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PanFranklin Squares of Order 6 and 7

Kenzi ODANI

Department of Mathematics Education, Aichi University of Education, Kariya 448-8542, Japan

1 PanFranklin Squares

Benjamin Franklin (1706–1790) was one of the Founding Fathers of the United States, a figure of 100 dollars bill. He is also famous as a scientist who prove that lightning is electricity by flying a kite. He also studied magic squares and left an interesting square to us. See the square shown in the right.

We can see that the sums of every 8 rows and every 8 columns are all equal. The sum comes to 260. Not only the sums of normal rows and columns, but also those of every 16 bent rows and 16 bent columns are all equal to 260. (However, it is not a magic square since the sums of diagonals are not equal to 260.) The meanings of bent rows and columns are as follows: The $\bullet$'s and $*$'s in Figure 2-1 respectively indicate the 1st and 5th downward bent rows, those in Figure 2-2 the 2nd and 6th upward bent rows, those in Figure 2-3 the 4th and 8th rightward bent columns, and those in Figure 2-4 the 3rd and 7th rightward bent columns.

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Figure 1

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Figure 2-1

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Figure 2-2

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Figure 2-3

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Figure 2-4

We call a square matrix of order n a *panFranklin square* if the sums of every n rows, n columns, $2n$ bent rows and $2n$ bent columns are equal. Especially, we call it *natural* if the entries are successive integers from 1 to n^2.

For a square matrix $[a_{ij}]$ of order n, we call it *horizontally and vertically asymmetric* (or *HV asymmetric*) if $a_{n+1-i,j} = -a_{ij}$ and $a_{i,n+1-j} = -a_{ij}$ for every i and j ($1 \leqq i, j \leqq n$). We call a matrix *equal-entries* if every entries are equal. The sum of an equal-entries matrix and an HV-asymmmetric matrix is a panFranklin square. We call such a square a *trivial* panFranklin square. The squares given below are the general forms of trivial panFranklin squares of order 4 and 5.

$$\begin{bmatrix} s+a_{11} & s+a_{12} & s-a_{12} & s-a_{11} \\ s+a_{21} & s+a_{22} & s-a_{22} & s-a_{21} \\ s-a_{21} & s-a_{22} & s+a_{22} & s+a_{21} \\ s-a_{11} & s-a_{12} & s+a_{12} & s+a_{11} \end{bmatrix}, \qquad \begin{bmatrix} s+a_{11} & s+a_{12} & s & s-a_{12} & s-a_{11} \\ s+a_{21} & s+a_{22} & s & s-a_{22} & s-a_{21} \\ s & s & s & s & s \\ s-a_{21} & s-a_{22} & s & s+a_{22} & s+a_{21} \\ s-a_{11} & s-a_{12} & s & s+a_{12} & s+a_{11} \end{bmatrix}.$$

Notice that every trivial panFranklin square has at least one pair of equal entries. For example, the $(1,1)$- and $(4,4)$-entries of a trivial panFranklin square of order 4 are equal, and so are the $(1,1)$- and $(5,5)$-entries of a trivial panFranklin square of order 5. Thus any trivial panFranklin square of order $\geqq 2$ can not be natural.

2 PanFranklin squares of order $\leqq 6$

In [2], Pasles claims that every panFranklin square of order $\leqq 6$ must be trivial, and therefore, that there are no natural panFranklin squares of order $\leqq 6$. However, the author found that his first claim is false for order 6. Since Pasles's second claim is based on the first claim, the case of order 6 of the second claim comes to reasonless. The author gives a proof of the case of order 6 of the second claim.

Theorem 1. (1) *Every panFranklin square of order* $\leqq 5$ *is trivial.*
(2) *There are no natural panFranklin squares of order* $\leqq 6$.

We can prove Theorem 1 (1) by finding a general solution of a system of linear equations. For example, we can prove the case of order 5 by finding a general solution of a system of 30 linear equations of with 25 unknowns. Since every trivial panFranklin squares must have same entries, it can not be a natural panFranklin square. So, we can deduce the cases of order $\leqq 5$ of Theorem 1 (2).

To prove the case of order 6, we prepare the following lemma.

Lemma 2. *Every panFranklin square of order 6 is a sum of a trivial panFranklin square and a square matrix of the following form:*

$$\begin{bmatrix} a+b & -a & a-b & -a & a & -a \\ -b & 0 & b & 0 & 0 & 0 \\ -a+b & a & -a-b & a & -a & a \\ -b & 0 & b & 0 & 0 & 0 \\ b & 0 & -b & 0 & 0 & 0 \\ -b & 0 & b & 0 & 0 & 0 \end{bmatrix}.$$

We can prove Lemma 2 by finding a general solution of a system of 36 linear equations with 36 unknowns. By Lemma 2, every panFranklin square of order 6 must have a pair of equal entries. For example, the $(2,2)$- and $(5,5)$-entries are equal. Therefore, it cannot be a natural panFranklin square. Hence we have proved the case of order 6 of Theorem 1 (2) .

3 PanFranklin square of order 7

As we have seen in Section 1, Franklin made a natural panFranklin square of order 8. On the other hand, in Section 2, we have proved that there are no natural panFranklin squares of order $\leqq 6$. Then the following problem arises.

Problem 3. *Are there natural panFranklin squares of order 7 ?*

To solve the above problem, the author found a general form of panFranklin squares of order 7.

Lemma 4. *Every panFranklin square of order 6 is a sum of a trivial panFranklin square and a square matrix of the following form:*

$$\begin{bmatrix} a+d & g & -g+b+c & -d & -c & -b & -a \\ -g & -a-b-c-d-e-f & -b-c+d+e+f & g-b & b+c & a+b+c & b \\ g+e+f & a+b+c-e-f & b+c+e+f & -g+b-e-f & -b-c & -a-b-c & -b \\ -a & -g-e & g-b-c+e & 0 & c & b & a \\ -f & e+f & -e-f & f & 0 & 0 & 0 \\ -e & d+e+f & -d-e-f & e & 0 & 0 & 0 \\ -d & e & -e & d & 0 & 0 & 0 \end{bmatrix}.$$

We can prove the above lemma by finding a general solution of a system of 42 linear equations with 49 unknowns. By Lemma 4, we can deduce that the $(4,4)$-entry of natural panFranklin squares of order 7 must be equal to 25. However, we leave Problem 3 an open problem.

References

[1] P. C. Pasles, The lost squares of Dr. Franklin, Amer. Math. Monthly **108** (2001), pp.489–511.

[2] —— , A bent for magic, Math. Magazine **79**, No.1 (2006), pp.3–13.

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