Bulletin of Aichi Univ. of Education, 60 (Natural Sciences), pp. 13–21, March, 2011

A Geometric Approach to Beatty Sequences in Higher Dimensions

Yukihiro HASHIMOTO

Department of Mathematics Education, Aichi University of Education, Kariya 448-8542, Japan

1. Introduction

In 1926, the following problem is proposed by Beatty[4], which has roots in astronomy. For a real number r, $[r]$ stands for the greatest integer which does not exceed r, and we put $\{r\} = r - [r]$.

Problem 1.1. *Let $\alpha, \beta > 1$ be irrational numbers with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Show that the sequences $\mathbf{N}_\alpha = \{[n\alpha] \mid n \in \mathbf{N}\}$ and $\mathbf{N}_\beta = \{[n\beta] \mid n \in \mathbf{N}\}$ form a partition of $\mathbf{N}$, that is,*

$$\mathbf{N} = \mathbf{N}_\alpha \cup \mathbf{N}_\beta, \ \mathbf{N}_\alpha \cap \mathbf{N}_\beta = \emptyset.$$

In other words, every natural number q is represented as either $q = [n\alpha]$ or $q = [n\beta]$. A lot of studies on the subject has been done, e.g., see the references in Stolarsky[16].

In this note, we consider the meaning of the equation

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

and give a geometric interpretation of the Beatty sequence, which tempts us to extend Beatty's statement in higher dimensions. Digressing from the original purpose, in the attempt to construct a Beatty-like sequence in 3-dimension, we obtain an asymptotic behaviour of

$$\left\{\frac{q}{\alpha}\right\} + \left\{\frac{q}{\beta}\right\} + \left\{\frac{q}{\gamma}\right\},$$

where α, β, γ are irrational number with $1/\alpha + 1/\beta + 1/\gamma = 1$ and β is sufficiently large (theorem 6.1).

2. Elementary proof

At first, we give an well-known proof for Beatty's original problem as follows.

Proof. Suppose that there exists a natural number $q \notin \mathbf{N}_\alpha \cup \mathbf{N}_\beta$. Then, we take $m, n \in \mathbf{N}$ such that

$$m\alpha < q < q + 1 < (m+1)\alpha \ \text{ and } \ n\beta < q < q + 1 < (n+1)\beta,$$

and thus

$$m + n < q\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = q < (q+1)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = q + 1 < m + n + 2,$$

hence $m + n < q < m + n + 1$, which contradict to the condition $q, m, n \in \mathbf{N}$.

Suppose there exist $m, n \in \mathbf{N}$ such that $q = [m\alpha] = [n\beta]$ holds. Then we have

$$q < m\alpha < q + 1 \ \text{ and } \ q < n\beta < q + 1,$$

hence

$$q = q\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) < m + n < (q+1)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = q + 1,$$

getting a contradiction again. □

3. A proof in terms of Sturmian word

Let $0 < \zeta < 1$ be an irrational number. It is well-known fact that the first difference

$$S(\zeta)_n = [(n+1)\zeta] - [n\zeta]$$

of the sequence $\mathbf{N}_\zeta$ gives a *Sturmian word*

$$S(\zeta) = S(\zeta)_1 S(\zeta)_2 S(\zeta)_3 \cdots$$

on alphabets $\{0,1\}$ (cf.[14]). (Note that $[(n+1)\zeta] - [n\zeta] \in \{0,1\}$ as $0 < \zeta < 1$.) For an infinite word σ on alphabets $\{0,1\}$, let $\overline{\sigma}$ be the *complement* of σ, defined by

$$\overline{\sigma}_n = 1 - \sigma_n.$$

It is known that if σ is Sturmian, then the complement $\overline{\sigma}$ is also Sturmian, and every Sturmian word is represented as a first difference of a sequence $\mathbf{N}_\xi$ of a suitable irrational number $0 < \xi < 1$. Moreover it is shown that $\zeta + \xi = 1$ (cf. Theorem 2.1.13, Corollary 2.2.19 and 2.2.20 of Lothaire[14]). The validity of the equation is equivalent to the Beatty's statement, as is shown below.

Lemma 3.1. *Let $0 < \zeta < 1$ be an irrational number, and put $\xi = 1 - \zeta$. Then for any natural number q, we have*

$$\{q\zeta\} + \{q\xi\} = 1 \quad \textit{and} \quad [q\zeta] + [q\xi] = q - 1.$$

Proof. Note that by the irrationality of ζ and ξ, $0 < \{q\zeta\}, \{q\xi\} < 1$, thus $0 < \{q\zeta\} + \{q\xi\} < 2$. Since $\zeta + \xi = 1$, we have

$$q = q\zeta + q\xi = \{q\zeta\} + \{q\xi\} + [q\zeta] + [q\xi],$$

showing that $\{q\zeta\} + \{q\xi\}$ is a natural number. Hence $\{q\zeta\} + \{q\xi\} = 1$. □

Proposition 3.2. *Let $0 < \zeta < 1$ be an irrational number. Then*

$$\overline{S(\zeta)} = S(1 - \zeta).$$

Proof. Set $\xi = 1 - \zeta$. It follows from Lemma 3.1 that

$$\begin{aligned}\overline{S(\zeta)}_q &= 1 - [(q+1)\zeta] + [q\zeta] \\ &= 1 - (q - [(q+1)\xi]) + (q - 1 - [q\xi]) \\ &= [(q+1)\xi] - [q\xi] = S(\xi)_q.\end{aligned}$$

□

The following key lemma is shown by Stolarsky[16].

Lemma 3.3. *Let $\alpha > 1$ be an irrational number. For any natural number q, there exists a natural number n such that $q = [n\alpha]$ if and only if*

$$S\left(\frac{1}{\alpha}\right)_q = 1.$$

Proof. Suppose that

$$S\left(\frac{1}{\alpha}\right)_q = 1, \quad \text{i.e.,} \quad \left[\frac{q+1}{\alpha}\right] = \left[\frac{q}{\alpha}\right] + 1.$$

Then there exists a natural number n such as

$$\frac{q}{\alpha} < n < \frac{q+1}{\alpha}, \quad \text{that is,} \quad q < n\alpha < q+1,$$

hence $q = [n\alpha]$. Since

$$\frac{q+1}{\alpha} - \frac{q}{\alpha} = \frac{1}{\alpha} < 1,$$

the converse is also shown by reversing the argument above. □

Now we give an answer to Beatty's problem in a more constructive way. Let $\alpha, \beta > 1$ be irrational numbers with $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$ (Here we call (α, β) a *Beatty pair*). By the definition of the complement of an infinite word and proposition 3.2, we see for any natural number q, either

$$S\left(\frac{1}{\alpha}\right)_q = 1 \quad \text{or} \quad S\left(\frac{1}{\beta}\right)_q = \overline{S\left(\frac{1}{\alpha}\right)_q} = 1$$

holds, showing that there exists a natural number n such that either

$$q = [n\alpha] \quad \text{or} \quad q = [n\beta]$$

holds via lemma 3.3, hence the answer.

4. **Geometric interpretation**

There are various kind of studies to extend or generalize the Beatty's problem. Lambek and Moser[13] gave a solution to a quite large class of "Beatty-like" partitions of natural numbers. Angel[1] constructed Beatty-like partitions of natural numbers into more than two subsets. In this section, we propose a geometric interpretation of the Beatty's problem, which may allows us an extension of our argument to higher dimensions, as is discussed in the following section.

For a pair of non-negative integers (m, n), $I(m, n)$ denotes a square in $\mathbf{R}^2$;

$$I(m, n) = \{(x, y) \mid m \le x \le m + 1,\ n \le y \le n + 1\}.$$

For a natural number q, π_q denotes a line $\{(x, y) \mid x + y = q\}$ in $\mathbf{R}^2$. Take a Beatty pair (α, β) and put $\boldsymbol{v} = \left(\dfrac{1}{\alpha}, \dfrac{1}{\beta}\right)$. Then we consider a half line $L_{\boldsymbol{v}} = \{t\boldsymbol{v} \mid t > 0\}$. Note that as (α, β) is a Beatty pair, we see

$$\frac{q}{\alpha} + \frac{q}{\beta} = q,$$

whence $L_{\boldsymbol{v}} \cap \pi_q = \{q\boldsymbol{v}\}$ for each $q \in \mathbf{N}$. Setting

$$\mathcal{I}_{\boldsymbol{v}} = \{(m, n) \in \mathbf{N}^2 \mid q\boldsymbol{v} \in I(m, n)^\circ \text{ for some } q \in \mathbf{N}\},$$

we have a following theorem: a geometric expression of the Beatty's statement.

Theorem 4.1. *The map*

$$\Omega_{\boldsymbol{v}} : \mathbf{N} \to \mathcal{I}_{\boldsymbol{v}}, \qquad \Omega_{\boldsymbol{v}}(q) = (m, n) \ \ \textit{if} \ \ q\boldsymbol{v} \in I(m, n)^\circ$$

is well defined and bijective. Moreover, when $\Omega_{\boldsymbol{v}}(q) = (m, n)$ we have

$$\Omega_{\boldsymbol{v}}(q+1) = \begin{cases} (m+1, n), & \textit{if and only if} \quad S\left(\dfrac{1}{\alpha}\right)_q = 1, \\[2ex] (m, n+1), & \textit{if and only if} \quad S\left(\dfrac{1}{\beta}\right)_q = 1. \end{cases}$$

Proof. Actually putting $m = [q/\alpha]$ and $n = [q/\beta]$, $m < q/\alpha < m + 1$ and $n < q/\beta < n + 1$ hold because of irrationality of α and β, and hence $q\boldsymbol{v} \in I(m, n)^\circ$. Since $I(m, n)^\circ$'s are disjoint, the open square $I(m, n)^\circ$ containing $q\boldsymbol{v}$ is unique, thus the map $\Omega_{\boldsymbol{v}}$ is well defined. As $m + n < x + y < m + n + 2$ whenever $(x, y) \in I(m, n)^\circ$, we see $I(m, n)^\circ \cap \pi_q \ne \emptyset$ if and only if $q = m + n + 1$. Then if we take a pair $(m, n) \in \mathcal{I}_{\boldsymbol{v}}$, $q\boldsymbol{v} \in I(m, n)^\circ$ if and only if $q = m + n + 1$, hence the bijection.

If $\Omega_{\boldsymbol{v}}(q) = (m, n)$ and $\Omega_{\boldsymbol{v}}(q + 1) = (m', n')$, we have $q = m + n + 1$ and $q + 1 = m' + n' + 1$, hence $m' + n' = m + n + 1$. Since $m' - m = [(q + 1)/\alpha] - [q/\alpha] \ge 0$ and $n' - n = [(q + 1)/\beta] - [q/\beta] \ge 0$, either

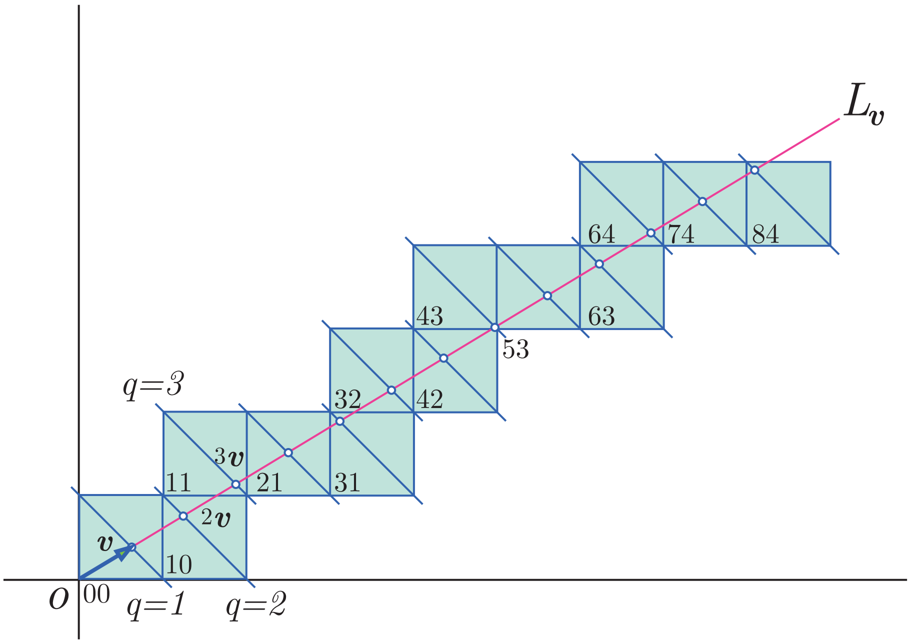

FIGURE 1. A geometric interpretation of the map $\Omega_{\boldsymbol{v}}$.

$m' = m+1$ or $n' = n+1$ occurs. We also see $m' = m+1$ (resp. $n' = n+1$) is equivalent to $S\left(\frac{1}{\alpha}\right)_q = 1$ (resp. $S\left(\frac{1}{\beta}\right)_q = 1$), hence the statement. □

In the figure 1, we see $\Omega_{\boldsymbol{v}}(1) = (0,0)$, $\Omega_{\boldsymbol{v}}(2) = (1,0)$, $\Omega_{\boldsymbol{v}}(3) = (1,1)$ and so on. Since the line $L_{\boldsymbol{v}}$ avoids integer points, a square that the line crosses should face next one with an edge, not a vertex, which indicates theorem 4.1.

5. An attempt to high dimensions

The essence of Theorem 4.1 is the following. Given a point $\boldsymbol{v} \in \mathbf{R}^d \setminus \{\mathbf{0}\}$, let $\{\pi_q\}$ be a one parameter family of hyperplanes disjoint each other in $\mathbf{R}^d$, and a line $L_{\boldsymbol{v}} = \{t\boldsymbol{v} \mid t \in \mathbf{R}\}$ transverse to π_q's, crossing π_q at $q\boldsymbol{v}$ for every $q \in \mathbf{Z}$. (more generally, one can consider a codimension-one foliation and a curve transverse to the foliation). Then, choose a lattice $\Lambda \subset \mathbf{R}^d$ with basis $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_d\}$, which fulfill the following conditions:

(1) Let Λ_π be a sublattice generated by $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_{d-1}\}$. Then π_q is invariant under the action of Λ_π for each $q \in \mathbf{Z}$, and $\pi_q + \boldsymbol{z}_d = \pi_{q+1}$.

(2) There exists a tiling $\mathcal{T}_q$ on each π_q (preferably a regular tiling), compatible with the Λ-action; for any two tiles $T \in \mathcal{T}_q$ and $T' \in \mathcal{T}_{q'}$, there exists a lattice point $\boldsymbol{w} \in \Lambda_\pi$ such that $T' = T + \boldsymbol{w} + (q'-q)\boldsymbol{z}_d$, and $T + \boldsymbol{w} \in \bigcup_{q\in\mathbf{Z}} \mathcal{T}_q$ for any tile $T \in \bigcup_{q\in\mathbf{Z}} \mathcal{T}_q$ and $\boldsymbol{w} \in \Lambda$.

(3) There exists a coding of tiles into Λ, that is, an injection $\Gamma : \bigcup_{q\in\mathbf{Z}} \mathcal{T}_q \to \Lambda$ compatible with the Λ-action;

$$\Gamma(T + \boldsymbol{w}) = \Gamma(T) + \boldsymbol{w}$$

holds for any tile $T \in \bigcup_{q\in\mathbf{Z}} \mathcal{T}_q$ and $\boldsymbol{w} \in \Lambda$.

(4) The crossing point $q\boldsymbol{v} \in \pi_q \cap L_{\boldsymbol{v}}$ is contained in the interior of a unique tile $T_q \in \bigcup_{p\in\mathbf{Z}} \mathcal{T}_p$ for every $q \in \mathbf{Z}$.

It comes from conditions (3) and (4) that we can define an injective map

$$\Omega_{\boldsymbol{v}} : \mathbf{N} \ni q \mapsto \Gamma(T_q) \in \Lambda.$$

Let $E = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_d\}$ be a set of basis of Λ, which may differ from the original one $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_d\}$, and $(p_1, p_2, \ldots, p_d)_E$ be an abbreviation of $\boldsymbol{p} = p_1\boldsymbol{e}_1 + p_2\boldsymbol{e}_2 + \cdots + p_d\boldsymbol{e}_d \in \Lambda$. Then we define a *Hamming distance* d_E in Λ with respect to E by

$$d_E(\boldsymbol{p}, \boldsymbol{q}) = \#\{i = 1, 2, \ldots, d \mid p_i \neq q_i\}$$

for $\boldsymbol{p} = (p_1, p_2, \ldots, p_d)_E$ and $\boldsymbol{q} = (q_1, q_2, \ldots, q_d)_E$. Here we come to a geometric reconstruction of the "Beatty-like" partition of natural numbers:

Definition 5.1. *The map $\Omega_{\boldsymbol{v}}$ is called* Beatty *with respect to basis E if*

$$d_E(\Omega_{\boldsymbol{v}}(q), \Omega_{\boldsymbol{v}}(q+1)) = 1$$

holds for every $q \in \mathbf{Z}$.

Indeed, whenever $\Omega_{\boldsymbol{v}}$ is Beatty, the subsets

$$\mathbf{N}_i = \{q \in \mathbf{N} \mid i\text{-'s coordinate of } \Omega_{\boldsymbol{v}}(q) \text{ and } \Omega_{\boldsymbol{v}}(q+1) \text{ with respect to } E \text{ are different}\}$$

give a partition of natural numbers into d parts.

5.1. **On the original Beatty sequence.** Let (α, β) be a Beatty pair in section 3 and set $\boldsymbol{v} = \left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$. For any $q \in \mathbf{Z}$, we put $\pi_q = \{(x, y) \in \mathbf{R} \mid x + y = q\}$. Then we see $L_{\boldsymbol{v}} \cap \pi_q = \{q\boldsymbol{v}\}$. We take the lattice $\Lambda = \mathbf{Z}\boldsymbol{e}_1 \oplus \mathbf{Z}\boldsymbol{e}_2$ with basis $E = \{\boldsymbol{e}_1 = (1, 0),\ \boldsymbol{e}_2 = (0, 1)\}$. Each π_q is invariant under the action of the sublattice $\Lambda_\pi = \mathbf{Z}\boldsymbol{z}_1$, and $\pi_q + \boldsymbol{z}_2 = \pi_{q+1}$, where we put $\boldsymbol{z}_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{z}_2 = \boldsymbol{e}_2$. The tiles of the tiling $\mathcal{T}_q$ on π_q are given as the line segments $T(m, q-m-1)$ connecting $(m, q-m)$ and $(m+1, q-m-1)$ ($m \in \mathbf{Z}$). In terms of section 4, $T(m, q-m-1) = I(m, q-m-1) \cap \pi_q$. As

$$T(m', q'-m') = T(m, q-m) + (m'-m)\boldsymbol{z}_1 + (q'-q)\boldsymbol{z}_2$$

and $T(m, n) + (a, b)_E = T(m+a, n+b)$, the tiling is compatible with the Λ-action. We define the coding map Γ with respect to E by $\Gamma(T(m, n)) = (m, n)_E$. Obviously we see

$$\Gamma(T(m, n) + (a, b)_E) = \Gamma(T(m+a, b+n)) = (m, n)_E + (a, b)_E = \Gamma(T(m, n)) + (a, b)_E,$$

hence Γ is compatible with the Λ-action. The irrationality of α and β shows $q\boldsymbol{v} \notin \mathbf{Z}^2$ for any $q \in \mathbf{Z} \setminus \{0\}$. Moreover we see

$$q\boldsymbol{v} \in T\left(\left[\frac{q}{\alpha}\right], \left[\frac{q}{\beta}\right]\right)^\circ,$$

then $\Omega_{\boldsymbol{v}}$ is defined by

$$\Omega_{\boldsymbol{v}}(q) = \left(\left[\frac{q}{\alpha}\right], \left[\frac{q}{\beta}\right]\right),$$

coincident with the map $\Omega_{\boldsymbol{v}}$ in theorem 4.1, which shows $\Omega_{\boldsymbol{v}}$ is Beatty.

Note that the orthogonal projection of unit squares $\{I(m, q-m-1)\}$ onto π_q induces the tiling $\mathcal{T}_q$, which gives an implication for the extension to higher dimensions.

5.2. **An attempt in 3-dimension.** We try to execute the geometric idea stated above in 3-dimension, taking standard lattice $\mathbf{Z}^3$. Let $\alpha, \beta, \gamma > 0$ be irrational numbers such that any two of their reciplocals are rational independent, satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.$$

Following the case of 2-dimension, we call the triplet (α, β, γ) *Beatty*, and we put $\boldsymbol{v} = \left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$. The family of planes in $\mathbf{R}^3$ are given by $\pi_q = \{(x, y, z) \in \mathbf{R}^3 \mid x + y + z = q\}$. Then the line $L_{\boldsymbol{v}} = \{t\boldsymbol{v} \mid t \in \mathbf{R}\}$ crosses each π_q at $q\boldsymbol{v}$. We adopt the lattice

$$\Lambda = \mathbf{Z}\boldsymbol{e}_1 \oplus \mathbf{Z}\boldsymbol{e}_2 \oplus \mathbf{Z}\boldsymbol{e}_3$$

with basis $E = \{\boldsymbol{e}_1 = (1, 0, 0),\ \boldsymbol{e}_2 = (0, 1, 0),\ \boldsymbol{e}_3 = (0, 0, 1)\}$. Putting $\boldsymbol{z}_1 = \boldsymbol{e}_1 - \boldsymbol{e}_3$, $\boldsymbol{z}_2 = \boldsymbol{e}_2 - \boldsymbol{e}_3$ and $\boldsymbol{z}_3 = \boldsymbol{e}_3$, the sublattice $\Lambda_\pi = \mathbf{Z}\boldsymbol{z}_1 \oplus \mathbf{Z}\boldsymbol{z}_2$ conserves each π_q, and $\pi_q + \boldsymbol{z}_3 = \pi_{q+1}$.

The tiling $\mathcal{T}_q$ is obtained as follows. For each lattice point $(a, b, c)_E \in \Lambda$ with $a + b + c = q - 1$, we define a tile $T(a, b, c) \in \mathcal{T}_q$ as an orthogonal projection of the cube

$$I(a, b, c) = \{(a + x, b + y, c + z) \in \mathbf{R}^3 \mid (x, y, z) \in [0, 1]^3\}$$

onto π_q, which is nothing but a hexagon tile.

Lemma 5.2. *The set*

$$\mathcal{T}_q = \{T(a, b, c) \mid a, b, c \in \mathbf{Z} \text{ and } a + b + c = q - 1\}$$

is a tiling on π_q compatible with the Λ-action.

Proof. Suppose two tiles $T(a, b, c)$ and $T(a', b', c')$ in $\mathcal{T}_q$ have a common internal point (x, y, z). Then there are $s > 0$ and $t > 0$ such that $(x + s, y + s, z + s) \in I(a, b, c)^\circ$ and $(x + t, y + t, z + t) \in I(a', b', c')^\circ$ hold. Suppose $s < t$, then $a < x + s < x + t < a' + 1$, hence $a \leq a'$. Similarly $b \leq b'$ and $c \leq c'$. Then $a' + b' + c' > a + b + c$ whenever $(a', b', c') \neq (a, b, c)$, which contradict to $a + b + c = a' + b' + c' = q - 1$. Thus $T(a, b, c)^\circ$'s are disjoint each other.

For any point $(x, y, z) \in \pi_q$, it is seen that $(x, y, z) \in I([x], [y], [z])$, and that, say when $x \in \mathbf{Z}$, $(x, y, z) \in I([x], [y], [z]) \cap I([x] - 1, [y], [z])$ holds. Putting $\rho = \{x\} + \{y\} + \{z\}$, we see $\rho = q - ([x] + [y] + [z]) \in \mathbf{Z}$ and $0 \leq \rho < 3$, hence $\rho = 0, 1, 2$. $\rho = 0$ means $x, y, z \in \mathbf{Z}$, then (x, y, z) is also contained in $I([x] - 1, [y], [z])$ with $[x] - 1 + [y] + [z] = q - 1$. $I([x], [y], [z])$ itself satisfies $[x] + [y] + [z] = q - 1$ when $\rho = 1$. In the case of $\rho = 2$, suppose $x \geq y \geq z$. Then we can take t such that $x + t = [x] + 1$, $y + t \leq [y] + 1$ and $z + t \leq [z] + 1$, hence $(x + t, y + t, z + t)$ is contained in $I([x] + 1, [y], [z])$ with $[x] + 1 + [y] + [z] = q - 1$. Consequently, $\mathcal{T}_q$ covers π_q. Therefore $\mathcal{T}_q$ is a tiling on π_q. The compatibility comes from the definition of $\mathcal{T}_q$. □

The coding $\Gamma : \bigcup_{q \in \mathbf{Z}} \mathcal{T}_q \to \Lambda$ is given by

$$\Gamma(T(a, b, c)) = (a, b, c)_E,$$

compatible with the Λ-action obviously. By the assumption of the Beatty triplet, $q\boldsymbol{v}$ is contained in the interior of a tile. Then we can define injective map $\Omega_{\boldsymbol{v}} : \mathbf{N} \to \Lambda$ by

$$\Omega_{\boldsymbol{v}}(q) = (a, b, c)_E, \tag{5.1}$$

where $q\boldsymbol{v} \in T(a, b, c)^\circ$ with $a + b + c = q - 1$.

Unfortunately one will see that $\Omega_{\boldsymbol{v}}$ is not Beatty. For sufficiently large q, there are tiles $T(a, b, c) \in \mathcal{T}_q$ and $T(a - 1, b + 1, c + 1) \in \mathcal{T}_{q+1}$ with $b > c > a \geq 1$, which are connected at one vertex only. As is mentioned in the next section, the sequence $\{(\{q/\alpha\}, \{q/\beta\}, \{q/\gamma\})\}$ is densely distributed on a tile $T(0, 0, 0)$. Then

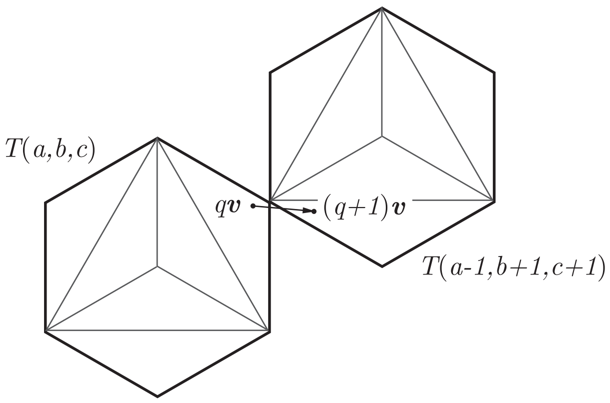

FIGURE 2. The case of the hamming distance 3.

for any Beatty triplet (α, β, γ) with $\beta < \gamma < \alpha$, one finds a number q such that $q\boldsymbol{v} \in T(a, b, c)$ and $(q+1)\boldsymbol{v} \in T(a-1, b+1, c+1)$, hence

$$d_E(\Omega_{\boldsymbol{v}}(q), \Omega_{\boldsymbol{v}}(q+1)) = 3$$

(see figure 2).

6. An asymptotic behaviour

Let (α, β, γ) be a Beatty triplet. We observe an asymptotic behaviour of $\rho_q = \{q/\alpha\} + \{q/\beta\} + \{q/\gamma\}$ when $\beta \to \infty$. We have seen that $\rho_q = 1$ or 2 in the previous section. As α and β are rationally independent, it is shown that the sequence $\{(q/\alpha, q/\beta)\}_{q\in\mathbf{N}}$ is uniformly distributed in $[0,1]^2$ by Weyl's theorem[17][10]. Therefore

$$\lim_{N\to\infty} \frac{\#\{1 \le q \le N \mid \rho_q = 1\}}{N} = \lim_{N\to\infty} \frac{\#\{1 \le q \le N \mid \rho_q = 2\}}{N} = \frac{1}{2}$$

holds. Moreover we see the following asymptotic behaviour.

Theorem 6.1. *Put*

$$x_n = \frac{1}{N}\#\{1 \le q \le n \mid \rho_q = 1\} \textit{ and } y_n = \frac{1}{N}\#\{1 \le q \le n \mid \rho_q = 2\},$$

where $N = [\beta]$. *If* β *is sufficiently large,* $x = x_n - \frac{l}{2}$ *and* $y = y_n - \frac{l}{2}$ *satisfy*

$$y^2 + 2(x-1)y + x^2 = 0, \quad 0 \le x \le \frac{1}{2} \tag{6.1}$$

asymptotically for $[l\beta] < n \le [(l+1)\beta]$ *and* $l = 0, 1, 2 \ldots$.

Proof. As $\rho_q = 1$ or 2, $\{q/\alpha\} + \{q/\beta\} > 1$ implies $\rho_q = 2$. Conversely, if $\rho_q = 2$, then $2 - \{q/\alpha\} - \{q/\beta\} = \{q/\gamma\} < 1$, hence $\{q/\alpha\} + \{q/\beta\} > 1$. Thus $\rho_q = 2$ if and only if $\{q/\alpha\} + \{q/\beta\} > 1$. According to Weyl's theorem, (or a direct observation of the 1-dimensional irrational rotation dynamics[12]), we can take $\{q/\alpha\}$ for a random variable X distributed as uniformly on $[0,1]$. In other words, for each q, we give the probability (or *weight*) $Prob(X < c) = c$ to the event $\{q/\alpha\} < c$. Thus for each $q \le N$, the event $\{q/\alpha\} > 1 - \{q/\beta\} = 1 - q/\beta$ has the probability

$$Prob\left(X > 1 - \frac{q}{\beta}\right) = \frac{q}{\beta},$$

and hence we have

$$\begin{aligned} y_n &= \frac{1}{N}\#\{1 \le q \le n \mid \rho_q = 2\} \\ &\sim \frac{1}{N}\sum_{q=1}^{n} Prob\left(X > 1 - \frac{q}{\beta}\right) = \frac{1}{N}\sum_{q=1}^{n}\frac{q}{\beta} = \frac{n(n+1)}{2\beta N}, \end{aligned} \tag{6.2}$$

when $N = [\beta]$ is sufficiently large. As $x_n + y_n = n/N$, we also have

$$x_n \sim \frac{n}{N} - \frac{n(n+1)}{2\beta N}.$$

Putting $t = n/N \sim (n+1)/\beta$, we come to

$$x_n \sim t - \frac{t^2}{2} \quad \text{and} \quad y_n \sim \frac{t^2}{2}.$$

By eliminating the variable t, we obtain (6.1). For the general $[k\beta] < q \le [(k+1)\beta]$, we just modify the probability of the event $\{q/\alpha\} > 1 - \{q/\beta\}$ as

$$Prob\left(X > 1 - \left\{\frac{q}{\beta}\right\}\right) = \frac{q}{\beta} - \left[\frac{q}{\beta}\right].$$

Since $k\beta < q < (k+1)\beta$, we have $[q/\beta] = k$. Thus for $[l\beta] < n \le [(l+1)\beta]$, we have

$$\begin{aligned} \frac{1}{N}\sum_{q=1}^{n}\left[\frac{q}{\beta}\right] &= \frac{1}{N}\left(\sum_{q=[l\beta]+1}^{n} l + \sum_{k=0}^{l-1}\sum_{q=[k\beta]+1}^{[(k+1)\beta]} k\right) \\ &= \frac{n-[l\beta]}{N}l + \frac{[l\beta]}{N}(l-1) - \sum_{k=1}^{l-1}\frac{[k\beta]}{N} = \frac{n}{N}l - \sum_{k=1}^{l}\frac{[k\beta]}{N} \\ &\sim lt - \sum_{k=1}^{l} k = lt - \frac{1}{2}l(l+1). \end{aligned}$$

Hence

$$y_n \sim \frac{1}{2}t^2 - lt + \frac{1}{2}l(l+1) = \frac{1}{2}(t-l)^2 + \frac{l}{2}.$$

As $x_n + y_n = t$, we have

$$y_n - \frac{l}{2} \sim \frac{1}{2}(x_n + y_n - l)^2 = \frac{1}{2}\left(x_n - \frac{l}{2} + y_n - \frac{l}{2}\right)^2.$$

Putting $x = x_n - l/2$ and $y = y_n - l/2$, we obtain (6.1). □

Note that the accuracy of (6.2) depends on the discrepancy of the sequence $\{q/\alpha\}$. The figure 3 illustrates the asymptotic behaviour of (x_n, y_n), where we take $\alpha = \sqrt{2}$ and $\beta = \sqrt{100001}$.

7. **Concluding remarks**

This study is motivated by an attempt to extend the previous study[12] to higher dimensions; a renormalization approach to the irrational rotation dynamics on $\mathbf{R}^d/\mathbf{Z}^d$, where we are to seek a natural generalization of Sturmian words over more than two alphabets, or a natural extension of the continued fractional expansion in high dimensions. On the other hand, the complexity of words generated by the billiard map in a cube has been analyzed in the pioneering research by Arnoux, Mauduit, Shiokawa and Tamura[2][3]. There are large amount of researches on combinatorics of words associated with billiard maps in high-dimensional polyhedra, called *billiard words*, e.g. Bedaride[5][6], Borel[9], where the words appear as cutting sequences, different from our approach. Finally we remark that, as another direction of generalization of the Beatty sequence, Berthé and Vuillon[7] has proposed the two-dimensional Beatty sequence, given as a double sequence. They define it by approximation of a plane by the integer lattice $\mathbf{Z}^3$ instead of our line $L_{\boldsymbol{v}}$.

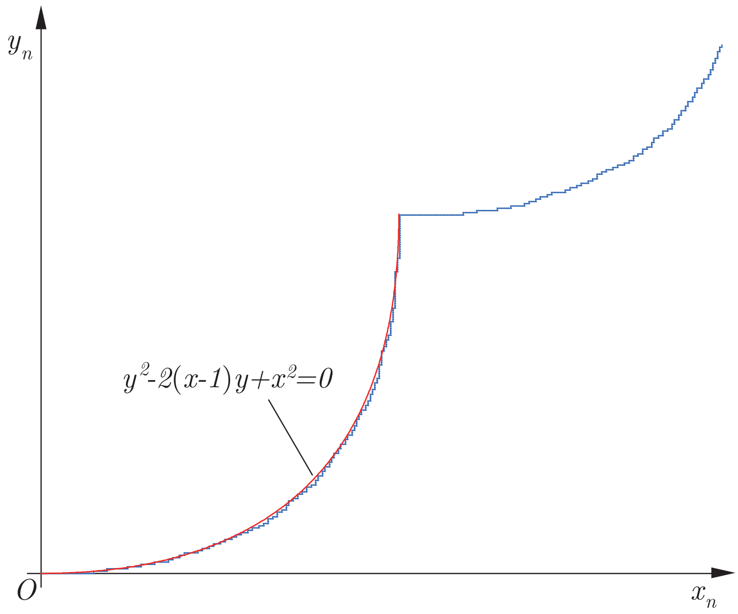

FIGURE 3. The behaviour of (x_n, y_n) and its asymptotic curve for $\alpha = \sqrt{2}$ and $\beta = \sqrt{100001}$.

REFERENCES

[1] M. Angel, *Partitions of the natural numbers*, Canad. Math. Bull. **7**, pp 219-236, 1964.

[2] P. Arnoux, C. Mauduit, I. Shiokawa and J. Tamura, *Complexity of sequences defined by billiard in the cube*, Bull. Soc. Math. France **122**, pp 1-12, 1994.

[3] P. Arnoux, C. Mauduit, I. Shiokawa and J. Tamura, *Rausy's conjecture on billiards in the cube*, Tokyo J. Math. **17**, pp 211-218, 1994.

[4] S. Beatty, *Problem 3173*, Amer. Math. Monthly **33**, pp 159, 1926.

[5] N. Bedaride, *Billiard complexity in rational polyhedra*, Regular and chaotic dynamics **8**, pp 97-104, 2002.

[6] N. Bedaride, *Directional complexity of the hypercubic billiard*, Discr. Math. **309**, pp 2053-2066, 2009.

[7] V. Berthé and L. Vuillon, *Tilings and rotations on the torus: a two-dimensional generalization of Sturmian sequences*, Discr. Math. **223**, pp 27-53, 2000.

[8] P. Bleher, *The energy level spacing for two harmonic oscillators with golden mean ratio of frequencies,* J. Stat. Phys. **61**, pp 869-876, 1990.

[9] J. P. Borel, *A geometrical characterization of factors of multidimensional Billiard words and some applications,* Theor. Comp. Sci. **380**, pp 286-303, 2007.

[10] J. Cassels, *An Introduction to Diophantine Approximation*, Cambridge, 1957.

[11] K. Dajani and C. Kraaikamp, *Ergodic theory of numbers*, Carus Mathematical Monographs **29**, Mathematical Association of America, 2002.

[12] Y. Hashimoto, *A renormalization approach to level statistics on 1-dimensional rotations,* Bull. of Aichi Univ. of Education, Natural Science **58**, pp 5-11, 2009.

[13] J. Lambek and L. Moser, *Inverse and complementary sequence of natural numbers*, Amer. Math. Monthly **61**, pp 454-458, 1954.

[14] M. Lothaire, *Algebraic combinatorics on words*, Encyclopedia of mathematics and its applications **90**, Cambridge University Press, 2002.

[15] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8**, pp 477-493, 1957.

[16] K. Stolarsky, *Beatty sequences, continued fractions, and certain shift operators*, Canad. Math. Bull. **19**, pp 473-482, 1976.

[17] H. Weyl, *Über die Gleichverteilungv on Zahlen modulo Eins*, Math. Ann. **77**, pp 313-352, 1916.

(Received September 14, 2010)