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Multisummabilitiy of Formal Solutions for Some Ordinary Differential Equations

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Abstract

We give an alternative proof of the multisummability property of divergent power series solutions of ordinary differential equations of irregular singular type, which is already known (cf. references).

1 Introduction

We consider the following nonlinear ordinary differential equation of irregular singular type

$$x^{p+1}\frac{d}{dx}u(x) = f(x, u(x)), \tag{1.1}$$

where $x \in \mathbb{C}, p \in \{1, 2, \cdots\}$ and $f(x, u)$ is a quadratic polynomial in u variable with coefficients that are multisummable formal power series in x variable.

We assume that $f(0, 0) = 0$ and $\frac{\partial f}{\partial u}(0, 0) = a \neq 0$. Then the equation has a unique formal power series solution $\hat{u}(x) = \sum_{n\geq 1} u_n x^n$ because of the assumption $a \neq 0$.

We shall prove the multisummability of this formal solution. We have to mention that for the multisummability of such formal solutions, B. Braaksma [3] gave a complete proof at the first time. Different proofs are obtained by many authors (cf. W. Balser [1], [2], J.-P. Ramis and Y. Sibuya [4] and their references). In the paper [3] by Braaksma, the key point of the proof is that he proved an analytic continuation property of solutions of the convolution equations which are obtained by Borel transformation of the ordinary differential equation. In the book [1] by Balser, he employed an iteration method for solutions of the convolution equations. In this paper, we shall define an approximation of solutions by series of functions for the convolution equations in Section 5, which enable us to give a different proof from theirs, but under a restricted assumption (3.4) on the type of multisummability.

The contents of this paper are as follows. In section 2, we give a brief review of the definition of multisummability. In section 3, we state the multisummability result (Theorem 2). In section 4, we prove our main theorem admitting Proposition 5, where we introduce the convolution equations derived from the original ordinary differential equation and study the existence and growth estimate of solutions in a sectorial domain. In section 5, we give the proof of Proposition 5. In this paper, we study only a single equation (1.1), but the result for system of equations is obtained similarly.

2 Definition of multisummability

We give notations and definitions by following Balser [1], [2].

• **Sector** For $d \in \mathbb{R}, \alpha > 0$ and ρ $(0 < \rho \leq \infty)$, we define a sector $S = S(d, \alpha, \rho)$ by

$$S(d, \alpha, \rho) := \left\{ x \in \mathbb{C};\ |d - \arg x| < \frac{\alpha}{2},\ 0 < |x| < \rho \right\}, \tag{2.1}$$

where d, α and ρ are called the direction, the opening angle and the radius of S, respectively. We write $S(d, \alpha, \infty) = S(d, \alpha)$.

• **Gevrey formal power series** We denote by $\mathbb{C}[[x]]$ the ring of formal power series of x with coefficients in $\mathbb{C}$. For $k > 0$, we define $\mathbb{C}[[x]]_{1/k}$, the ring of formal power series of Gevrey order $1/k$ in the following way: $\hat{f}(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{C}[[x]]_{1/k}$ if there exist some positive constants C and K such that for any n, we have

$$|f_n| \leq CK^n \Gamma(1 + n/k), \tag{2.2}$$

where Γ denotes the Gamma function.

• **Gevrey asymptotic expansion** Let $k > 0$, $\hat{f}(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{C}[[x]]_{1/k}$ and $f(x)$ be analytic in $S(d, \alpha, \rho)$. Then we define that

$$f(x) \sim_k \hat{f}(x) \quad \text{in } S(d, \alpha, \rho), \tag{2.3}$$

if for any closed subsector S' of $S(d, \alpha, \rho)$, there exist some positive constants C and K such that for any N, we have

$$\left| f(x) - \sum_{n=0}^{N-1} f_n x^n \right| \leq CK^N |x|^N \Gamma(1 + N/k), \quad x \in S'. \tag{2.4}$$

• **Exponential growth function** Let $k > 0$ and $f(x)$ be analytic in $S(d, \alpha)$. Then we define that $f(x) \in \mathrm{Exp}(k, S(d, \alpha))$ if for any closed subsector S' of $S(d, \alpha)$, there exist some positive constants C and δ such that we have

$$|f(x)| \leq Ce^{\delta |x|^k}, \quad x \in S'. \tag{2.5}$$

• **Laplace transformation** Let $f(x) \in \mathrm{Exp}(k, S(d, \alpha))$ and bounded at the origin. Then we define the k-Laplace transformation $(\mathcal{L}_{k,d} f)(\xi)$ by

$$(\mathcal{L}_{k,d} f)(\xi) := \frac{1}{\xi^k} \int_0^{\infty(d)} \exp\left\{ -(x/\xi)^k \right\} f(x) dx^k, \tag{2.6}$$

where the path of integration runs from 0 to ∞ along $\arg x = d$. Then $(\mathcal{L}_{k,d} f)(\xi)$ is analytic in $S(d, \alpha' + \pi/k, \rho)$ for any $\alpha' < \alpha$ and some positive ρ which depends on α'.

Let $\hat{f}(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{C}[[x]]$. Then we define the formal k-Laplace transformation $(\hat{\mathcal{L}}_k \hat{f})(\xi)$ by

$$(\hat{\mathcal{L}}_k \hat{f})(\xi) := \sum_{n=0}^{\infty} f_n \Gamma(1 + n/k) \xi^n \in \mathbb{C}[[\xi]]. \tag{2.7}$$

Let $k_1 > 0$. If $f(x) \sim_{k_1} \hat{f}(x)$ in $S(d, \alpha)$ and $f(x) \in \mathrm{Exp}(k, S(d, \alpha))$, then $(\mathcal{L}_{k,d} f)(\xi) \sim_{k_2} (\hat{\mathcal{L}}_k \hat{f})(\xi)$ in $S(d, \alpha' + \pi/k, \rho)$ with $k_2 = (k^{-1} + k_1^{-1})^{-1}$.

• **Borel transformation** Let $\varepsilon > 0$, and let $g(\xi)$ be analytic in $S(d,\alpha,\rho)$ with $\alpha > \pi/k$ and bounded at the origin. Then we define the k-Borel transformation $(\mathcal{B}_{k,d}g)(x)$ by

$$(\mathcal{B}_{k,d}g)(x) := \frac{1}{2\pi i}\int_{\gamma_k} \xi^k g(\xi)\exp\left\{(x/\xi)^k\right\} d\xi^{-k}, \tag{2.8}$$

where the path $\gamma_k(\subset S(d,\alpha,\rho))$ runs from the origin along $\arg\xi = d + (\pi+\varepsilon)/(2k)$ to some finite point ξ_1, then along the circle $|\xi| = |\xi_1|$ to the ray $\arg\xi = d - (\pi+\varepsilon)/(2k)$, and back to the origin along this ray. Then $(\mathcal{B}_{k,d}g)(x) \in \mathrm{Exp}(k, S(d,\alpha-\pi/k))$.

Let $\hat{g}(\xi) = \sum_{n=0}^{\infty} g_n\xi^n \in \mathbb{C}[[\xi]]$. Then we define the formal k-Borel transformation $(\hat{\mathcal{B}}_k\hat{g})(x)$ by

$$(\hat{\mathcal{B}}_k\hat{g})(x) := \sum_{n=0}^{\infty}\frac{g_n}{\Gamma(1+n/k)}x^n \in \mathbb{C}[[x]]. \tag{2.9}$$

Let $k_1 > 0$. If $g(\xi) \sim_{k_1} \hat{g}(\xi)$ in $S(d,\alpha,\rho)$ with $\alpha > \pi/k$, then $(\mathcal{B}_{k,d}g)(x) \sim_{k_2} (\hat{\mathcal{B}}_k\hat{g})(x)$ in $S(d,\alpha-\pi/k)$, where $k_2 = (k_1^{-1}-k^{-1})^{-1}$ if $k_1 < k$, and $k_2 = \infty$ if $k_1 \geq k$, respectively.

Moreover , if $f(x)$ and $g(\xi)$ are satisfied the assumptions above, we have $(\mathcal{B}_{k,d}\mathcal{L}_{k,d}f)(x) = f(x)$, and $(\mathcal{L}_{k,d}\mathcal{B}_{k,d}g)(\xi) = g(\xi)$.

• **Convolution** Let $f(\xi)$ and $g(\xi)$ be analytic in $S(d,\alpha,\rho)$, and bounded at the origin. Then k-convolution of $f(\xi)$ and $g(\xi)$ is defined by

$$(f *_k g)(\xi) = \int_0^{\xi} f((\xi^k-\eta^k)^{1/k})\frac{d}{d\eta}g(\eta)d\eta. \tag{2.10}$$

We remark that this k-convolution is not commutative in general. In fact, by integration by parts, we get the following.

$$(f *_k g)(\xi) = g(\xi)f(0) - f(\xi)g(0) + (g *_k f)(\xi).$$

If $f(0) = g(0) = 0$, the k-convolution is commutative.

We give the relationship between k-Laplace transformation, k-Borel transformation and k-convolution.

If $f, g \in \mathrm{Exp}(k, S(d,\alpha))$ with $f(0) = g(0) = 0$, then we have $\mathcal{L}_{k,d}(f *_k g) = (\mathcal{L}_{k,d}f)\cdot(\mathcal{L}_{k,d}g)$ on $S(d,\alpha'+\pi/k,\rho)$. Similarly if f and g are analytic in $S(d,\alpha,\rho)$ with $\alpha > \pi/k$ and $f(0) = g(0) = 0$, then we have $\mathcal{B}_{k,d}(f\cdot g) = (\mathcal{B}_{k,d}f) *_k (\mathcal{B}_{k,d}g)$ on $S(d,\alpha-\pi/k)$.

• **Acceleration** We define the $(\tilde{k},k)$-acceleration operator in a direction d denoted by $\mathcal{A}_{\tilde{k},k;d}$. Let $\tilde{k} > k > 0$ and $\kappa = (k^{-1}-\tilde{k}^{-1})^{-1}$. Let $f(x) \in \mathrm{Exp}(\kappa, S(d,\alpha))$ and bounded at the origin. Then we define the $(\tilde{k},k)$-accerelation of f by

$$(\mathcal{A}_{\tilde{k},k;d}f)(\xi) := \frac{1}{\xi^k}\int_0^{\infty(d)} f(x)C_{\tilde{k}/k}\left((x/\xi)^k\right)dx^k. \tag{2.11}$$

Here for $\alpha > 1$, the kernel function $C_\alpha(z)$ is given by

$$C_\alpha(z) = \frac{1}{2\pi i}\int_\gamma u^{1/\alpha-1}\exp\{u - zu^{1/\alpha}\}du, \tag{2.12}$$

where the path of integration γ is Hankel's integral for the inverse Gamma function: from ∞ along $\arg u = -\pi$ to some $u_0(<0)$, then on the circle $|u| = |u_0|$ to $\arg u = \pi$, and back to ∞ along the ray of argument π.

Then $(\mathcal{A}_{\tilde{k},k;d}f)(\xi)$ is analytic in $S(d,\alpha'+\pi/\kappa,\rho)$ for any $\alpha' < \alpha$ and some positive ρ which depends on α'. Moreover, if $f(x) \sim_{k_1} \hat{f}(x)$ in $S(d,\alpha)$, then

$$(\mathcal{A}_{\tilde{k},k;d}f)(\xi) \sim_{k_2} \sum_{n=0}^{\infty} f_n\frac{\Gamma(1+n/k)}{\Gamma(1+n/\tilde{k})}\xi^n \quad \text{in } S(d,\alpha'+\pi/k,\rho), \quad \text{where } k_2 = (k_1^{-1}+\kappa^{-1})^{-1}. \tag{2.13}$$

If $f, g \in \mathrm{Exp}(\kappa, S(d,\alpha))$ with $f(0) = g(0) = 0$, then we have $\mathcal{A}_{\tilde{k},k;d}(f *_k g) = (\mathcal{A}_{\tilde{k},k;d} f) *_{\tilde{k}} (\mathcal{A}_{\tilde{k},k;d} g)$.

• **Definition of multisummability** Let $q \in \mathbb{N}$ and $\hat{f}(x) \in \mathbb{C}[[x]]$. Let $+\infty = k_0 > k_1 > k_2 > \cdots > k_{q-1} > k_q > 0$ and define κ_i by $\kappa_i = (k_i^{-1} - k_{i-1}^{-1})^{-1}$ for $i = 1, 2, \cdots, q$. For $i = 1, 2, \cdots, q$, let $d_i \in \mathbb{R}$ and $S_i = S(d_i, \pi/k_i + \varepsilon_i)$ $(\varepsilon_i > 0)$ and $S_i' = S(d_i, \varepsilon_i)$ be sectors such that $S_{j-1} \subset S_j$, $j = 2, 3, \cdots, q$. Then $\hat{f}(x)$ is $\boldsymbol{k}$-summable ($\boldsymbol{k} = (k_1, k_2, \cdots, k_q)$) in multidirection $\boldsymbol{d} = (d_1, d_2, \cdots, d_q)$, if the following conditions are satisfied:

a) $f_q(\xi) := (\hat{\mathcal{B}}_{k_q} \hat{f})(\xi)$ is convergent in a neighborhood of the origin.

b) For $j = q, q-1, \cdots, 1$, the function f_j can be continued analytically on S_j' and $f_j \in \mathrm{Exp}(\kappa_j, S_j')$, and if $j \neq 1$, we define $f_{j-1} := \mathcal{A}_{k_{j-1},k_j;d_j} f_j$ which is analytic in $S(d_j, \pi/\kappa_j + \varepsilon_j', \rho_j)$ with $\varepsilon_j' \leq \varepsilon_j - \varepsilon_{j-1}$ and $\rho_j > 0$. In this case, we have $f_1 \in \mathrm{Exp}(\kappa_1, S_i')$ $(\kappa_1 = (k_1^{-1} - k_0^{-1})^{-1} = k_1)$.

Then the $\boldsymbol{k}$-sum of $\hat{f}(x)$ in multidirection $\boldsymbol{d}$ is defined by $\mathcal{L}_{k_1,d_1} f_1$ and denoted by $f_{\boldsymbol{k},\boldsymbol{d}}(x)$. We notice that $f_{\boldsymbol{k},\boldsymbol{d}}(x) \sim_{k_q} \hat{f}(x)$ in $S(d_1, \pi/k_1 + \varepsilon_1', \rho_1)$.

We may omit the direction d in the operators $\mathcal{L}_{k,d}, \mathcal{B}_{k,d}, \mathcal{A}_{\tilde{k},k;d}$ and $f_{\boldsymbol{k},\boldsymbol{d}}$.

3 Result

In the equation (1.1), let the form of $f(x,u)$ be

$$f(x,u) = f^{[0]}(x) + f^{[1]}(x)u(x) + f^{[2]}(x)u^2(x). \tag{3.1}$$

Here without loss of generality, we may assume that $f^{[2]}(0) = 0$. In fact, by putting $\tilde{u}(x) = u(x) - u_1 x = \sum_{n \geq 2} u_n x^n$, we get an equation of $\tilde{u}$ of the form

$$x^{p+1} \frac{d}{dx} \tilde{u} = \tilde{f}^{[0]}(x) + \tilde{f}^{[1]}(x)\tilde{u}(x) + \tilde{f}^{[2]}(x)\tilde{u}^2(x), \tag{3.2}$$

where $\tilde{f}^{[1]}(0) = \frac{\partial f}{\partial u}(0,0) = a$ and $\tilde{f}^{[2]}(x) = O(x)$. We put $\tilde{f}^{[1]}(x) = a + \tilde{\tilde{f}}^{[1]}(x)$ $(\tilde{\tilde{f}}^{[1]}(0) = 0)$. After deleting "tilde", we obtain the following form

$$x^{p+1} \frac{d}{dx} u(x) = f^{[0]}(x) + au(x) + f^{[1]}(x)u(x) + f^{[2]}(x)u^2(x). \tag{3.3}$$

For the multisummability result, we assume the following conditions.

Inhomogeneous part $f^{[0]}(x)$ and the coefficients $f^{[1]}(x)$, $f^{[2]}(x)$ are $\boldsymbol{k}$-summable in multidirection $\boldsymbol{d}$.

Moreover, we assume that

$$p \geq k_1. \tag{3.4}$$

We remark that in the following we interpret for the estimates of solutions that $\kappa_0 = p$ if $p > k_1$, and

$$\kappa_1 = \begin{cases} k_1 & \text{if } p = k_1, \\ \tilde{\kappa}_1 = (k_1^{-1} - p^{-1})^{-1} & \text{if } p > k_1. \end{cases}$$

To formulate the result we use the following definition.

Definition 1 *The set of singular directions D for the operator $x^{p+1}\frac{d}{dx} - a$ is defined by*

$$D = \{(\arg a + 2\pi n)/p; n = 0, 1, 2, \cdots, p-1\}. \tag{3.5}$$

Under the above preparations, we can prove the following result.

Theorem 2 *Let $u(x)$ be the formal solution of (3.3).*

When $p = k_1$, the formal solution $u(x)$ is $\boldsymbol{k}$-summable in multidirection $(\tau_1, d_2, \cdots, d_q)$, where $\tau_1 \notin D$ and $S(\tau_1, \delta_1) \subset S(d_1, \varepsilon_1)$.

When $p > k_1$, the formal solution $u(x)$ is $(p, k_1, \cdots, k_q)$-summable in multidirection $(d_0, d_1, \cdots, d_q)$, where $d_0 \notin D$ and $S(d_0, \pi/p + \varepsilon_0) \subset S(d_1, \pi/k_1 + \varepsilon_1)$.

In the case where $p = k_1$, its multisum $u_{\boldsymbol{k}}(x)$ is the analytic solution of (3.3) in $\tilde{S}_1 = S(\tau_1, \pi/k_1 + \delta_1', \rho_1')$ for any $\delta_1' < \delta_1$ and $\rho_1' > 0$, and $u_{\boldsymbol{k}}(x) \sim_{k_q} u(x)$ in $\tilde{S}_1$. In the case where $p > k_1$, its multisum $u_{(p,\boldsymbol{k})}(x)$ is the analytic solution of (3.3) in $\tilde{S}_0 = S(d_0, \pi/p + \varepsilon_0', \rho_0')$ for any $\varepsilon_0' < \varepsilon_0$ and $\rho_0' > 0$, and $u_{(p,\boldsymbol{k})}(x) \sim_{k_q} u(x)$ in $\tilde{S}_0$.

4 Convolution equations

We recall the equation (3.3) as follows.

$$x^{p+1}\frac{d}{dx}u(x) = f^{[0]}(x) + a\,u(x) + f^{[1]}(x)u(x) + f^{[2]}(x)u^2(x), \tag{4.1}$$

where $a \neq 0$ and $f^{[\ell]}(x) = O(x)$ as $x \to 0$ for $\ell = 0, 1, 2$.

First, we take formal k_i-Borel and p-Borel transformation of the equation (4.1).

For $i = 1, 2, \cdots, q$,

$$\frac{\xi^p}{\Gamma(1+p/k_i)} *_{k_i} \xi\frac{d}{d\xi}v_i(\xi) = f_i^{[0]}(\xi) + av_i(\xi) + (f_i^{[1]} *_{k_i} v_i)(\xi) + (f_i^{[2]} *_{k_i} v_i *_{k_i} v_i)(\xi), \tag{4.2}$$

and for $i = 0, 1$,

$$pD_\xi^{-1}\xi^p\frac{d}{d\xi}v_i(\xi) = f_i^{[0]}(\xi) + av_i(\xi) + (f_i^{[1]} *_p v_i)(\xi) + (f_i^{[2]} *_p v_i *_p v_i)(\xi), \tag{4.3}$$

where $D_\xi^{-1} = \int_0^\xi$, and $f_i^{[\ell]}$ are given as follows: Let $S_i' = S(d_i, \varepsilon_i)$ for $i = 1, 2, \cdots, q$.

1) $f_q^{[\ell]} := \hat{\mathcal{B}}_{k_q} f^{[\ell]}$, which is holomorphic in a neighborhood of the origin and belongs to $\mathrm{Exp}(\kappa_q, S_q')$.

2) $f_i^{[\ell]} := \mathcal{A}_{k_i,k_{i+1}} f_{i+1}^{[\ell]}$, which is analytic in $S(d_{i+1}, \pi/\kappa_i + \varepsilon_{i+1}', \rho_{i+1})$ and belongs to $\mathrm{Exp}(\kappa_i, S_i')$ for $i = q-1, \cdots, 2, 1$.

3) $f_0^{[\ell]} := \mathcal{A}_{p,k_1} f_1^{[\ell]} = \mathcal{B}_p f_{\boldsymbol{k}}^{[\ell]} \in \mathrm{Exp}(p, S(d_1, \pi/\tilde{\kappa}_1 + \varepsilon_1'))$ with $\tilde{\kappa}_1 = (k_1^{-1} - p^{-1})^{-1}$ if $p > k_1$.

Here we use the relation $(\mathcal{B}_p x^{p+1}\frac{d}{dx}u)(\xi) = pD_\xi^{-1}\xi^p\frac{d}{d\xi}(\mathcal{B}_p u)(\xi)$ in the expression (4.3).

We differentiate the convolution equations (4.2) and (4.3) with respect to ξ and put $w_i(\xi) = d/(d\xi)v_i(\xi) \Leftrightarrow v_i(\xi) = (D_\xi^{-1}w_i)(\xi)$. By dividing by a in (4.2), and $p\xi^p - a$ in (4.3) respectively, we obtain the following expressions.

For $i = 1, 2, \cdots, q$,

$$\begin{aligned} w_i(\xi) \;=\; & \frac{1}{a}\frac{d}{d\xi}\Bigg\{\frac{\xi^p}{\Gamma(1+p/k_i)} *_{k_i} \xi w_i(\xi) - f_i^{[0]}(\xi) \\ & -\Big(f_i^{[1]} *_{k_i} (D_\xi^{-1}w_i)\Big)(\xi) - \Big(f_i^{[2]} *_{k_i} (D_\xi^{-1}w_i) *_{k_i} (D_\xi^{-1}w_i)\Big)(\xi)\Bigg\} \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&= \frac{1}{a}\Bigg[\frac{p(p-k_i)}{\Gamma(1+p/k_i)}\xi^{k_i-1}\Big(\xi^{p-2k_i} *_{k_i} D_\xi^{-1}(\xi^{k_i}w_i)\Big)(\xi) - \frac{d}{d\xi}f_i^{[0]}(\xi) \\
&\qquad -\xi^{k_i-1}\Big(\frac{d}{d\xi}f_i^{[1]}(\xi)\cdot\xi^{1-k_i} *_{k_i} (D_\xi^{-1}w_i)\Big)(\xi) \\
&\qquad\quad -\xi^{k_i-1}\Big(\frac{d}{d\xi}f_i^{[2]}(\xi)\cdot\xi^{1-k_i} *_{k_i} (D_\xi^{-1}w_i)(\xi) *_{k_i} (D_\xi^{-1}w_i)(\xi)\Big)\Bigg] \\
&=: T_i(D_\xi^{-1}w_i)(\xi),
\end{aligned}
$$

and for $i = 0, 1$,

$$
\begin{aligned}
(4.5)\qquad w_i(\xi) &= \frac{1}{p\xi^p - a}\frac{d}{d\xi}\Big[f_i^{[0]}(\xi) + \Big(f_i^{[1]} *_p (D_\xi^{-1}w_i)\Big)(\xi) + \Big(f_i^{[2]} *_p (D_\xi^{-1}w_i) *_p (D_\xi^{-1}w_i)\Big)(\xi)\Big] \\
&= \frac{1}{p\xi^p - a}\Bigg[\frac{d}{d\xi}f_i^{[0]}(\xi) + \xi^{p-1}\Big(\frac{d}{d\xi}f_i^{[1]}(\xi)\cdot\xi^{1-p} *_p (D_\xi^{-1}w_i)(\xi)\Big) \\
&\qquad +\xi^{p-1}\Big(\frac{d}{d\xi}f_i^{[2]}(\xi)\cdot\xi^{1-p} *_p (D_\xi^{-1}w_i)(\xi) *_p (D_\xi^{-1}w_i)(\xi)\Big)\Bigg] \\
&=: T_i(D_\xi^{-1}w_i)(\xi),
\end{aligned}
$$

where we use the relation

$$
\begin{aligned}
\frac{d}{d\xi}(u *_k v)(\xi) &= u(0)v'(\xi) + \xi^{k-1}\int_0^\xi u'((\xi^k-\eta^k)^{1/k})\cdot(\xi^k-\eta^k)^{(1-k)/k}v'(\eta)d\eta \\
&= u(0)v'(\xi) + \xi^{k-1}(u'(\xi)\cdot\xi^{1-k} *_k v(\xi))
\end{aligned}
$$

and $w_i(0) = 0$.

We shall analyze the above convolution equations (4.4) and (4.5).

Let U_i' be a closed subsector of $S_i' = S(d_i, \varepsilon_i)$ and $U(r) := U_i' \cap \{|\xi| \le r\}$ for some $r > 0$. Let $B(r)$ be the space of continuous functions $u : U(r) \to \mathbb{C}$ such that u is analytic in the interior of $U(r)$ and

$$
(4.6)\qquad \|u\|_r := \sup_{\xi\in U(r)}|u(\xi)| < \infty.
$$

Lemma 3 1) *Let $k > 0$. For $u, v \in B(r)$, we have the following inequality.*

$$
(4.7)\qquad \left|\frac{d}{d\xi}\Big((D_\xi^{-1}u) *_k (D_\xi^{-1}v)\Big)(\xi)\right| = \left|\xi^{k-1}\Big(u(\xi)\cdot\xi^{1-k} *_k (D_\xi^{-1}v)(\xi)\Big)\right| \le \frac{1}{k}B(1/k, 1/k)\|u\|_r\|v\|_r|\xi|,
$$

where $B(\alpha,\beta)$ denotes the Beta function.

2) *Let $\kappa \ge k > 0$. Let u and v be analytic in $S(d,\varepsilon)$ and satisfy*

$$
(4.8)\qquad |u(\xi)| \le U\frac{|\xi|^\ell}{\Gamma((1+\ell)/k)}e^{\delta|\xi|^\kappa}, \quad |v(\xi)| \le V\frac{|\xi|^m}{\Gamma((1+m)/k)}e^{\delta|\xi|^\kappa}, \quad \xi \in S(d,\varepsilon),
$$

where $U, V, \delta > 0$ and $\ell, m \ge 0$. Then we have

$$
(4.9)\qquad \left|\frac{d}{d\xi}\Big((D_\xi^{-1}u) *_k (D_\xi^{-1}v)\Big)(\xi)\right| = \left|\xi^{k-1}\Big(u(\xi)\cdot\xi^{1-k} *_k (D_\xi^{-1}v)(\xi)\Big)\right| \le \frac{1}{k}\frac{UV|\xi|^{\ell+m+1}}{\Gamma((\ell+m+2)/k)}e^{\delta|\xi|^\kappa}.
$$

Lemma 3 1) implies that the operator T_i is well-defined on $B(s)$ for some positive s. In fact, we have the following estimates.

$$
(4.10)\qquad \left|\frac{d}{d\xi}f_i^{[\ell]}(\xi)\right| \le F, \quad \left|\frac{1}{p\xi^p - a}\right| \le A, \ \xi \in U(s).
$$

Therefore we get the following estimates by Lemma 3. For $w_i \in B(s)$, we have

$$|T_i(D_\xi^{-1}w_i)(\xi)| \le C_1(k)\|w_i\|_s s^p + C_2 + C_3(k)\|w_i\|_s s + C_4(k)\|w_i\|_s^2 s^2 := f_i(s;\|w_i\|_s) < \infty, \tag{4.11}$$

for $i = 1, 2, \cdots, q$ and

$$|T_i(D_\xi^{-1}w_i)(\xi)| \le C_1' + C_2'(k)\|w_i\|_s s + C_3'(k)\|w_i\|_s^2 s^2 := f_i(s;\|w_i\|_s) < \infty, \tag{4.12}$$

for $i = 0, 1$, where C_m and C_m' are positive constants.

Proposition 4 *The equation $T_q(D_\xi^{-1}w_q) = w_q$ has a unique holomorphic solution in a neighborhood of origin.*

Proposition 4 follows from above estimates for T_i. More precisely, we can prove that the operator T_q is a contraction map on a proper Banach space by taking r sufficiently small as follows:
⟨ Proof of Proposition 4 ⟩ Let $M > 0$. We define a closed ball $B(r, M) := \{u \in B(r); \|u\|_r \le M\}$, where we put $U(r) = \{|\xi| \le r\}$ in the above definition.

First, let $M > C_2$ and let r_0 be a smallest positive root of $f_q(r; M) = M$. For any M with $M > C_2$ if we take $r > 0$ such that $r \le \min\{r_0, s\}$, then the operator T_q is well-defined on $B(r, M)$.

Next, let $u, v \in B(r, M)$. Then we have

$$|T_q(D_\xi^{-1}u) - T_q(D_\xi^{-1}v)| \le (C_1(k)r^p + C_3(k)r + 2MC_4(k)r^2)\|u - v\|_r =: g(r)\|u - v\|_r. \tag{4.13}$$

If we take $r_1 \le s$ such that $g(r_1) < 1$, then T_q becomes a contraction map on $B(r_1, M)$.

Consequently, a closed ball $B(r, M)$ $(M > C_2)$ is defined as follows:

First, we take and fix $r_1 > 0$ such that $g(r_1) < 1$.

Next, we take $r > 0$ such that $r \le \min\{s, r_0, r_1\}$.

By taking the closed ball $B(r, M)$ like this, we see that the equation $T_q(D_\xi^{-1}w_q) = w_q$ has a unique solution in $B(r, M)$.

Proposition 5 *The case $p = k_1$. Let $i \in \{1, 2, \cdots, q\}$. Let $S_1' = S(\tau_1, \delta_1)$ and $S_i' = S(d_i, \varepsilon_i)$ $(i \ge 2)$ be sectors and $S_1' \cap D = \phi$. Let $w_i(\xi)$ be an analytic solution of (4.4)-(i) or (4.5)-(1) on $S_i' \cap \{|\xi| < s\}$ for some positive s. Then w_i can be continued analytically on S_i' and $w_i \in \mathrm{Exp}(\kappa_i, S_i')$.*

The case $p > k_1$. Let $i \in \{0, 1, \cdots, q\}$. Let $S_i' = S(d_i, \varepsilon_i)$ be sectors and $S_0' \cap D = \phi$. Let $w_i(\xi)$ be an analytic solution of (4.4)-(i) or (4.5)-(0) on $S_i' \cap \{|\xi| < s\}$ for some positive s. Then w_i can be continued analytically on S_i' and $w_i \in \mathrm{Exp}(\kappa_i, S_i')$, where we write $\kappa_1 = \tilde{\kappa}_1 (= (k_1^{-1} - p^{-1})^{-1})$ and $\kappa_0 = p$.

Remark 6 *Let $i \in \{2, 3, \cdots, q\}$. Let $w_i \in \mathrm{Exp}(\kappa_i, S_i')$ be a solution of (4.4)-(i). Then $w_{i-1} = \frac{d}{d\xi}(\mathcal{A}_{k_{i-1},k_i} D_\xi^{-1} w_i)$ is a solution of (4.4)-$(i-1)$ or (4.5)-(1) in $S(d_i, \pi/\kappa_i + \varepsilon_i', \rho_i')$ for any $\varepsilon_i' < \varepsilon_i$ and some positive ρ_i'.*

Moreover, in case where $p > k_1$, if $w_1 \in \mathrm{Exp}(\tilde{\kappa}_1, S_1')$ is a solution of (4.4)-(1), then $w_0 = \frac{d}{d\xi}(\mathcal{A}_{p,k_1} D_\xi^{-1} w_1)$ is a solution of (4.5)-(0) in $S(d_1, \pi/\tilde{\kappa}_1 + \varepsilon_1', \rho_1')$ for any $\varepsilon_1' < \varepsilon_1$ and some positive ρ_1'.

We postpone the proof of Proposition 5 and give the proof of Theorem 2.

⟨ Proof of Theorem 2 ⟩ We only give the proof for the case where $p = k_1$ because the proof for the case where $p > k_1$ is obtained similarly.

Let $S_i' = S(d_i, \varepsilon_i)$ for $i \in \{2, 3, \cdots, q\}$ and $S_1' = S(\tau_1, \delta_1)$. Let $u(x)$ be a formal solution of (3.3) and set $w_q(\xi) = \frac{d}{d\xi}(\hat{\mathcal{B}}_{k_q} u)(\xi)$. Then w_q is a solution of (4.4)-(q) by Proposition 4 and it can be continued

analytically on S'_q and $w_q \in \mathrm{Exp}(\kappa_q, S'_q)$ by Proposition 5. Therefore $w_{q-1}(\xi) = \frac{d}{d\xi}(\mathcal{A}_{k_{q-1},k_q} D_\xi^{-1} w_q)(\xi)$ is well-defined and it is a solution of (4.4)-$(q-1)$ by Remark 6. Moreover w_{q-1} is analytic in S'_{q-1} and belongs to $\mathrm{Exp}(\kappa_{q-1}, S'_{q-1})$ by Proposition 5. Inductively we see that $w_i(\xi) = \frac{d}{d\xi}(\mathcal{A}_{k_i,k_{i+1}} D_\xi^{-1} w_{i+1})(\xi) \in \mathrm{Exp}(\kappa_i, S'_i)$ and w_i is a solution of (4.4)-(i). Finally, $w_1(\xi) = \frac{d}{d\xi}(\mathcal{A}_{p,k_2} D_\xi^{-1} w_2)(\xi) \in \mathrm{Exp}(\kappa_1, S'_1)$, where $\kappa_1 = k_1 = p$ and it is a solution of (4.5)-(1). Hence $u(x)$ is $\boldsymbol{k}$-summable in multidirection $(\tau_1, d_2, \cdots, d_q)$ and its multisum is given by $u_{\boldsymbol{k}}(x) = (\mathcal{L}_{k_1} D_\xi^{-1} w_1)(x)$, which is analytic in $S(\tau_1, \pi/k_1 + \delta'_1, \rho'_1)$ for any $\delta'_1 < \delta_1$ and some positive ρ'_1.

5 Proof of Proposition 5

We shall prove Proposition 5. First, we define the sequence $\{w_i^n(\xi)\}_{n\geq 0}$ by the following recurrence formulas:
(A): For $i = 1, 2, \cdots, q$,

$$\begin{aligned} w_i^0(\xi) &= -\frac{1}{a}\frac{d}{d\xi} f_i^{[0]}(\xi), \\ w_i^n(\xi) &= -\frac{1}{a}\frac{d}{d\xi}\Bigg[\Big(f_i^{[1]}(\xi) *_{k_i} (D_\xi^{-1} w_i^{n-1})\Big)(\xi) \\ &\quad + \sum_{\ell+m=n-2}\Big(f_i^{[2]} *_{k_i} (D_\xi^{-1} w_i^{\ell}) *_{k_i} (D_\xi^{-1} w_i^{m})\Big)(\xi) - c_i(\xi^p *_{k_i} \xi w_i^{n-p})(\xi)\Bigg], \quad (n \geq 1) \end{aligned}$$

where $w_i^{-k}(\xi) \equiv 0$ if $k > 0$ and $c_i = \frac{p(p-k_i)}{\Gamma(1+p/k_i)}$.
(B): For $i = 0, 1$,

$$\begin{aligned} w_i^0(\xi) &= \frac{1}{p\xi^p - a}\frac{d}{d\xi} f_i^{[0]}(\xi), \\ w_i^n(\xi) &= \frac{1}{p\xi^p - a}\frac{d}{d\xi}\left[\Big(f_i^{[1]} *_p (D_\xi^{-1} w_i^{n-1})\Big)(\xi) + \sum_{\ell+m=n-2}\Big(f_i^{[2]} *_p (D_\xi^{-1} w_i^{\ell}) *_p (D_\xi^{-1} w_i^{m})\Big)(\xi)\right], \ (n \geq 1) \end{aligned}$$

where $w_i^{-k}(\xi) \equiv 0$ if $k > 0$.

We put $W_i(\xi) := \sum_{n\geq 0} w_i^n(\xi)$. We see that $W_i(\xi)$ is nothing but a formal power series solution of (4.4, 4.5)-(i) from above recurrence formulas and in particular, the formal power series solution $W_q(\xi)$ is convergent in a neighborhood of the origin by Proposition 4. Therefore $W_q(\xi)$ coincides with $w_q(\xi) = d/(d\xi)(\hat{\mathcal{B}}_{k_q} u)(\xi)$. Then we can prove the following proposition.

Proposition 7 *The formal solution $W_i(\xi)$ of (4.4, 4.5)-(i) is an analytic function on S'_i, and belongs to* $\mathrm{Exp}(\kappa_i, S'_i)$*, where $S'_1 = S(\tau_1, \delta_1)$ and $S'_i = S(d_i, \varepsilon_i)$ $(i \geq 2)$ if $p = k_1$, and $S'_i = S(d_i, \varepsilon_i)$ if $p > k_1$.*

In order to prove Proposition 7, we recall estimates of the coefficients and inhomogeneous part, that is, $f_i^{[\ell]}(\xi) \in \mathrm{Exp}(\kappa_i, S(d_i, \varepsilon_i))$ for $i = 0, 1, \cdots, q$, where $\kappa_1 = k_1$ and $\kappa_0 = p$. Therefore we have the following estimates.

$$\left|\frac{d}{d\xi} f_i^{[\ell]}(\xi)\right| \leq \frac{L}{\Gamma(1/k_i)} \exp\{\delta|\xi|^{\kappa_i}\}, \quad \xi \in S'_i, \quad (\ell = 0, 1, 2) \tag{5.1}$$

with some positive constants L and δ. Moreover, it is assumed

$$\left|\frac{1}{p\xi^p - a}\right| \leq A, \quad \xi \in S'_i \ (i = 0, 1). \tag{5.2}$$

Proposition 7 is derived from the following lemma.

Lemma 8 *Let $i \in \{0, 1, 2, \cdots, q\}$. For any n, each function $w_i^n(\xi)$, which is defined by recurrence formula (A) or (B), is analytic in the sector S_i' and there exists a positive constant $B_{i,n}$ such that the following inequalities hold for $\xi \in S_i'$.*
For $i = 1, 2, \cdots, q$,

$$|w_i^n(\xi)| \leq B_{i,n} \frac{|\xi|^n}{\Gamma((1+n)/k_i)} \exp\{\delta|\xi|^{\kappa_i}\}. \tag{5.3}$$

For $i = 0$,

$$|w_0^n(\xi)| \leq B_{0,n} \frac{|\xi|^n}{\Gamma((1+n)/p)} \exp\{\delta|\xi|^{p}\}. \tag{5.4}$$

Moreover, there are some positive constants C and K such that for all n the following inequalities hold. For $i = 2, 3, \cdots, q$, or $i = 1$ and $p > k_1$,

$$B_{i,n} \leq CK^n n!^{1/p}. \tag{5.5}$$

For $i = 1$ and $p = k_1$, or $i = 0$,

$$B_{i,n} \leq CK^n. \tag{5.6}$$

By taking the integral path in the convolution as the segment $[0, \xi]$, we may see that each $w_i^n(\xi)$ is analytic in S_i'. The inequalities (5.3) and (5.4) follow from Lemma 3 2). In fact, we shall prove the inequality (5.3) for the case where $i \geq 2$, or $i = 1$ and $p > k_1$.

We consider the recurrence formula (A).

For $n = 0$, since we have $|w_i^0(\xi)| = (1/|a|)|d/(d\xi)f_i^{[0]}(\xi)| \leq (1/|a|)(L/\Gamma(1/k_i))e^{\delta|\xi|^{\kappa_i}}$, we may put $B_{i,0} = L/|a|$.

We assume that the inequality (5.3) holds up to $n-1$. Then we have from Lemma 3 2)

$$|w_i^n(\xi)| \leq \frac{|\xi|^n}{\Gamma((1+n)/k_i)} e^{\delta|xi|^{\kappa_i}} \frac{L}{|a|} \left\{ \frac{1}{k_i} B_{i,n-1} + \frac{1}{k_i^2} \sum_{\ell+m=n-2} B_{i,\ell} B_{i,m} + \frac{n-p+1}{L} B_{i,n-p} \right\}, \tag{5.7}$$

where $B_{i,-k} = 0$ if $k > 0$. Therefore we may put

$$B_{i,n} = \frac{L}{|a|} \left\{ \frac{1}{k_i} B_{i,n-1} + \frac{1}{k_i^2} \sum_{\ell+m=n-2} B_{i,\ell} B_{i,m} + \frac{n-p+1}{L} B_{i,n-p} \right\}. \tag{5.8}$$

This means that the desired inequality (5.3) is obtained.

In order to prove the inequalities (5.5) and (5.6), we use the majorant method effectively for recurrence formulas which are satisfied by $B_{i,n}$. We only consider the case where $i \geq 2$. For the recurrence formula (5.8), we insert $B_{i,n} = A_n n!^{1/p}$ and divide both hand sides by $n!^{1/p}$.

$$\begin{aligned} A_n &= b_1 \frac{(n-1)!^{1/p}}{n!^{1/p}} A_{n-1} + b_2 \sum_{\ell+m=n-2} A_\ell A_m \frac{\ell!^{1/p} m!^{1/p}}{n!^{1/p}} + b_3(n-p+1) \frac{(n-p)!^{1/p}}{n!^{1/p}} A_{n-p} \\ &\leq b_1 A_{n-1} + b_2 \sum_{\ell+m=n-2} A_\ell A_m + b_3 A_{n-p}, \end{aligned}$$

where $A_{-k} = 0$ if $k > 0$ and b_i are some constants. We consider the following recurrence formula for $\{C_n\}$.

$$C_n = b_1 C_{n-1} + b_2 \sum_{\ell+m=n-2} C_\ell C_m + b_3 C_{n-p}, \quad C_0 = B_{i,0}, \tag{5.9}$$

where $C_{-k} = 0$ if $k > 0$. Then we can see that $C_n \geq A_n = B_{i,n}/n!^{1/p}$ and the generating function of $\{C_n\}$ is holomorphic in a neighborhood of the origin. Therefore we obtain $B_{i,n} \leq C_n n!^{1/p} \leq CK^n n!^{1/p}$ by some positive constants C and K.

Now, we can prove Proposition 7 by Lemma 8 immediately as follows. For $i \geq 2$,

$$|W_i(\xi)| \leq \sum_{n\geq 0} |w_i^n(\xi)| \leq C \sum_{n\geq 0} \frac{(K|\xi|)^n n!^{1/p}}{\Gamma((1+n)/k_i)} e^{\delta|\xi|^{\kappa_i}}. \tag{5.10}$$

Since by Stirling's formula we have

$$\frac{n!^{1/p}}{\Gamma((1+n)/k_i)} \leq \frac{ck^n}{\Gamma(1+n/\tilde{\kappa}_i)},$$

with some positive constants c and k, where $\tilde{\kappa}_i = (k_i^{-1} - p^{-1})^{-1} (\leq \kappa_i)$, we obtain the desired exponential estimate.

$$|W_i(\xi)| \leq \tilde{C} \sum_{n\geq 0} \frac{(\tilde{K}|\xi|)^n}{\Gamma(1+n/\tilde{\kappa}_i)} e^{\delta|\xi|^{\kappa_i}} \leq C_1 \exp(K_1|\xi|^{\tilde{\kappa}_i} + \delta|\xi|^{\kappa_i}). \tag{5.11}$$

We can prove the estimates for the case $i = 0, 1$ similarly. We omit the detail.

Finally, we have to show that $W_i(\xi)$ is a unique analytic solution of (4.4, 4.5)-(i) in $U(r) := U_i' \cap \{|\xi| \leq r\}$, where U_i' is a closed subsector of S_i' for some r small. As we proved Proposition 4, we can prove the uniqueness of solution of (4.4, 4.5)-(i) on $U(r)$ by proving that T_i is a contraction map on a proper Banach space if r is chosen sufficiently small. Hence the proof of Proposition 5 is finished.

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