Polynomial properties on large symmetric association schemes

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Abstract. In this paper we characterize "large" regular graphs using certain entries in the projection matrices onto the eigenspaces of the graph. As a corollary of this result, we show that "large" association schemes become P-polynomial association schemes. Our results are summarized as follows. Let $G = (V, E)$ be a connected k-regular graph with $d + 1$ distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$. Since the diameter of G is at most d, we have the Moore bound

$$|V| \leq M(k, d) = 1 + k \sum_{i=0}^{d-1} (k-1)^i.$$

Note that if $|V| > M(k, d-1)$ holds, the diameter of G is equal to d. Let E_i be the orthogonal projection matrix onto the eigenspace corresponding to θ_i. Let $\partial(u, v)$ be the path distance of $u, v \in V$.

Theorem. *Assume $|V| > M(k, d-1)$ holds. Then for $x, y \in V$ with $\partial(x, y) = d$, the (x, y)-entry of E_i is equal to*

$$-\frac{1}{|V|} \prod_{j=1,2,\ldots,d, j \neq i} \frac{\theta_0 - \theta_j}{\theta_i - \theta_j}.$$

If a symmetric association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ has a relation R_i such that the graph (X, R_i) satisfies the above condition, then $\mathfrak{X}$ is P-polynomial. Moreover we show the "dual" version of this theorem for spherical sets and Q-polynomial association schemes.

1. Introduction

A *symmetric association scheme* of class d is a pair $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$, where X is a finite set and $\{R_i\}_{i=0}^d$ is a set of binary relations on X satisfying

(1) $R_0=\{(x,x)\mid x\in X\}$,
(2) $X\times X=\bigcup_{i=0}^d R_i$, and $R_i\cap R_j$ is empty if $i\neq j$,
(3) ${}^tR_i=R_i$ for any $i\in\{0,1,\ldots,d\}$, where ${}^tR_i=\{(y,x)\mid (x,y)\in R_i\}$,
(4) for any $i,j,k\in\{0,1,\ldots,d\}$, there exists an integer p_{ij}^k such that for any pair $x,y\in X$ with $(x,y)\in R_k$, it holds that $p_{ij}^k=|\{z\in X\mid (x,z)\in R_i,(z,y)\in R_j\}|$.

The i-th *adjacency matrix* A_i of $\mathfrak{X}$ is the matrix indexed by X with the entry

$$(A_i)_{xy}=\begin{cases}1 \text{ if } (x,y)\in R_i,\\ 0 \text{ otherwise.}\end{cases}$$

The *Bose–Mesner algebra* $\mathfrak{A}$ of $\mathfrak{X}$ is the algebra generated by the adjacency matrices $A_0,A_1,\ldots,A_d$ over the complex field $\mathbb{C}$. Then $\{A_i\}_{i=0}^d$ is a natural basis of $\mathfrak{A}$, and $\mathfrak{A}$ is also closed under the Hadamard product, i.e., the entry-wise matrix product. $\mathfrak{A}$ has another remarkable basis which consists of primitive idempotents $E_0,E_1,\ldots,E_d$ [2, Section II.3]. We define $P_i(j)$, $Q_i(j)$ by the following equalities:

$$A_i=\sum_{j=0}^d P_i(j)E_j,\qquad E_i=\frac{1}{|X|}\sum_{j=0}^d Q_i(j)A_j.$$

The values $P_i(j)$, $Q_i(j)$ are called the *parameters* of an association scheme. We use the notation $k_i=P_i(0)$ (degrees), and $m_i=Q_i(0)$ (multiplicities) for $i=0,1,\ldots,d$.

A symmetric association scheme is called a *P-polynomial scheme* with respect to the ordering $A_0,A_1,\ldots,A_d$ if there exists a polynomial v_i of degree i such that $A_i=v_i(A_1)$ for each $i\in\{0,1,\ldots,d\}$. We say a symmetric association scheme is a P-polynomial scheme with respect to A_i if it has the P-polynomial property with respect to some ordering $A_0,A_i,A_{i_2},A_{i_3},\ldots,A_{i_d}$. A symmetric association scheme is called a *Q-polynomial scheme* with respect to the ordering $E_0,E_1,\ldots,E_d$ if there exists a polynomial v_i^* of degree i such that $E_i=v_i^*(E_1^\circ)$ for each $i\in\{0,1,\ldots,d\}$, where $\circ$ means the multiplicity is the Hadamard product. Moreover a symmetric association scheme is called a Q-polynomial scheme with respect to E_i if it has the Q-polynomial property with respect to some ordering $E_0,E_i,E_{i_2},E_{i_3},\ldots,E_{i_d}$.

The P-polynomial schemes and Q-polynomial schemes are interpreted as discrete cases of two-point homogeneous spaces and rank-1 symmetric spaces, respectively [2, Section III.6], [5, Chapter 9]. Wang [16] showed that compact two-point homogeneous spaces are compact rank-1 symmetric spaces and vise versa. Bannai and Ito [2, Section III.6] conjectured that if class d is sufficiently large, a primitive association scheme is P-polynomial if and only if it is Q-polynomial. Here a symmetric association scheme $(X,\{R_i\}_{i=0}^d)$ is said to be *primitive* if the graph (X,R_i) is connected for each $i=1,\ldots,d$. One of the main contributions is a sufficient condition for association schemes to have polynomial properties. Almost characterizations of the polynomial properties are proved by the relationship among the parameters on the scheme, see

[2, 3, 7, 9, 11, 10]. In this paper, we just focus on the size, and prove that a sufficiently large association scheme has the polynomial property.

There are two upper bounds for the size of a symmetric association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$. We have two interpretations of $\mathfrak{X}$ as regular graphs A_i with eigenvalues $\{P_i(j)\}_{j=0}^d$, and spherical sets E_i with inner products $\{Q_i(j)/|X|\}_{j=0}^d$. If A_i is connected, then A_i is of diameter at most d, and we have the Moore bound. Namely if $P_i(0)$ is distinct from $P_i(1), P_i(2), \ldots, P_i(d)$, then we have

$$|X| \leq M(k_i, d) = 1 + k_i \sum_{j=0}^{d-1} (k_i - 1)^j.$$

On the other hand, if the diagonal entries of E_i are distinct from the others, E_i becomes the Gram matrix of some spherical finite set with at most d distances between distinct points. We have an upper bound for the cardinality of a spherical finite set with only s distances [6]. Namely if $Q_i(0)$ is distinct from $Q_i(1), Q_i(2), \ldots, Q_i(d)$, then

$$|X| \leq N(m_i, d) = \binom{m_i + d - 1}{d} + \binom{m_i + d - 2}{d - 1}.$$

In the present paper, we show the following:

(i) If $P_i(0)$ is distinct from $P_i(1), P_i(2), \ldots, P_i(d)$ and $|X| > M(k_i, d-1)$, then $(X, \{R_i\}_{i=0}^d)$ has the P-polynomial property with respect to A_i.

(ii) If $Q_i(0)$ is distinct from $Q_i(1), Q_i(2), \ldots, Q_i(d)$ and $|X| > N(m_i, d-1)$, then $(X, \{R_i\}_{i=0}^d)$ has the Q-polynomial property with respect to E_i.

Though these sufficient conditions are fairly simple, there are many examples including strongly regular graphs, Johnson or Hamming schemes of sufficiently large degree or multiplicity.

2. Regular graphs and P-polynomial schemes

Let $G = (V, E)$ be a connected k-regular graph with at most $d + 1$ distinct eigenvalues, and A the adjacency matrix. Since the diameter of G is at most d, we have the Moore bound

$$|V| \leq M(k, d) = 1 + k \sum_{j=0}^{d-1} (k - 1)^j.$$

Let $k = \theta_0 > \theta_1 > \theta_2 > \cdots > \theta_s$ be distinct eigenvalues of G, where $s \leq d$. Let E_i be the orthogonal projection matrix onto the eigenspace corresponding to θ_i. In particular $E_0 = (1/|V|)J$, where J is the all-ones matrix. Let $\partial(x, y)$ be the path distance of $x \in V$ and $y \in V$. Let $R_i = \{(x, y) \mid x, y \in V, \partial(x, y) = i\}$ and $R_i(x) = \{y \mid y \in V, \partial(x, y) = i\}$. For each $i \in \{1, \ldots, d\}$, we define

$$K_i = \prod_{j=1,2,\ldots,s, j \neq i} \frac{\theta_0 - \theta_j}{\theta_i - \theta_j}.$$

We begin with the following result.

Theorem 2.1. *Let $G = (V, E)$ be a connected k-regular graph with diameter d. If G has precisely $d + 1$ distinct eigenvalues then, for $(x, y) \in R_d$, the (x, y)-entry of E_i is $-K_i/|V|$ for each $i \in \{1, 2, \dots, d\}$.*

Proof. Define

$$f_i(t) = \prod_{j=1,2,\dots,d, j\neq i} \frac{t - \theta_j}{\theta_i - \theta_j}$$

for each $i \in \{1, 2, \dots, d\}$. Then we have

$$f_i(A) = \sum_{j=0}^{d} f_i(\theta_j)E_j = K_iE_0 + E_i.$$

Because the degree of $f_i(t)$ is $d-1$ and G is of diameter d, the (x, y)-entry of $f_i(A)$ is equal to 0 for $(x, y) \in R_d$. Therefore the (x, y)-entry of E_i is equal to $-K_i/|V|$ for each $i \in \{1, 2, \dots, d\}$. □

We now use the Moore bound to show that a large connected regular graph satisfies the assumption of Theorem 2.1.

Theorem 2.2. *Let $G = (V, E)$ be a connected k-regular graph with at most $d + 1$ distinct eigenvalues. Assume $|V| > M(k, d - 1)$ holds. Then we have the following.*

1. *G is of diameter d.*
2. *G has $d + 1$ distinct eigenvalues.*
3. *For $(x, y) \in R_d$, the (x, y)-entry of E_i is $-K_i/|V|$ for each $i \in \{1, \dots, d\}$.*
4. *For each $x \in V$, the number of entries $-K_i/|V|$ in the x-th row of E_i is at least $|V| - M(k, d - 1)$.*

Proof. (1): The diameter of G is at most d because G has at most $d + 1$ distinct eigenvalues. If the diameter of G is smaller than d, then we have $|V| \leq M(k, d - 1)$, a contradiction. Therefore the diameter of G is d.

(2): From (1), G has at least $d+1$ distinct eigenvalues. Therefore G has exactly $d + 1$ distinct eigenvalues.

(3): This follows immediately from (1), (2), and Theorem 2.1.

(4): For each $x \in V$, we have $|\bigcup_{i=0}^{d-1} R_i(x)| \leq M(k, d - 1)$. Therefore $|R_d(x)| \geq |V| - M(k, d - 1)$ holds. This implies that the number of entries $-K_i/|V|$ in the x-th row of E_i is at least $|V| - M(k, d - 1)$. □

We apply the above theorem to symmetric association schemes. First we recall the following easy fact.

Lemma 2.3 ([8, Lemma 3.2, page 229] **or** [11, Lemma 3.2]**).** *Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^{d})$ be a symmetric association scheme of class d. Suppose that $P_j(0)$ is distinct from $P_j(i)$ for $i = 1, \dots, d$. Then (X, R_j) has diameter d if and only if $\mathfrak{X}$ is P-polynomial with respect to A_j*

By Theorem 2.2 and Lemma 2.3, we immediately obtain the following theorem which shows the P-polynomial property of a large association scheme.

Theorem 2.4. *Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme. Suppose $P_j(0)$ is distinct from $P_j(1), \dots, P_j(d)$. If $|X| > M(k_j, d-1)$ holds, then $\mathfrak{X}$ is a P-polynomial association scheme with respect to A_j.*

We also remark that Theorem 2.1, together with Lemma 2.3, generalizes the implication (1) $\Rightarrow$ (2) of the following known characterization of P-polynomial association schemes.

Theorem 2.5 ([11, 12])**.** *Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme of class d. Suppose that $P_j(0), P_j(1), \dots, P_j(d)$ are mutually distinct. Then the following are equivalent:*

(1) *$\mathfrak{X}$ is a P-polynomial association scheme with respect to A_j.*
(2) *There exists $l \in \{0, 1, \dots, d\}$ such that for each $h \in \{1, 2, \dots, d\}$,*

$$\prod_{i=1,2,\dots,d, i\neq h} \frac{P_j(0) - P_j(i)}{P_j(h) - P_j(i)} = -Q_h(l).$$

Moreover if (2) holds, then A_l is the d-th matrix with respect to the resulting polynomial ordering.

Though the condition in Theorem 2.4 is fairly simple, there are many examples.

Example 2.6. Every connected strongly regular graph with v vertices satisfies the assumption in Theorem 2.4 because $v > M(k, 1) = 1 + k$.

Example 2.7. Let G be a connected regular graph of girth g, with $d+1$ distinct eigenvalues, and v vertices, which satisfies $g \geq 2d-1$. It is known that G is distance-regular [14, 1]. We have the lower bound $v \geq M(k, d-1)$ from the assumption of girth [15]. If G attains this bound, then G is a Moore graph with only d distinct eigenvalues. Therefore $v > M(k, d-1)$ holds, and G satisfies the condition in Theorem 2.4. Known examples are listed in [14]. For instance, a Moore graph satisfies $g \geq 2d-1$.

Example 2.8. Infinite families of distance-regular graphs of diameter d with unbounded degree k satisfy the condition in Theorem 2.4 for sufficiently large k. Indeed the number of the vertices can be expressed by the polynomial $\sum_{i=0}^d v_i(k)$ in k of degree d. For example, the Johnson scheme $J(n, 3)$ satisfies the condition for every $n \geq 51$, and the Hamming scheme $H(3, q)$ satisfies the condition for every $q \geq 7$.

3. Spherical sets and Q-polynomial schemes

The dual results of those in Section 2 will be obtained in this section. Let $A(X) = \{\langle x, y\rangle \mid x, y \in X, x \neq y\}$ for X in the unit sphere S^{m-1}, where $\langle x, y\rangle$ is the usual inner product of x and y. Let X be a finite subset in S^{m-1} which satisfies $|A(X)| \leq d$, where d is not as in the previous section.

Let $\theta_1^* > \cdots > \theta_s^*$ be the elements in $A(X)$, and $\theta_0^* = 1$, where $s \le d$ and $\theta_0^* > \theta_1^* > \cdots > \theta_s^*$. Then we have the absolute bound [6]:

$$|X| \le N(m,d) = \binom{m+d-1}{d} + \binom{m+d-2}{d-1}.$$

We can obtain the graph $G_i = (X, R_i)$, where $R_i = \{(x,y) \mid x, y \in X, \langle x,y \rangle = \theta_i^*\}$ for each $i \in \{1,2,\ldots,d\}$. Let A_i be the adjacency matrix of G_i. For each $i \in \{1,2,\ldots,d\}$, we define

$$K_i^* = \prod_{j=1,2,\ldots,s, j\ne i} \frac{\theta_0^* - \theta_j^*}{\theta_i^* - \theta_j^*}.$$

We say an $n \times n$ matrix E is *Schur-connected* if there is a polynomial q such that $q(E^\circ)$ has rank n [8], where $\circ$ means the Hadamard product. Note that the Gram matrix M of a finite set X in S^{m-1} is Schur-connected by taking $q(x)$ as the annihilator $\prod_{\alpha \in A(X)}(x-\alpha)$. The *Schur-diameter* of E is the least integer d such that $q(E^\circ)$ has rank n for some polynomial q of degree d [8]. If the Schur-diameter of M is d, then $|A(X)| \ge d$. Let $P_i(S^{m-1})$ denote the linear space of the restrictions of polynomials of degree at most i, in m variables, to S^{m-1}.

We begin with the following result.

Theorem 3.1. *Let X be a finite set in S^{m-1} which satisfies that the Schur-diameter of the Gram matrix M is equal to d. If $|A(X)| = d$ then $-K_i^*$ is an eigenvalue of A_i for each $i \in \{1,2,\ldots,d\}$. Moreover the multiplicity of $-K_i^*$ is at least $|X| - N(m,d-1)$.*

Proof. Define

$$f_i^*(t) = \prod_{j=1,2,\ldots,d, j\ne i} \frac{t - \theta_j^*}{\theta_i^* - \theta_j^*}$$

for each $i \in \{1,2,\ldots,d\}$. Note that every diagonal entry of M is 1. Then we have

$$f_i^*(M^\circ) = K_i^* I + A_i. \tag{3.1}$$

For each $x \in X$ and m variables $\xi = (\xi_1, \ldots, \xi_m)$, we consider a polynomial $f_i^*(\langle x, \xi \rangle)$. By Lemma 2.2 in [13], the rank of $K_i^* I + A_i = (f_i^*(\langle x,y \rangle))_{x,y \in X}$ is bounded above by $N(m,d-1) = \dim P_{d-1}(S^{m-1})$. Therefore the matrix (3.1) has eigenvalue zero with multiplicity at least $|X| - N(m,d-1)$. Thus A_i has the eigenvalue $-K_i^*$ with multiplicity at least $|X| - N(m,d-1)$. $\square$

We use the absolute bound to show that a large spherical set satisfies the assumption of Theorem 3.1.

Theorem 3.2. *Let X be a finite set in S^{m-1} which satisfies $|A(X)| \le d$. Assume $|X| > N(m,d-1)$ holds. Then we have the following.*

(1) *The Schur-diameter of the Gram matrix M of X is equal to d.*
(2) *X has d inner products, namely $|A(X)| = d$.*
(3) *$-K_i^*$ is an eigenvalue of A_i for each $i \in \{1,2,\ldots,d\}$.*

(4) *The multiplicity of* $-K_i^*$ *is at least* $|X| - N(m, d-1)$.

Proof. (1): Since $|A(X)| \leq d$ holds, the Schur-diameter of M is at most d. If the Schur-diameter of M is less than d, then there exists a polynomial $q(x)$ of degree less than d such that the rank of $q(M^\circ)$ is $|X|$. Since $q(\langle x, \xi \rangle) \in P_{d-1}(S^{m-1})$ for $x \in S^{m-1}$, in variables $\xi = (\xi_1, \ldots, \xi_m)$, we have $|X| \leq N(m, d-1) = \dim P_{d-1}(S^{m-1})$ by Lemma 2.2 in [13], a contradiction. Therefore the Schur-diameter of the Gram matrix M of X is equal to d.

(2): From (1), we have $|A(X)| \geq d$, hence $|A(X)| = d$.

(3),(4): These follow immediately from (1), (2), and Theorem 3.1. □

We apply the above theorems to symmetric association schemes. First we recall the following fact.

Lemma 3.3 ([10, Lemma 2.2])**.** *Let* $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ *be a symmetric association scheme of class d. Suppose that* $Q_j(0)$ *is distinct from* $Q_j(i)$ *for* $i = 1, \ldots, d$. *Then* E_j *has Schur-diameter d if and only if* $\mathfrak{X}$ *is Q-polynomial with respect to* E_j.

By Theorem 3.2 and Lemma 3.3, we immediately obtain the following theorem which shows the Q-polynomial property of a large symmetric association scheme.

Theorem 3.4. *Let* $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ *be a symmetric association scheme. Assume* $Q_j(0)$ *is distinct from* $Q_j(1), \ldots, Q_j(d)$. *If* $|X| > N(m_j, d-1)$, *then* $\mathfrak{X}$ *is a Q-polynomial scheme with respect to* E_j.

We also remark that Theorem 3.1, together with Lemma 3.3, generalizes the implication (1) ⇒ (2) of the following known characterization of Q-polynomial association schemes.

Theorem 3.5 ([10, 12])**.** *Let* $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ *be a symmetric association scheme of class d. Suppose that* $Q_j(0), Q_j(1), \ldots, Q_j(d)$ *are mutually distinct. Then the following are equivalent:*

(1) $\mathfrak{X}$ *is a Q-polynomial association scheme with respect to* E_j.

(2) *There exists* $l \in \{0, 1, \ldots, d\}$ *such that for each* $h \in \{1, 2, \ldots, d\}$,

$$\prod_{i=1,2,\ldots,d, i \neq h} \frac{Q_j(0) - Q_j(i)}{Q_j(h) - Q_j(i)} = -P_h(l).$$

Moreover if (2) holds, then E_l *is the d-th matrix with respect to the resulting polynomial ordering.*

The following are examples satisfying the condition in Theorem 3.4.

Example 3.6. Every connected strongly regular graph with v vertices satisfies the assumption in Theorem 3.4 because $v > N(m_1, 1) = 1 + m_1$.

Example 3.7. Let $X \subset S^{m-1}$ be a spherical t-design [6] with d distances which satisfies $t \geq 2d-2$. Then the set X has the structure of a Q-polynomial association scheme [6]. From the inequality $t \geq 2d-2$, we have $|X| \geq N(m, d-1)$ [6], which is called an absolute bound for spherical designs. If $|X| = N(m, d-1)$ holds, then X has only $d-1$ distances [6]. Thus $|X| > N(m, d-1)$ holds. The association scheme obtained from X satisfies the assumption in Theorem 3.4. Known examples are listed in [4]. For instance, a tight spherical design satisfies $t \geq 2d-2$.

Example 3.8. Infinite families of Q-polynomial association schemes with unbounded multiplicity m_1 satisfy the condition in Theorem 3.4 for sufficiently large m_1. Indeed the number of the vertices can be expressed by the polynomial $\sum_{i=0}^{d} v_i^*(|X|m_1)$ in m_1 of degree d. For example, the Johnson scheme $J(n,3)$ satisfies the condition for every $n \geq 7$, and the Hamming scheme $H(3,q)$ satisfies the condition for every $q \geq 4$.

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