3n+1)|^k$ while

$$(3.1) \qquad (3n+1)|^{k+1} = (3n|^{k+1}+1)|^{k+1} = \Big(3(a_k \cdot 2^k + n|^k) + 1\Big)\Big|^{k+1}$$
$$= \Big((3a_k + \tau_k(n)) \cdot 2^k + (3n+1)|^k\Big)\Big|^{k+1} = \big(3a_k + \tau_k(n)\big)\big|^1 \cdot 2^k + (3n+1)|^k.$$

Therefore $c_k = \big(3a_k + \tau_k(n)\big)\big|^1 = \big(a_k + \tau_k(n)\big)\big|^1$.
(3) The case $\tau_{k+1}(n)$ is shown by definition. We also see $3n|^k + 1 = \tau_k(n) \cdot 2^k + r$ where $r = (3n|^k+1)|^k < 2^k$. Then we have

$$2^{k+1} < 3(n|^k + 2^k) + 1 = (\tau_k(n) + 3) \cdot 2^k + r < (\tau_k(n) + 4) \cdot 2^k.$$

When $\tau_k(n) = 0$, it holds that $2^{k+1} < 3(n|^k + 2^k) + 1 < 2^{k+2}$, hence $\tau_{k+1}(n|^k + 2^k) = 1$. When $\tau_k(n) = 1, 2$, we have

$$2^{k+2} = (1+3) \cdot 2^k \le (\tau_k(n) + 3) \cdot 2^k + r < (2+4) \cdot 2^k = 3 \cdot 2^{k+1},$$

hence $\tau_{k+1}(n|^k + 2^k) = 2$.
(4) Taking n for $n|^k + a_k \cdot 2^k$, it comes from (3.1) that

$$\beta(3(n|^k + a_k \cdot 2^k) + 1)|_{k+1} = \beta(3n|^k + 1)|_k + \frac{1}{2^{k+1}}\beta(3a_k + \tau_k(n))|_1.$$

When $\tau_k(n) = 0$ or 2, we have $(3a_k + \tau_k(n))|^1 = a_k$, and when $\tau_k(n) = 1$, we have $(3a_k + \tau_k(n))|^1 = 1 - a_k$, hence the assertion. □

Since $\big[\ \beta(n|^k)\ \big)_k = \big[\ \beta(n|^k)\ \big)_{k+1} \oplus \big[\ \beta(n|^k + 2^k)\ \big)_{k+1}$, Proposition 3.1 (4) means that F flips the images of $\big[\ \beta(n|^k)\ \big)_{k+1}$ and $\big[\ \beta(n|^k + 2^k)\ \big)_{k+1}$ if and only if $\tau_k(n) = 1$.

3.2. **Substitution dynamics and automaton associated with the approximant F_k's.** According to Proposition 3.1, the behavior of τ_k's brings a substitution dynamics and an automaton.

For an odd number n and $k \in \mathbf{N}$, we call n *stationary* at k-th order when $\tau_k(n) = 0$, *exchanging* when $\tau_k(n) = 1$ and *unfixed* when $\tau_k(n) = 2$ respectively. We label each segment $[\ \beta(n)\ )_k$ as S, E and U when n is stationary, exchanging and unfixed at k-th order respectively. The original segment $[\ \beta(1)\ )_1 = [1/2, 1)$ is labeled U.

Then combining Proposition 3.1 (3) and (4), to increment the approximation order k by 1 causes a division of each segment, and induces a substitution

$$\sigma : S \to SE \qquad E \to SU \qquad U \to EU,$$

which are mapped by F as

$$F(\sigma) : F(S) \to F(S)F(E) \qquad F(E) \to F(U)F(S) \qquad F(U) \to F(E)F(U).$$

Accordingly we associate each $\beta(n)$ with a string $\boldsymbol{w} = w_1 w_2 \cdots$ consists of S, E and U; we take w_k for S, E or U when $[\ \beta(n)\ )_k$ is labeled as S, E or U, in another words, when $\tau_k(n) = 0, 1,$ or 2 respectively. Note that $w_k = S$ if $k > \mathrm{ord}(n) + 2$ as $3n + 1 < 2^{\mathrm{ord}(n)+3}$. The sequence $\boldsymbol{w}$ is given by calculating $\tau_k(n)$ for each k, however, Proposition 3.1 (3) brings us another way to obtain $\boldsymbol{w}$ directly, without help of τ_k. Actually, suppose that $(0.b_1 b_2 \cdots b_k)_2$ is stationary S at k-th order for instance, then $(0.b_1 b_2 \cdots b_k b_{k+1})_2$ is again stationary S if $b_{k+1} = 0$ and exchanging E if $b_{k+1} = 1$.

Thus we define a deterministic finite state automaton $M = (\{S, E, U\}, \{0, 1\}, \delta, E, \{S\})$ with states S, E, U, an alphabet $\{0, 1\}$ of input symbols, an initial state E, an accepting state S and a transition function δ given as

$$\delta(S,0)=S,\quad \delta(E,0)=S,\quad \delta(U,0)=E,$$
$$\delta(S,1)=E,\quad \delta(E,1)=U,\quad \delta(U,1)=U.$$

The automaton M encodes a binary $(0.b_1b_2\cdots b_k)_2$ to a string $\boldsymbol{w}=w_1w_2\cdots w_k$ over $\{S,E,U\}$, by scanning each term of $b_1b_2\cdots b_k$ from left to right. Apparently we see the automaton M comes to the final state S whenever M has read binaries of the form $(0.b_1b_2\cdots b_k00)_2$ completely; any input data of the form $b_1\cdots b_k00$ is accepted by M

Conversely, we restore the binary $F(\beta(n))=\beta(3n+1)$ from $\beta(n)$ and its encode $\boldsymbol{w}$.

Corollary 3.2. *Consider a binary expression $\beta(n)=(0.b_1b_2\cdots b_k)_2$ with $b_{k-1}=b_k=0$ for any odd number n, and its encode $\boldsymbol{w}=w_1w_2\cdots w_k$ that the automaton M outputs. Let $c_1c_2\cdots c_k$ be a string consist of* 0 *and* 1*, defined as*

$$c_i=\begin{cases} b_i, & \text{if } w_{i-1}\neq E,\\ 1-b_i, & \text{if } w_{i-1}=E,\end{cases}$$

where we take w_0 for E. Then we have

$$\beta(3n+1)=(0.c_1c_2\cdots c_k)_2.$$

Proof. Applying Proposition 3.1 (2) for $n=(b_kb_{k-1}\cdots b_1)_2$ and $3n+1=(c_kc_{k-1}\cdots c_1)_2$, we have

$$c_i=\big(b_i+\tau_{i-1}(n)\big)\big|^1=\begin{cases} b_i, & \text{if } \tau_{i-1}(n)=0,2,\\ 1-b_i, & \text{if } \tau_{i-1}(n)=1,\end{cases}$$

hence the assertion. □

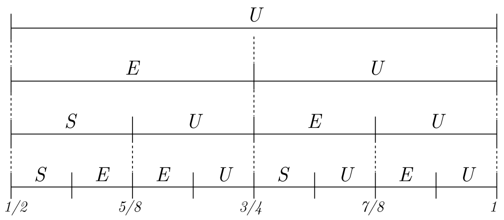

FIGURE 2. Substitution σ on segments

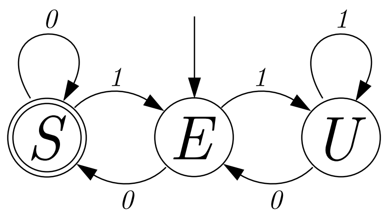

FIGURE 3. Transitive diagram of M

3.3. **An attracting property and a $3x+1$ problem on F_k's.** It easily comes from the definition of F_k that any finitely long binary sequence $\beta(n)$ arrives at a certain k-bit binary sequence by a sufficiently large number of iterations of F_k.

Proposition 3.3. *Consider the k-th approximant F_k. Then for any natural number n, there exists $t\in\mathbf{N}$ that*

$$\operatorname{ord}(\beta^{-1}(F_k^t(\beta(n))))\leq k.$$

Proof. Denote $n_l=\beta^{-1}(F_k^l(\beta(n)))$. When n_l is odd, decompose $n_l=A_l+B_l$ with $B_l=n_l|^k$ and $A_l=n_l-B_l$. By definition, we see $\beta(n_l)|_k=\beta(B_l)$ and $\beta(n_l)=\beta(A_l)+\beta(B_l)$. Thus $F_k(\beta(n_l))=\beta(A_l)+F(\beta(B_l))|_k$ by the definition of F_k. As $F_k(\beta(n_l))\in[0,1/2)$, we see

$$F_k^2(\beta(n_l))=2\big(\beta(A_l)+F(\beta(B_l))|_k\big)=\beta(A_l/2)+2F(\beta(B_l))|_k.$$

As $\operatorname{ord}(A_l/2)=\operatorname{ord}(A_l)-1$, there exists $t\in\mathbf{N}$ such that $A_t=0$, hence the assertion. □

Thus the orbit of any finite binary sequence $\beta(n)$ eventually joins that of a k bit one, meaning that we concentrate our attention on k bit binaries to answer the following 'Collatz-like' problem on F_k:

Problem 3.4 ($3x+1$ problem on F_k). *Show that for any natural number n, there exists $t \in \mathbf{N}$ such that*

$$F_k^t(\beta(n)) = 0 \text{ or } 1/2.$$

Note that finite bit approximation causes the fixed point $n = 0$, e.g., for F_3 we have

$$(0.101)_2 \quad \mapsto \quad (0.000)_2,$$

by eliminating binaries smaller than 2^{-3}. Problem 3.4 moderates the original $3x+1$ problem, where only finite numbers smaller than 2^k are considered. Thus any orbit of $\beta(n)$ are eventually periodic.

4. Interval preserving approximant of the conjugacy F

The approximant F_k behaves as an interval exchange transformation on $[1/2, 1)$, while F_k expands the interval $[0, 1/2)$, which seems to cause some difficulties to close the true nature of the $3x+1$ problem on F_k. Accordingly, we consider another series of approximants G_k's defined as

$$\begin{aligned} G_k(x) &= x - x|_k + F(x|_k)|_k, && \text{for } x \in [0,1), \\ &= \begin{cases} x + x|_k, & \text{for } x \in [0, 1/2), \\ x - x|_k + \beta(3\beta^{-1}(x|_k)+1)|_k, & \text{for } x \in [1/2, 1), \end{cases} \end{aligned}$$

which are interval preserving and right continuous, approximating F uniformly on $[0, 1)$;

$$|F(x) - G_k(x)| < 2^{-k} \text{ for all } x \in [0,1).$$

Since G_k translates only each segment $[\ \beta(n)\ )_k$ to another by definition, the orbit of any point $x \in [0, 1)$ are eventually periodic, described completely by the orbit of the k bit sequence $x|_k$. Thus the corresponding $3x+1$ problem is stated as follows, which reduces Problem 3.4 to a finite combinatorial one:

Problem 4.1 ($3x+1$ problem on G_k). *Show that for any $x \in [0, 1)$, there exists $t \in \mathbf{N}$ such that*

$$G_k^t(x) \in [\ 0\ )_k \cup [\ \beta(1)\ )_k = \left[\ 0,\ 1/2^k\ \right) \cup \left[\ 1/2,\ 1/2 + 1/2^k\ \right).$$

Again we note that any $x \in [\ 0\ )_k$ is fixed under G_k. In spite of restricting our argument to finite bits, Problem 4.1 seems to be not trivial. If we consider $5x+1$ map instead of $3x+1$ for instance, we find lots of periodic orbits as increasing the approximation order k. Problem 4.1 appears to be presenting the unique characteristics of $3x+1$ map.

To attack the original $3x+1$ problem, we have to face the fact that for any initial point $\beta(n)$, it seems to exist sufficiently large approximation order k where the orbit of $\beta(n)$ under G_k pass through only stationary segments. Consider $n = 15 = (1111)_2$. Table 1 shows the corresponding orbits under G_7 and G_8 (omitting even numbers). We see that the orbit pass through stationary segments for $k \geq 8$; each segment label is finished at S. Thus the orbit is determined independently of the approximation order $k \geq 8$, meaning that the orbit under G_k coincides with the original one under F. See the graphs of G_1 to G_8 and their orbits of $(0.1111)_2$ in Figures 4 and 5.

Orbit under G_7:

n	$\beta(n)$	segment label
15	.1111000	$UUUUESS$
23	.1110100	$UUUEUES$
35	.1100010	$UUESSES$
53	.1010110	$UEUEUUE$
1	.1000000	$UESSSSS$

Orbit under G_8:

n	$\beta(n)$	segment label
15	.1111000	$UUUUESSS$
23	.1110100	$UUUEUESS$
35	.1100010	$UUESSESS$
53	.1010110	$UEUEUUES$
5	.1010000	$UEUESSSS$
1	.1000000	$UESSSSSS$

TABLE 1. Orbits of $n = 15$ under G_7 and G_8

5. A conjecture arisen from a graph symmetric to F

In view of the definition (2.2) of the conjugacy F, we also consider a map H symmetrical about $(1/2, 1/2)$ with F on $\beta(\mathbf{N})$:

$$H(x) = \begin{cases} \lim_{k\to\infty} \beta(3\beta^{-1}(x|_k) + 1), & x \in [0, 1/2), \\ 2x - 1, & x \in [1/2, 1). \end{cases}$$

In fact, the topological closure of

$$\{(\beta(n), \beta(3n + 1) \mid n \in \mathbf{N}\}$$

is symmetrical about $(1/2, 1/2)$. For any natural numbers n and k, consider the *one's-complement* $\overline{n|^k}$ of k-bit; $\overline{n|^k} = 2^k - 1 - n|^k$. Then we have $\beta(n|^k) + \beta(\overline{n|^k}) = 1 - 2^{-k}$. It is easily seen that $\overline{(3n + 1)|^k} = (3\overline{n|^k} + 1)|^k$. Then we see $1 - 2^{-k} = \beta((3n + 1)|^k) + \beta(\overline{3n + 1}|^k) = \beta(3n + 1)|_k + \beta(3\overline{n}|^k + 1)|_k$. As a result, we show

$$\lim_{k\to\infty} (\beta(\overline{n|^k}), \beta(3\overline{n|^k} + 1)|_k)$$

is symmetrical to $(\beta(n), \beta(3n + 1))$ about $(1/2, 1/2)$.

We also note that H is the conjugacy of the following arithmetic procedure:

$$h(n) = \begin{cases} 3n + 1, & \text{if } n \text{ is even}, \\ (n - 1)/2, & \text{if } n \text{ is odd}. \end{cases}$$

It is easy to find that h has at least three periodic orbits, which pass through $n = 0$, $n = 4$ and $n = 16$ respectively. However observation of the dynamics of H leads us to another conjecture:

Conjecture 5.1. *For each natural number n, there exists a finite number t such that $h^t(n) = 0, 4$ or 16.*

Note that H is also right continuous on $[0, 1)$, where we are to consider, so to speak, the 'left continuous version' of F.

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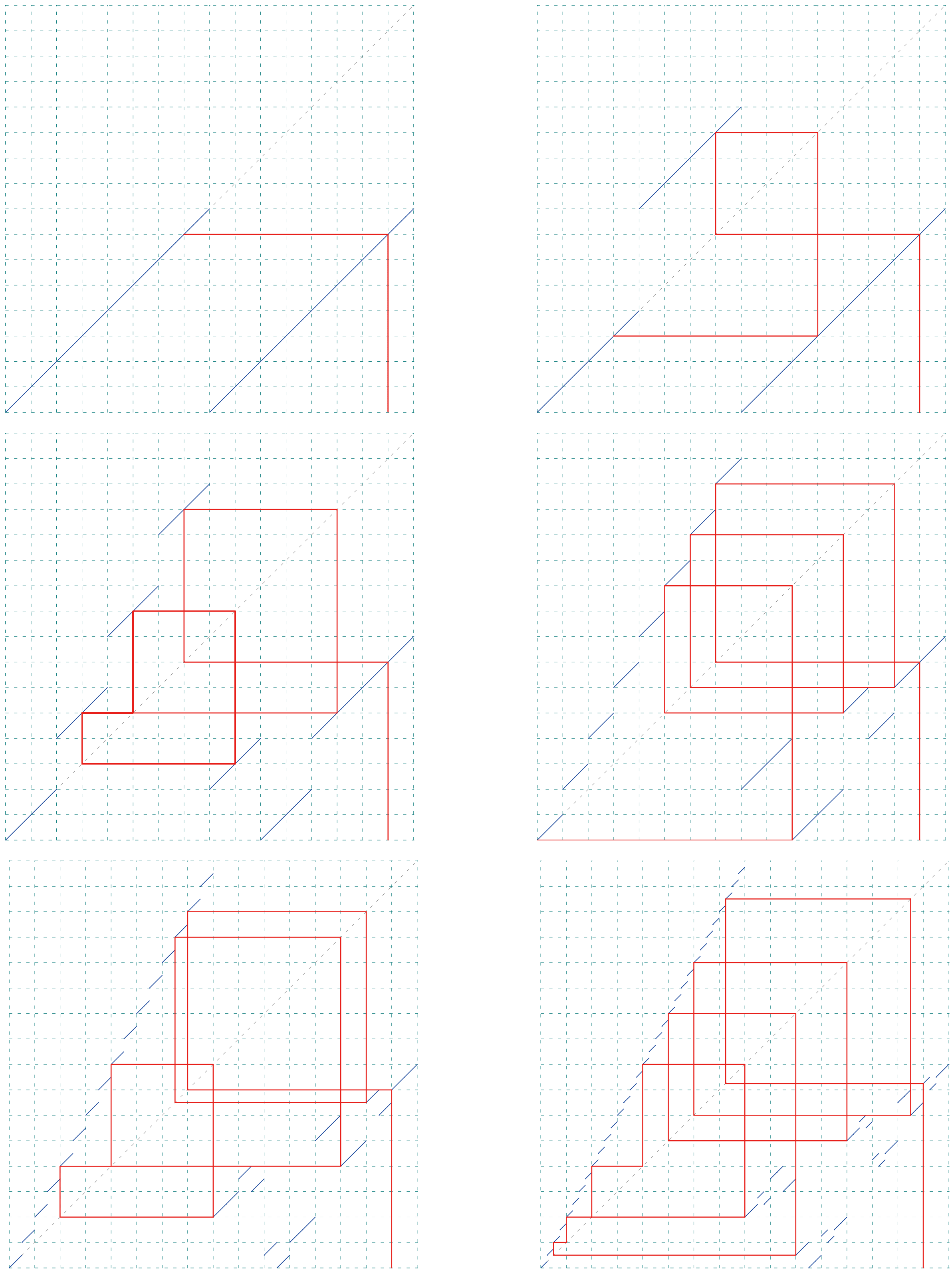

FIGURE 4. Graphs of G_1 to G_6 with their orbit start from $(0.1111)_2$

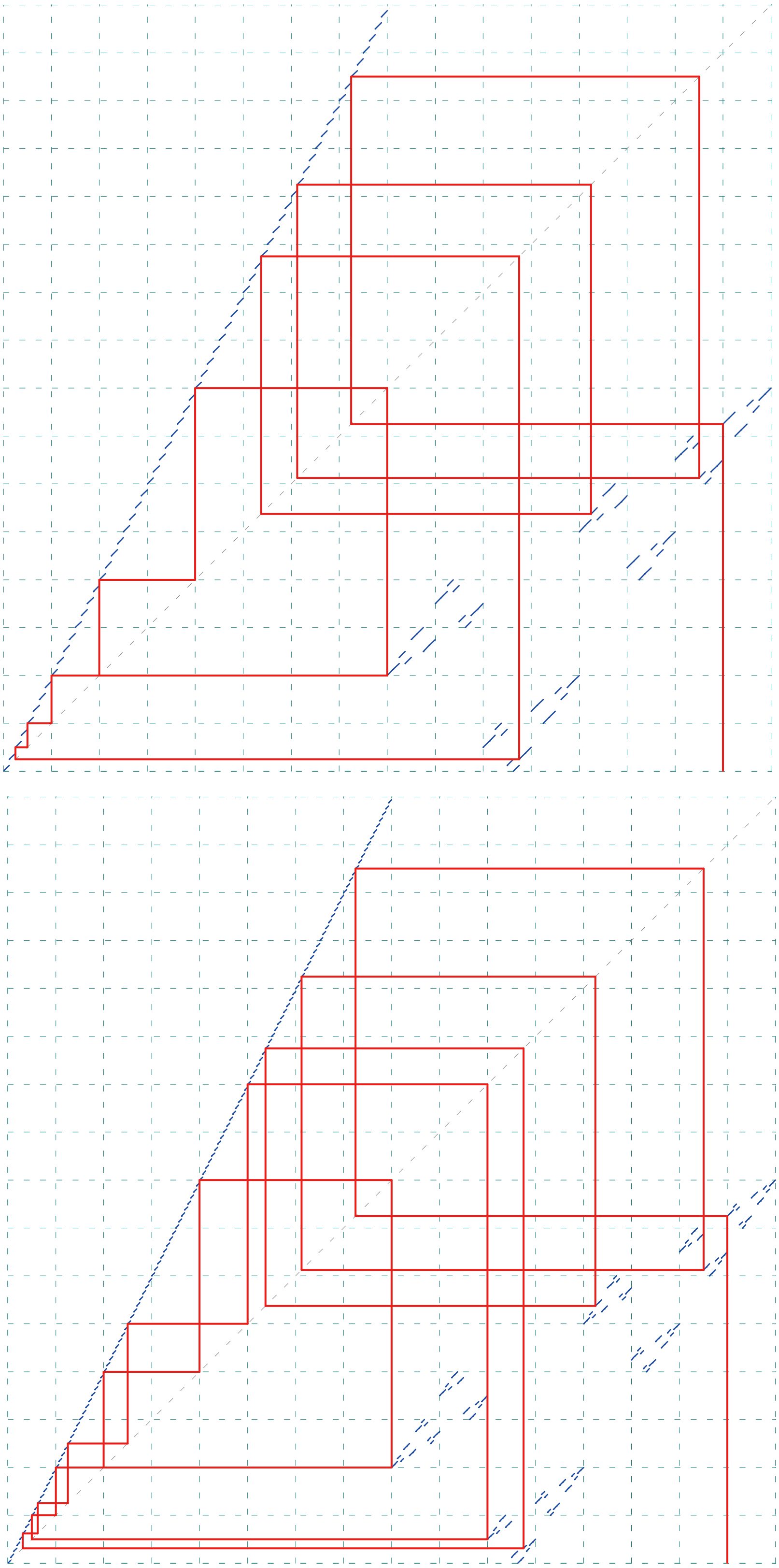

FIGURE 5. Graphs of G_7 and G_8 with their orbit start from $(0.1111)_2$

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