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On a Relation between Sine Formula and Radii of Circumcircles for Spherical Triangles

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Introduction

For a triangle $\triangle ABC$ on a plane, we know well *the cosine and the sine formulae*:

$$a^2 = b^2 + c^2 - 2bc\cos A, \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \tag{1}$$

On the other hand, for a spherical triangle $\triangle ABC$ on a unit sphere, there are also *the cosine and the sine formulae*:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{2}$$

In the above formulae, we denote by a, b, c the length of the edges BC, CA, AB, respectively, by A, B, C the inner angles at the vertices A, B, C, respectively, and by R the radius of the circumcircle. Since every spherical triangle can be considered to lie in a half sphere, these values must satisfy $0 < a, b, c, A, B, C < \pi$ and $0 < R <= \pi/2$.

A question arises to the author. For plane triangles, there is a relation that $a / \sin A = 2R$. Then, *for spherical triangles, is $f = \sin a / \sin A$ a function of R?* — However, the answer is negative. Figure 1 and 2 show the trace of (R, f) for spherical triangles which are generated at random. Spherical triangles of Figure 1 are generated to contain its circumcenter, and those of Figure 2 not to contain it. We find from the figures that each trace is not a curve!

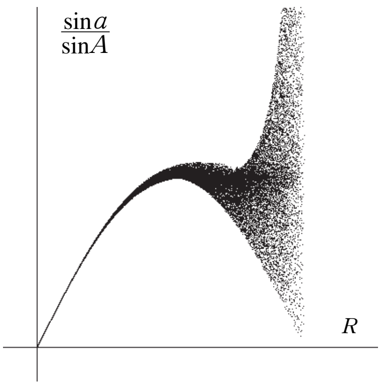

Figure 1 : $O \in \triangle ABC$

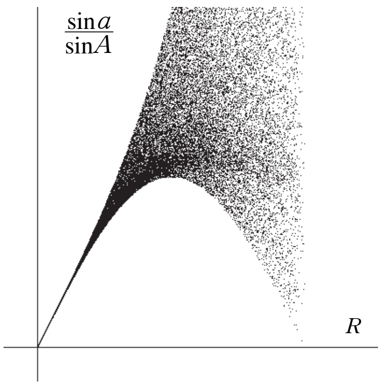

Figure 2 : $O \notin \triangle ABC$

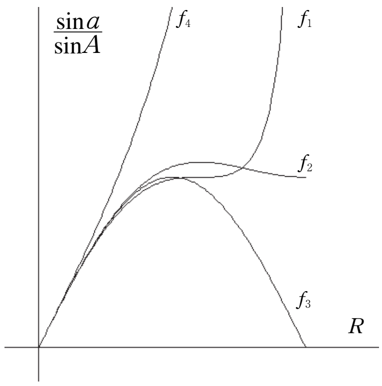

Figure 3

A further question arises to the author. *What curves bound the traced areas?* — On the question, Figure 3 shows the graphs of the following functions:

$$\begin{aligned} f_1(R) &= 2\tan R\left(1 - \frac{3}{4}\sin^2 R\right)^{3/2}, & f_2(R) &= 2\sin R\left(1 - \frac{1}{2}\sin^2 R\right), \\ f_3(R) &= \sin 2R, & f_4(R) &= 2\tan R. \end{aligned} \tag{3}$$

It seems that the traced area of Figure 1 is bounded by f_1, f_2 and f_3, and that of Figure 2 by f_3 and f_4. The following are the answer to the question.

Theorem 1. *If the circumcenter $O \in \triangle ABC$, then the following hold:*

(1) $f_1(R) \leqq \dfrac{\sin a}{\sin A} \leqq f_2(R)$ *when* $0 < \sin^2 R \leqq \dfrac{22-2\sqrt{13}}{27}$.

(2) $f_3(R) < \dfrac{\sin a}{\sin A} \leqq f_2(R)$ *when* $\dfrac{22-2\sqrt{13}}{27} < \sin^2 R \leqq \dfrac{14-2\sqrt{5}}{11}$.

(3) $f_3(R) < \dfrac{\sin a}{\sin A} \leqq f_1(R)$ *when* $\dfrac{14-2\sqrt{5}}{11} < \sin^2 R \leqq 1$.

Theorem 2. *If the circumcenter $O \notin \triangle ABC$, then* $f_3(R) < \dfrac{\sin a}{\sin A} < f_4(R)$.

In the paper, we will prove the above theorems by applying the method of the maximum and minimum problem to a two-variable function.

Proof of theorems

The following formula is the key to prove Theorems 1 and 2 :

$$\left(\frac{\sin a}{\sin A}\right)^2 = 4\tan^2 R\cos^2(a/2)\cos^2(b/2)\cos^2(c/2) \tag{4}$$

Proof of (4). We denote by L, M, N the midpoints of the edges BC, CA, AB, respectively, and by α, β, γ the angles $\angle BOL$, $\angle COM$, $\angle AON$, respectively. Since we have $\angle LOC = \alpha$, $\angle MOA = \beta$ and $\angle NOB = \gamma$, we obtain that $\alpha + \beta + \gamma = \pi$. By applying the sine formula to three triangles $\triangle OBL$, $\triangle OCM$ and $\triangle OAN$, we obtain that

$$\frac{\sin(a/2)}{\sin\alpha} = \frac{\sin(b/2)}{\sin\beta} = \frac{\sin(c/2)}{\sin\gamma} = \sin R. \tag{5}$$

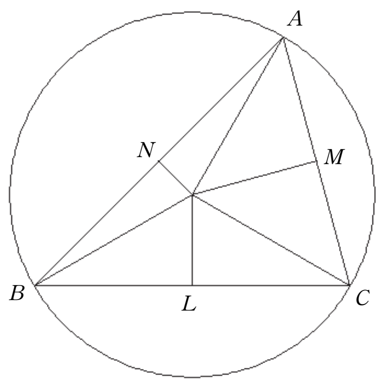

Figure 4

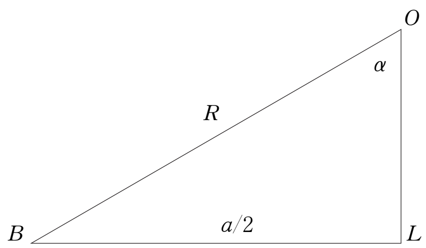

Figure 5

By using $\beta = \pi - \gamma - \alpha$, we obtain that

$$\begin{aligned}\sin^2\gamma - \sin^2\alpha &= \sin^2\gamma(1-\sin^2\alpha) - (1-\sin^2\gamma)\sin^2\alpha \\ &= \sin^2\gamma\cos^2\alpha - (\cos\gamma\sin\alpha)^2 \\ &= \sin^2\gamma\cos^2\alpha - (\sin\beta - \sin\gamma\cos\alpha)^2 \\ &= 2\sin\beta\sin\gamma\cos\alpha - \sin^2\beta. \end{aligned}\tag{6}$$

By using (6) and (5), we obtain that

$$\cos\alpha = \frac{\sin^2\beta + \sin^2\gamma + \sin^2\alpha}{2\sin\beta\sin\gamma} = \frac{q^2+r^2-p^2}{2qr}, \tag{7}$$

where $p = \sin(a/2)$, $q = \sin(b/2)$, $r = \sin(c/2)$. By using (5) and (7), we obtain that

$$\sin^2 R = \frac{\sin^2(a/2)}{1-\cos^2\alpha} = \frac{p^2(2qr)^2}{(2qr)^2-(q^2+r^2-p^2)^2}$$
$$= \frac{4p^2q^2r^2}{2p^2q^2+2q^2r^2+2r^2p^2-p^4-q^4-r^4}. \tag{8}$$

By using it, we obtain that

$$\tan^2 R = \frac{4p^2q^2r^2}{2p^2q^2+2q^2r^2+2r^2p^2-p^4-q^4-r^4-4p^2q^2r^2}. \tag{9}$$

On the other hand, by using the cosine formula, we obtain that

$$\left(\frac{\sin a}{\sin A}\right)^2 = \frac{\sin^2 a}{1-\cos^2 A} = \frac{\sin^2 a \cdot (\sin b \sin c)^2}{(\sin b \sin c)^2 - (\cos a - \cos b \cos c)^2}$$
$$= \frac{4q^2(1-p^2)\cdot 4q^2(1-q^2)\cdot 4r^2(1-r^2)}{4q^2(1-q^2)\cdot 4r^2(1-r^2) - \{(1-2p^2)-(1-2q^2)(1-2r^2)\}^2}$$
$$= \frac{16p^2q^2r^2(1-p^2)(1-q^2)(1-r^2)}{2p^2q^2+2q^2r^2+2r^2p^2-p^4-q^4-r^4-4p^2q^2r^2}. \tag{10}$$

By putting (9) to (10), we have proved (4). □

Proof of Theorem 1. We fix the radius R of the circumcircle, and put $k=\sin^2 R$. Then the pair (α, β) of the angles determines the shape of the triangle. Since $O \in \triangle ABC$, all of α, β, γ are acute or right. So we assume that the variables (α, β) vary in the following domain:

$$D_1 = \{(\alpha, \beta) \in [0, \pi/2]^2 \mid \alpha+\beta \geqq \pi/2\}. \tag{11}$$

By putting (5) to (4), and by using $\gamma = \pi - \alpha - \beta$, we obtain that

$$\left(\frac{\sin a}{\sin A}\right)^2 = \frac{4k}{1-k}(1-k\sin^2\alpha)(1-k\sin^2\beta)(1-k\sin^2(\alpha+\beta)). \tag{12}$$

We can regard $f=\sin a/\sin A$ as a function of (α, β), and so we denote it by $f(\alpha, \beta)$. So we can reduce the problem to that of finding the maximum and the minimum of $f(\alpha, \beta)$ on D_1.

To do it, we partially differentiate (12) and obtain that

$$\frac{\partial}{\partial\alpha}\left(f(\alpha,\beta)^2\right) = \frac{4k}{1-k}(1-k\sin^2\beta)\frac{\partial}{\partial\alpha}\left\{\left(1-k\frac{1-\cos 2\alpha}{2}\right)\left(1-k\frac{1-\cos 2(\alpha+\beta)}{2}\right)\right\}$$
$$= \frac{-2k^2}{1-k}(1-k\sin^2\beta)\left\{(2-k)\left(\sin 2\alpha + \sin 2(\alpha+\beta)\right) + k\left(\sin 2\alpha \cos 2(\alpha+\beta) + \cos 2\alpha \sin 2(\alpha+\beta)\right)\right\}$$
$$= \frac{-2k^2}{1-k}(1-k\sin^2\beta)\left\{2(2-k)\sin(2\alpha+\beta)\cos\beta + k\sin 2(2\alpha+\beta)\right\}$$
$$= \frac{-4k^2}{1-k}(1-k\sin^2\beta)\sin(2\alpha+\beta)\left\{2\cos\beta - k\left(\cos\beta - \cos(2\alpha+\beta)\right)\right\}$$
$$= \frac{-8k^2}{1-k}(1-k\sin^2\beta)\sin(2\alpha+\beta)\{\cos\beta - k\sin(\alpha+\beta)\sin\alpha\}. \tag{13}$$

Similarly, we obtain that

$$\frac{\partial}{\partial\beta}\left(f(\alpha,\beta)^2\right) = \frac{-8k^2}{1-k}(1-k\sin^2\alpha)\sin(\alpha+2\beta)\{\cos\alpha - k\sin(\alpha+\beta)\sin\beta\}. \tag{14}$$

To get the maximum or the minimum of $f(\alpha, \beta)$, we must find the points (α, β) at which both (13) and (14) vanish. We obtain the following four cases:

(i) $\sin(2\alpha+\beta)=0, \sin(\alpha+2\beta)=0.$
(ii) $\sin(2\alpha+\beta)=0, \cos\alpha=k\sin(\alpha+\beta)\sin\beta.$
(iii) $\cos\beta=k\sin(\alpha+\beta)\sin\alpha, \sin(\alpha+2\beta)=0.$
(iv) $\cos\beta=k\sin(\alpha+\beta)\sin\alpha, \cos\alpha=k\sin(\alpha+\beta)\sin\beta.$

We can reduce the case (i) to $(\alpha, \beta)=(\pi/3, \pi/3)$ at which $f(\alpha, \beta)=f_1(R)$. We can also reduce the cases (ii), (iii) and (iv) respectively to $(\alpha, \beta)=(\lambda, \pi-2\lambda)$, $(\pi-2\lambda, \lambda)$ and (λ, λ) at which $f(\alpha, \beta)=1$, where $\lambda=\sin^{-1}(1/\sqrt{2k})$. Remark that, in the case, it is required that $1/\sqrt{2k}\leqq 1$, and so $R\geqq \pi/4$.

On the other hand, the function $f(\alpha, \beta)$ possibly takes the maximum or the minimum on the boundary of D_1, which consists of the following three line segments:

$$l_1=\{(\alpha, \pi/2) \mid 0\leqq\alpha\leqq\pi/2\},\ l_2=\{(\pi/2, \beta) \mid 0\leqq\beta\leqq\pi/2\},\\ l_3=\{(\alpha, \pi/2-\alpha) \mid 0\leqq\alpha\leqq\pi/2\}. \tag{15}$$

For $(\alpha, \pi/2)\in l_1$, we can calculate as follows:

$$f(\alpha, \pi/2)^2=4k(1-k\sin^2\alpha)(1-k\cos^2\alpha)=4k(1-k+\frac{1}{4}k^2\sin^2 2\alpha). \tag{16}$$

So the function $f(\alpha, \beta)$ on l_1 takes the maximum $f_2(R)$ at $(\pi/4, \pi/2)$ and the minimum $f_3(R)$ at $(0, \pi/2)$ and $(\pi/2, \pi/2)$. Similar conclusions hold on l_2 and l_3.

Therefore we have three values $f_1(R)$, $f_2(R)$, $f_3(R)$ and 1 (only when $R\geqq\pi/4$) as the candidates for the maximum and the minimum. By comparing these values, we have proved the inequalities. Finally, we remark that, since each of $(0, \pi/2)$, $(\pi/2, \pi/2)$ and $(\pi/2, 0)$ can not make any triangles, the left inequalities of (2) and (3) must be strict. □

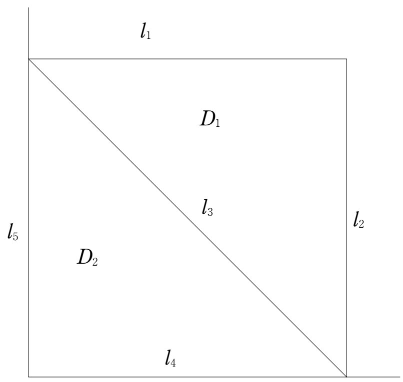

Figure 6

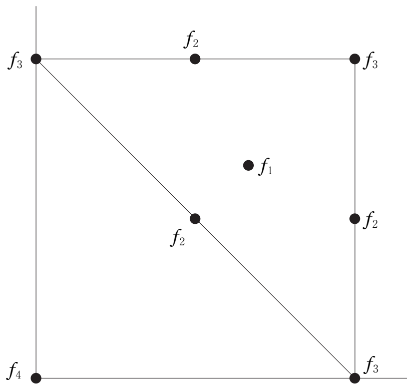

Figure 7

Proof of Theorem 2. Since $O\notin\triangle ABC$, one of α, β, γ is obtuse, say, $\gamma=\pi-\alpha-\beta\geqq\pi/2$. So we assume that the variables (α, β) vary in the following domain:

$$D_2=\{(\alpha, \beta)\in[0, \pi/2]^2 \mid \alpha+\beta\leqq\pi/2\}. \tag{17}$$

There are no points in D_2 at which both (13) and (14) vanish. So the function $f(\alpha, \beta)$ must take the maximum and the minimum on the boundary of D_2, which consists of the following three line segments:

$$l_3,\ l_4=\{(\alpha, 0) \mid 0\leqq\alpha\leqq\pi/2\},\ l_5=\{(0, \beta) \mid 0\leqq\beta\leqq\pi/2\}. \tag{18}$$

For $(\alpha, 0)\in l_4$, we can calculate as follows:

$$f(\alpha, 0)^2=\frac{4k}{1-k}(1-k\sin^2\alpha)^2. \tag{19}$$

So the function $f(\alpha, \beta)$ on l_4 takes the maximum $f_4(R)$ at $(0, 0)$ and the minimum $f_3(R)$ at $(\pi/2, 0)$. A similar conclusion holds on l_5. Moreover, the function $f(\alpha, \beta)$ on l_3 takes the maximum $f_2(R)$ at $(\pi/4, \pi/4)$ and the minimum $f_3(R)$ at $(0, \pi/2)$ and $(\pi/2, 0)$.

Therefore we have three values $f_3(R)$, $f_4(R)$ and $f_5(R)$ as the candidates for the maximum and the minimum. By comparing these values, we have proved the inequalities. Finally, we remark that, since each of $(0, 0)$, $(\pi/2, 0)$ and $(0, \pi/2)$ can not make any triangles, the right and the left inequalities must be strict. $\square$

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