Bulletin of Aichi Univ. of Education, 60（Natural Sciences）, pp. 23–28, March, 2011

A Remark on Cyclic Vector for Singular System of Linear Ordinary Diffrential Equations

Masatake MIYAKE*, Kunio ICHINOBE** and Hiroshi SUZUKI*

** Graduate School of Mathematics, Nagoya University, Nagoya 464 - 8602, Japan*

*** Department of Mathematics Education, Aichi University of Education, Kariya 448-8542, Japan*

Abstract

It is known that a singular system of linear ordinary differential equations in a complex plane is reducible into a single equation by a meromorphic transformation of unknown functions by using Deligne's cyclic vector. In this paper we characterize the case when it is possible by a holomorphic transformation.

1 Introduction

Let $\mathbb{C}$ be the set of complex numbers or the complex variable z. We denote by $\mathbb{C}\{z\}$ and $K\{z\}$ the set of holomorphic functions and meromorphic functions at the origin of $\mathbb{C}$, respectively.

Let a singular N-system at $z = 0$ of linear ordinary differential equations be given by

$$L\left(z, \frac{d}{dz}\right)\vec{u}(z) \equiv z\frac{d}{dz}\vec{u}(z) - A(z)\vec{u}(z) = 0, \tag{1.1}$$

where $A(z) = (a_{ij}(z)) \in M_N(\mathbb{C}\{z\})$, the set of $N \times N$ matrices with entries in $\mathbb{C}\{z\}$. Here we assume that $A(0) \neq 0$, which means the system (1.1) is a singular system of first kind.

By using Deligne's cyclic vector [Del, Lemma 1.3], we can find a matrix $P(z) \in GL_N(K\{z\})$, the set of invertible matrices with entries in $K\{z\}$, such that by a change of unknown functions $\vec{u}(z) = P(z)\vec{v}(z)$ the system (1.1) is reduced into the following form

$$z\frac{d}{dz}\vec{v}(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ a_N(z) & a_{N-1}(z) & \cdots & a_2(z) & a_1(z) \end{pmatrix}\vec{v}(z), \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}. \tag{1.2}$$

We call such a matrix $P(z)$ a transformation matrix.

We note that this system is equivalent with the following single equation for the first component $v_1(z)$ of $\vec{v}(z)$,

$$\delta^N v_1(z) = \sum_{j=1}^{N} a_j(z)\delta^{N-j}v_1(z), \quad \delta = z\frac{d}{dz}. \tag{1.3}$$

Such a reduction of a system into a single equation makes it possible to convert the study of properties of solutions for systems to those for single equations (cf. [Mal], [Sib] and [Hsi-Sib]).

For example we note that, by the above change of the unknown functions, in the reduced system (1.2) the coefficients $\{a_j(z)\}$ are expected to be meromorphic functions, but the fact that the system (1.1) is regular singular at $z = 0$ and hence a single equation (1.3) is regular singular at $z = 0$ implies that $\{a_j(z)\}$ are all holomorphic at $z = 0$ (compare the characterization of regular singularity for single equations in [Cod-Lev, Chapter 4, Theorems 5.1 and 5.2] and for systems in [Kit]). Whereas the transformation matrix $P(z)$ can not be taken in $GL_N(\mathbb{C}\{z\})$, in general (cf. [Hsi-Sib, Lemma XIII, 5-1]). By their proof we know that the assumption $A(z) \in M_N(\mathbb{C}\{z\})$ can be replaced by $A(z) \in M_N(K\{z\})$ to obtain a transformation matrix $P(z) \in GL_N(K\{z\})$.

In this paper we characterize the case when the transformation matrix $P(z)$ is taken in $GL_N(\mathbb{C}\{z\})$, which is stated as follows.

Theorem 1. *In order that the transformation matrix $P(z)$ is taken in $GL_N(\mathbb{C}\{z\})$ in the above reduction, it is necessary and sufficient that the eigenspaces of $A(0)$ are all one dimension, that is, the minimal polynomial of $A(0)$ is nothing but the characteristic polynomial of $A(0)$.*

Before we proceed to the proof, we give a brief summary for construction of the transformation matrix $P(z)$. Let

$$\mathcal{L}\left(z, \frac{d}{dz}\right) = z\frac{d}{dz} + {}^{\mathrm{t}}A(z)$$

be the transposed operator of $L(z, d/dz)$. A vector function $\vec{q}(z) \in K\{z\}^N$ is called a cyclic vector of $\mathcal{L}(z, d/dz)$ if

$$(1.4) \qquad Q(z) := (\vec{q}(z), \mathcal{L}\vec{q}(z), \cdots, \mathcal{L}^{N-1}\vec{q}(z)) \in GL_N(K\{z\}),$$

where $(\vec{a}_1, \cdots, \vec{a}_N)$ denotes the matrix with j-th column vector $\vec{a}_j$. The existence of a cyclic vector $\vec{q}(z) \in K\{z\}^N$ or $Q(z) \in GL_N(K\{z\})$ is assured for every $\mathcal{L}(z, d/dz)$ even when $A(z) \in M_N(K\{z\})$ by P. Deligne [Del] (cf. [Hsi-Sib]). Then the transformation matrix $P(z)$ is obtained by $P(z) = {}^{\mathrm{t}}Q(z)$.

Our purpose is to find a condition under which we assure that $P(z) \in GL_N(\mathbb{C}\{z\})$. This means that $Q(0) \in GL_N(\mathbb{C})$. By restricting the relation (1.4) at $z = 0$, we get

$$(1.5) \qquad Q(0) = (\vec{q}(0), {}^{\mathrm{t}}A(0)\vec{q}(0), \cdots, {}^{\mathrm{t}}A(0)^{N-1}\vec{q}(0)) \in GL_N(\mathbb{C}).$$

Conversely, if a vector $\vec{q}_0 \in \mathbb{C}^N$ satisfies this relation we see that $\vec{q}_0$ is also a cyclic vector of $\mathcal{L}(z, d/dz)$. We call again such a constant vector $\vec{q}_0$ a cyclic vector of ${}^{\mathrm{t}}A(0)$.

Thus, Theorem 1 is reduced to the following,

Theorem 2. *For a constant matrix $A \in M_N(\mathbb{C})$, a cyclic vector $\vec{p} \in \mathbb{C}^N$ exists if and only if all eigenspaces of A are one dimension.*

Theorem 2 is proved only by an elementary theory of matrices, but in the procedure we meet a generalized form of Vandermonde's determinant which seems to be interesting.

2 Proof of Theorem 2

For disjoint eigenvalues $\{\lambda_j\}_{j=1}^k$ of A, we denote each eigenspace by $E(\lambda_j)$ and we put $n_j = \dim E(\lambda_j)$.

Proof of Necessity.

The condition that $\dim E(\lambda_j) = 1$ for all j is equivalent that the Jordan canonical form of A has only one Jordan cell $J(\lambda_j)$ for each eigenvalue λ_j. We take the Jordan canonical form of the form.

$$A \sim \mathrm{Diag}\,[J(\lambda_1), \cdots, J(\lambda_k)], \quad J(\lambda_j) = \begin{pmatrix} \lambda_j & 0 & 0 & \cdots & 0 \\ 1 & \lambda_j & 0 & \cdots & 0 \\ 0 & 1 & \lambda_j & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_j \end{pmatrix}, \tag{2.1}$$

where $A \sim B$ means the equivalence of two matrices A and B, and $\mathrm{Diag}[J_1, \cdots, J_k]$ denotes the block diagonal matrix with i-th diagonal block J_i.

We show that it is impossible to exist a cyclic vector $\vec{p} \in \mathbb{C}^N$ if $\dim E(\lambda_{j_0}) \geq 2$ for $\exists j_0$. Say $j_0 = 1$. We assume that A is already a Jordan canonical form and $\{J_i(\lambda_1)\}_{i=1}^{\ell}$ $(\ell \geq 2)$ are the Jordan cells associated with λ_1. We put $J_i(\lambda_1) \in M_{m_i}(\mathbb{C})$ $(m_1 + \cdots + m_\ell \leq N)$. For $\vec{p} = {}^{t}(x_1, x_2, \cdots, x_N) \in \mathbb{C}^N$ we put

$$A^k \vec{p} = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_N^{(k)} \end{pmatrix} \quad (k \geq 0).$$

Then we have $x_1^{(k)} = x_1 \lambda_1^k$, $x_{m_1+1}^{(k)} = x_{m_1+1}\lambda_1^k$, $\cdots$ $(\forall k \geq 0)$. This shows that $\det(\vec{p}, A\vec{p}, \cdots, A^{N-1}\vec{p}) \neq 0$ is impossible for every $\vec{p} \in \mathbb{C}^N$, since the first row vector and the (m_1+1)-th row vector are parallel. (Q.E.D.)

Proof of Sufficiency.

We may assume that the matrix A is already a Jordan canonical form (2.1). Then we shall prove that a vector $\vec{p}$ given below becomes a cyclic vector of A,

$$\vec{p} = \vec{e}_1 + \vec{e}_{n_1+1} + \cdots + \vec{e}_{n_1+\cdots+n_{k-1}+1}, \tag{2.2}$$

where $\vec{e}_j = {}^{t}(0, \cdots, 0, \overset{j}{\overbrace{1}}, 0, \cdots, 0)$ and n_j denotes the size of Jordan cell $J(\lambda_j)$ with $n_1 + n_2 + \cdots + n_k = N$. Now let us prove that $\det\left(\vec{p}, A\vec{p}, \cdots, A^{N-1}\vec{p}\right) \neq 0$. By an easy calculation we see that

$$\left(\vec{p}, A\vec{p}, \cdots, A^{N-1}\vec{p}\right) = \begin{bmatrix} M(\lambda_1; n_1 \times N) \\ M(\lambda_2; n_2 \times N) \\ \vdots \\ M(\lambda_k; n_k \times N) \end{bmatrix}, \tag{2.3}$$

where the $n \times N$ matrix $M(\lambda; n \times N)$ is defined by

$$M(\lambda; n \times N) := \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \cdots & \lambda^{N-2} & \lambda^{N-1} \\ 0 & 1 & 2\lambda & \cdots & \cdots & (N-2)\lambda^{N-3} & (N-1)\lambda^{N-2} \\ 0 & 0 & 1 & \cdots & \cdots & {}_{N-2}C_2\lambda^{N-4} & {}_{N-1}C_2\lambda^{N-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & {}_{N-1}C_{n-1}\lambda^{N-n} \end{bmatrix}, \tag{2.4}$$

where (i,j) entry $c_{ij}(\lambda)$ $(1 \leq i \leq n,\ 1 \leq j \leq N)$ is written by binomial coefficients,

$$c_{ij}(\lambda) = {}_{j-1}C_{i-1}\lambda^{j-i}, \quad c_{ij}(\lambda) := 0 \text{ if } i > j. \tag{2.5}$$

Here we understand that ${}_{j-1}C_{i-1} = 0$ if $i > j$. When $k = 1$, we have $\det M(\lambda_1; N \times N) = 1$, and there is nothing to prove. In the following we assume that $k \geq 2$. Note that if $n_j = 1$ for all j, then the matrix (2.3) is nothing but Vandermonde's matrix. By this reason, we call the matrix (2.3) a generalized Vandermonde's matrix. For the generalized Vandermonde's determinant, we can prove

Lemma 1 (Generalized Vandermonde's determinant) *The determinant of generalized Vandermonde's matrix* (2.3) *is obtained by*

$$\det \begin{bmatrix} M(\lambda_1; n_1 \times N) \\ M(\lambda_2; n_2 \times N) \\ \vdots \\ M(\lambda_k; n_k \times N) \end{bmatrix} = \prod_{1 \leq i < j \leq k} (\lambda_j - \lambda_i)^{n_i \times n_j}. \tag{2.6}$$

Thus the sufficiency is reduced to prove the formula (2.6), which will be proved in the next section. (Q.E.D.)

3 Proof of Lemma 1

The formula (2.6) is proved by reducing the sizes $\{n_j\}$ and the numbers of eigenvalues $\{\lambda_j\}$. In fact, the following sub-lemma makes it possible.

Sub-Lemma. *It holds that*

$$\det \begin{bmatrix} M(\lambda_1; n_1 \times N) \\ M(\lambda_2; n_2 \times N) \\ \vdots \\ M(\lambda_k; n_k \times N) \end{bmatrix} = \prod_{1 < j \leq k} (\lambda_j - \lambda_1)^{n_j} \times \det \begin{bmatrix} M(\lambda_1; n_1 - 1 \times (N-1)) \\ M(\lambda_2; n_2 \times (N-1)) \\ \vdots \\ M(\lambda_k; n_k \times (N-1)) \end{bmatrix}. \tag{3.1}$$

Proof. We apply the following matrix of size N from the right to the matrix (2.3).

$$\begin{bmatrix} 1 & -\lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -\lambda_1 \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix}. \tag{3.2}$$

Then the block $M(\lambda_1; n_1 \times N)$ is changed into

$$\begin{bmatrix} 1 & 0 & \cdots\cdots & 0 \\ 0 & & & \\ \vdots & & M(\lambda_1; (n_1-1) \times (N-1)) & \\ 0 & & & \end{bmatrix}. \tag{3.3}$$

In fact, it is sufficient to notice the following relation of binomial coefficients,

$$(3.4) \qquad {}_{j-1}C_{i-1} = {}_{j-2}C_{i-1} + {}_{j-2}C_{i-2}, \quad \text{i.e.,} \quad {}_{j-2}C_{i-2} = {}_{j-1}C_{i-1} - {}_{j-2}C_{i-1}.$$

Next we examine how the block $M(\lambda_2; n_2 \times N)$ is changed.

1) The first row vector becomes

$$\begin{aligned}(1, \lambda_2 - \lambda_1, \cdots, \lambda_2^{j-1} - \lambda_2^{j-2}\lambda_1, \cdots) =& (1, 0, \cdots, 0) + \\ &+ (\lambda_2 - \lambda_1)\,(0, 1, \lambda_2, \lambda_2^2, \cdots, \lambda_2^{N-2}).\end{aligned}$$

2) The second row vector becomes

$$\begin{aligned}&(0, 1, 2\lambda_2 - \lambda_1, 3\lambda_2^2 - 2\lambda_2\lambda_1, \cdots, {}_{j-1}C_1\lambda_2^{j-1} - {}_{j-2}C_1\lambda_2^{j-i-1}\lambda_1, \cdots) \\ =&(0, 1, \lambda_2, \lambda_2^2, \cdots, \lambda_2^{N-2}) \ + \ (\lambda_2 - \lambda_1)\,(0, 0, 1, 2\lambda_2, \cdots, (N-2)\lambda_2^{N-3}),\end{aligned}$$

since ${}_{j-1}C_1 - {}_{j-2}C_1 = {}_{j-2}C_0 = 1$.

3) Generally, (i, j) entry becomes

$$\begin{aligned}c_{ij}(\lambda_2) - c_{i,j-1}(\lambda_2)\lambda_1 &= {}_{j-1}C_{i-1}\lambda_2^{j-i} - {}_{j-2}C_{i-1}\lambda_2^{j-i-1}\lambda_1 \\ &= {}_{j-2}C_{i-2}\lambda_2^{j-i} + (\lambda_2 - \lambda_1) \times {}_{j-2}C_{i-1}\lambda_2^{j-i-1} \\ &= c_{i-1,j-1}(\lambda_2) + (\lambda_2 - \lambda_1)c_{i,j-1}(\lambda_2).\end{aligned}$$

These observations show that the block $M(\lambda_2; n_2 \times N)$ is changed into

$$(3.5) \qquad \left[\vec{e}_1, \ \tilde{M}(\lambda_2; n_2 \times (N-1))\right],$$

where

$$\tilde{M}(\lambda_2; n_2 \times (N-1)) = \begin{pmatrix} \lambda_2 - \lambda_1 & \cdots & \cdots & 0 \\ 1 & \lambda_2 - \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \lambda_2 - \lambda_1 \end{pmatrix} M(\lambda_2; n_2 \times (N-1)).$$

Since the other blocks $M(\lambda_i; n_i \times N)$ $(i \geq 3)$ are changed into matrices like (3.5), we get the formula (3.1) by expanding the determinant for the changed matrix by the first row.

(Q.E.D.)

References

[Cod-Lev] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, Inc., 1955.

[Del] P. Deligne, *Equations différentielles à point singuliers réguliers*, Lecture Notes in Math. No. 163, Springer-Verlag, New York, 1970.

[Hsi-Sib] P.-F. Hsieh an Y. Sibuya, *Basic Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1999.

[Kit] K. Kitagawa, L'irregularité en un point singulier d'un systèmes d'équations différentielles linéaires d'ordre 1, J. Math. Kyoto Univ., Vol. 23 (1983), 427 - 440.

[Mal] B. Malgrange, Sur les points singuliers des équations différentielles, Enseign. Math.,20 (1974), 147 - 176.

[Sib] Y. Sibuya, *Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation*, Translations of Math. Monographs 82, American Mathematical Society, Providence, RI, 1990.

(Received September 14, 2010)