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Smoothness of Voice Leadings under the Maximal Evenness Ansatz

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1. Tonality and maximally evenness ansatz

In the previous paper [4], we have tried to describe the tonality, which characterize the traditional European music, from a viewpoint of the maximal evenness ansatz. The crucial point is that a piece of tonal music is fundamentally composed over the diatonic scale (7 tones) which are maximally even sub-collection of the chromatic scale (12 tones), and the harmony is governed mainly by the chords stacking in thirds which are maximally even sub-collection of the diatonic scale. So, the maximal evenness ansatz seems to brings us a clue to create a purely mathematical model for tonal music theory.

The concept of maximal evenness has been firstly introduced into mathematical music theory by Clough and Douthett[1], where they have realized maximally even collections by mechanical sequences associated with fractions, described by J-functions (Definition 2.1). As is well-known, mechanical sequences is 'balanced', called Myhill's property. In this paper, we observe influence of the maximally evenness on chord progressions, and derive a smoothness of voice leadings among maximally-even chords (Theorem 2.18) from Myhill's property.

2. Maximal evenness induces smoothness of voice leadings

2.1. **Maximal evenness and diatonic chords.** Firstly, we define the J-function introduced by Clough and Douthett[1].

Definition 2.1. *For $c, d, m \in \mathbf{Z}$ with $c > d$ and $d \neq 0$, the* J-function *on $\mathbf{Z}$ is defined as*

$$J^m_{c,d}(k) = \left\lfloor \frac{ck+m}{d} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Lemma 2.2. *Suppose c and d are prime to each other. Then the sequence $(J^m_{c,d}(k+1) - J^m_{c,d}(k))_{k\in\mathbf{Z}}$ consists of $\lfloor c/d \rfloor$ and $\lfloor c/d \rfloor + 1$ with the period d. called a* mechanical sequence *associated with the fraction c/d.*

Proof. As one see

$$\lfloor \alpha(x+1) + \beta \rfloor - \lfloor \alpha x + \beta \rfloor = \alpha + \{\alpha x + \beta\} - \{\alpha(x+1) + \beta\} \in \{\lfloor \alpha \rfloor, \lfloor \alpha \rfloor + 1\}$$

for $\alpha, \beta, x \in \mathbf{R}$, where $\{x\} = x - \lfloor x \rfloor$, we have $J^m_{c,d}(k+1) - J^m_{c,d}(k) \in \{\lfloor c/d \rfloor, \lfloor c/d \rfloor + 1\}$. □

It is known that mechanical words have Myhill's property[1)], an embodiment of maximally evenness[3][4].

Definition 2.3 (Myhill's property for periodic sequences)**.** *Let $\boldsymbol{w} = (w_1, w_2, \dots, w_d)$ be a finite sequence over some alphabet $\{a, b\}$. $|\boldsymbol{w}|$ and $|\boldsymbol{w}|_a$ denote the length d of $\boldsymbol{w}$ and the number of entries w_k equals to a respectively. Let $\boldsymbol{s} = (s_k)_{k\in\mathbf{Z}}$ be a sequence over the alphabet $\{a, b\}$ with the period p, that is, $s_{k+p} = s_k$ for any $k \in \mathbf{Z}$. Then $\boldsymbol{s}$ has* Myhill's property *if for any subsequences $\boldsymbol{x} = (s_k, s_{k+1}, \dots)$ and $\boldsymbol{y} = (s_l, s_{l+1} \dots)$ of $\boldsymbol{s}$,*

$$||\boldsymbol{x}|_a - |\boldsymbol{y}|_a| \leq 1 \text{ holds whenever } |\boldsymbol{x}| = |\boldsymbol{y}|.$$

We use notations $\mathcal{J}^m_{c,d}$ as the tuple $(J^m_{c,d}(k))_{k=0,\dots,d-1}$ and $|\mathcal{J}^m_{c,d}|$ as the set $\{t \in \mathcal{J}^m_{c,d}\}$ of entries of $\mathcal{J}^m_{c,d}$.

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1)This property is known as a *balanced word* in combinatorics on words. See [7].

Example 2.4. Adopting a semitone encoding θ for the chromatic scale as

(2.1) $$C=0, C^\sharp=1, D=2, D^\sharp=3, E=4, F=5, F^\sharp=6, G=7, G^\sharp=8, A=9, A^\sharp=10, B=11,$$

the diatonic scale, say C-major $CDEFGAB$ is expressed as the tuple $(0,2,4,5,7,9,11)=(J^5_{12,7}(k))_{k=0,\dots,6}=\mathcal{J}^5_{12,7}$. Hence the diatonic scale is embedded into the chromatic scale 'maximally even way'.

Example 2.5. Diatonic chords composed of three notes stacked in 'thirds' in the C-major (diatonic) scale are

$$CEG,\ DFA,\ EGB,\ FAC,\ GBD,\ ACE,\ BDF,$$

called (diatonic) triads. Adopting a wholetone encoding η_C for the C-major scale as

(2.2) $$C=0, D=1, E=2, F=3, G=4, A=5, B=6,$$

the triads are expressed as the tuples

$$CEG=\mathcal{J}^0_{7,3},\ DFA=\mathcal{J}^3_{7,3},\ EGB=\mathcal{J}^6_{7,3},\ FAC=\mathcal{J}^9_{7,3},\ GBD=\mathcal{J}^{12}_{7,3},\ ACE=\mathcal{J}^{15}_{7,3},\ BDF=\mathcal{J}^{18}_{7,3}.$$

We also see that the 7-th chords that consist of four diatonic tones are expressed as

$$CEGB=\mathcal{J}^3_{7,4}, DFAC=\mathcal{J}^7_{7,4}, EGBD=\mathcal{J}^{11}_{7,4}, FACE=\mathcal{J}^{15}_{7,4}, GBDF=\mathcal{J}^{19}_{7,4}, ACEG=\mathcal{J}^{23}_{7,3}, BDFA=\mathcal{J}^{27}_{7,3}.$$

Thus the triads and 7-th chords are also embedded into the diatonic scale 'maximally even way'.

Therefore the diatonic triads or 7-th chords are understood as 'multi-order maximally even sets', pointed out by Douthett[2]. Indeed, using an abbreviation

$$\mathcal{J}^{m,n}_{c,d,e}=J^m_{c,d}(\mathcal{J}^n_{d,e})=(J^m_{c,d}\circ J^n_{d,e}(k))_{k=0,\dots,e-1}$$

for $c>d>e>0$, the triads or 7-th chords are described such as

$$CEG=\mathcal{J}^{5,0}_{12,7,3},\ CEGB=\mathcal{J}^{5,3}_{12,7,4}.$$

Traditional European music is basically performed on the twelve tone system, called *chromatic scale*

$$Ch=(C,C^\sharp,D,D^\sharp,E,F,F^\sharp,G,G^\sharp,A,A^\sharp,B),$$

however, we treat the chromatic scale more abstract way. We note that the tones with basic frequencies f and $2^n f, n\in\mathbf{Z}$ sound 'same' for our ears. Such tones are called *octave equivalent*. Hereafter, for a tuple $A=(a_1,a_2,\dots)$, $|A|$ denotes the set $\{a_0,a_1,\dots\}$ consists of entries of A.

Definition 2.6 (General chromatic scale)**.** *A* general chromatic scale *Ch is a tuple $(f_0,f_1,\dots,f_{c-1})$ of tones with basic frequencies $f_0<f_1<\cdots<f_{c-1}$ satisfying $f_{c-1}<2f_0$. A* general semitone encoding θ *associated with the chromatic scale Ch is a map $|Ch|\ni f_k\mapsto k\in\mathbf{Z}$. By octave equivalence, we extend the chromatic scale Ch periodically, denoting $\overline{Ch}$. The semitone encoding θ is naturally extended to a bijection $\theta:|\overline{Ch}|\to\mathbf{Z}$, as $\theta(f)\equiv\theta(f')\pmod c$ if and only if f and f' are octave equivalent.*

More explicitly, the extended chromatic scale is an infinite tuple of frequencies

$$\overline{Ch}=(\dots,2^{-1}f_0,2^{-1}f_1,\dots,2^{-1}f_{c-1},f_0,f_1,\dots,f_{c-1},2f_0,2f_1,\dots,2f_{c-1},2^2f_0,\dots),$$

and its semitone encoding is given by $\theta(2^n f_k)=cn+k$ for $n\in\mathbf{Z}, 0\le k<c$.

Definition 2.7 (General scales and chords)**.** *Given a general chromatic scale Ch and its extension $\overline{Ch}$. A* scale *is a tuple $S=(t_0,t_1,\dots,t_{d-1})$ of tones selected from $|\overline{Ch}|$, of which elements are arranged in ascending order $\theta(t_0)<\theta(t_1)<\cdots<\theta(t_{d-1})$ and $\theta(t_{d-1})-\theta(t_0)<c$. $\overline{S}$ denotes the extension of S by octave equivalence. The* general wholetone encoding η_S *associated with the scale S is a map $|\overline{S}|\ni t_k\mapsto k\in\mathbf{Z}$. A*

chord T *in the extended scale* $\overline{S}$ *is a tuple of tones*[2] $T = (t_{i_1}, \ldots, t_{i_n})$ *selected from* $|\overline{S}| = \{\ldots, t_{-1}, t_0, t_1, \ldots\}$, *arranged in ascending order* $\theta(t_{i_1}) < \theta(t_{i_2}) < \cdots < \theta(t_{i_n})$. *We call* $oct(T) = \lfloor(\theta(t_{i_n}) - \theta(t_{i_1}))/c\rfloor + 1$ *the* octave range *of the chord* T.

By octave equivalence, we often identify the chromatic scale Ch with $\mathbf{Z}/c\mathbf{Z}$. In this sense, the extension $\overline{Ch}$ corresponds to the covering space $\mathbf{Z}$ of $\mathbf{Z}/c\mathbf{Z}$. Similarly, a scale S consists of d tones is identified with $\mathbf{Z}/d\mathbf{Z}$, and its extension $\overline{S}$ corresponds to the covering space $\mathbf{Z}$ of $\mathbf{Z}/d\mathbf{Z}$.

$$\begin{array}{ccccccc} \mathbf{Z} & \xleftarrow{\eta_S} & \overline{S} & \hookrightarrow & \overline{Ch} & \xrightarrow{\theta} & \mathbf{Z} \\ \downarrow \pi & \circlearrowright & \downarrow & \text{oct.eq.} \downarrow & & \circlearrowright & \downarrow \pi \\ \mathbf{Z}/d\mathbf{Z} & \xleftarrow{\eta_S} & S & \hookrightarrow & Ch & \xrightarrow{\theta} & \mathbf{Z}/c\mathbf{Z} \end{array}$$

In the definition of chords, we take $\overline{S}$ instead of S because we are to treat chords like $G7^{(9)} = GBDFA = (7, 11, 14, 17, 21)$ (in the semitone encoding) $= (5, 7, 9, 11, 13)$ (in the wholetone encoding) in C-major diatonic scale. Such a chord exceeds an octave: its octave range is $\lfloor(21 - 7)/12\rfloor + 1 = 2$. Considering a maximally-even chord $\mathcal{J}^m_{hd,e}$ with $h \geq e$ will be nonsense because the distance of adjacent two voices in the chord becomes more than one octave. So hereafter we assume $h < e$. As $hd/e > 1$ and $(h-1)e < h(e-1)$ for $h, d, e \in \mathbf{N}$ with $d > e > h$, one see that

$$(h-1)d = \left\lfloor \frac{(h-1)de + m}{e} \right\rfloor - \left\lfloor \frac{m}{e} \right\rfloor \leq \left\lfloor \frac{hd(e-1) + m}{e} \right\rfloor - \left\lfloor \frac{m}{e} \right\rfloor < \left\lfloor \frac{hde + m}{e} \right\rfloor - \left\lfloor \frac{m}{e} \right\rfloor = hd.$$

Thus we have $oct(\mathcal{J}^m_{hd,e}) = h$.

Definition 2.8. *Consider a chord* $T = (t_{i_1}, \ldots, t_{i_n})$ *in the extended scale* $\overline{S} = (\ldots, t_{-1}, t_0, t_1, \ldots)$ *of a scale* S *of* d *tones. The* translation $T + l$ *of* T *in* $\overline{S}$ *is a chord in* $\overline{S}$ *given by* $T + l = (t_{i_1+l}, \ldots, t_{i_n+l})$ *for any* $l \in \mathbf{Z}$. *The* first inversion $T(1)$ *of* T *is a chord in* $\overline{S}$ *given by* $T(1) = (t_{i_2}, \ldots, t_{i_n}, t_{i_1+hd})$, *where* $h = oct(T)$. *Inductively, the* p-th inversion $T(p)$ *of* T *is given as the first inversion of* $T(p-1)$.

For the 'maximally-even chord' such as diatonic chords in a diatonic scale, its translation and inversion are expressed naturally in terms of J-functions.

Proposition 2.9 ([4])**.** *Given a scale* S *of* d *tones. Consider a maximally-even chord* $\mathcal{J}^m_{hd,e}$ *in* $\overline{S}$ *with the octave range* $h = oct(T)$. *Then the translation* $\mathcal{J}^m_{hd,e} + l$ *for any integer* l *is expressed by*

$$\mathcal{J}^m_{hd,e} + l = \mathcal{J}^{m+le}_{hd,e}, \tag{2.3}$$

and the p*-th inversion* $\mathcal{J}^m_{hd,e}(p)$ *for any integer* p *is expressed by*

$$\mathcal{J}^m_{hd,e}(p) = \mathcal{J}^{m+phd}_{hd,e}. \tag{2.4}$$

Thus transpositions and inversions of maximally-even chords are maximally-even chords.

Proof. (2.3) comes from

$$J^{m+le}_{hd,e}(k) = \left\lfloor \frac{hdk + m + le}{e} \right\rfloor = \left\lfloor \frac{hdk + m}{e} \right\rfloor + l = J^m_{hd,e}(k) + l.$$

By definition, we see p-th inversion of $\mathcal{J}^m_{hd,e}$ is expressed as

$$\begin{aligned} \mathcal{J}^m_{hd,e}(p) &= \left(J^m_{hd,e}(p), J^m_{hd,e}(p+1), \ldots, J^m_{hd,e}(p+e-1), J^m_{hd,e}(0) + hd, \ldots, J^m_{hd,e}(p-1) + hd\right) \\ &= \left(J^m_{hd,e}(p), J^m_{hd,e}(p+1), \ldots, J^m_{hd,e}(e-1), J^m_{hd,e}(e), \ldots, J^m_{hd,e}(e-1+p)\right) \\ &= \left(J^m_{hd,e}(k+p)\right)_{k=0,\ldots,e-1}. \end{aligned}$$

[2] We often call the entry t_{i_k} of a chord T the 'voice'.

Then (2.4) comes from

$$J_{hd,e}^{m}(k+p) = \left\lfloor \frac{hd(k+p)+m}{e} \right\rfloor = \left\lfloor \frac{hdk+m+phd}{e} \right\rfloor = J_{hd,e}^{m+phd}(k).$$

□

2.2. **Distances of chords.** A sequence of chords is called the 'chord progression' or the 'voice leading': the chords are connected carefully and harmoniously. It is preferable that each voice in the chord progression moves smoothly, that is, the music line of each voice moves from one note to the next by a jump as small as possible. In this subsection, all chords are identified with tuples of integers through the wholetone encoding η_S associated with a given scale S.

Definition 2.10 (Distances of chords)**.** *Given a scale S of d tones and two chords $P = (p_1, \ldots, p_e)$ and $Q = (q_1, \ldots, q_e)$ of e voices in $\overline{S}$. We define a distance D_S of the chords associated with the scale S by*

$$D_S(P,Q) = \max_{1\le k\le e} |p_k - q_k|,$$

and its inverted version ID_S by

$$ID_S(P,Q) = \min_{\alpha,\beta\in\mathbf{Z}} D_S(P(\alpha), Q(\beta)).$$

Lemma 2.11. *Let $P = (p_k)$ and $Q = (q_k)$ be chords of e voices in a given scale S with $oct(P) = oct(Q)$. Then*

$$D_S(P(\alpha), Q(\beta)) = D_S(P, Q(\beta-\alpha))$$

holds for any $\alpha, \beta \in \mathbf{Z}$, hence

$$ID_S(P,Q) = \min_{\alpha\in\mathbf{Z}} D_S(P, Q(\alpha)).$$

Proof. We write p_k^α as the entries of the inversion $P(\alpha)$, thus $P(\alpha) = (p_k^\alpha)$. By definition, we see $p_k^\alpha = p_{k+1}^{\alpha-1}$ and $q_k^\beta = q_{k+1}^{\beta-1}$ for $k \le e-1$, and $p_e^\alpha = p_1^{\alpha-1} + hd$ and $q_e^\beta = q_1^{\beta-1} + hd$, where $d = \#|S|$ and $h = oct(P) = oct(Q)$. Then we have $|p_k^\alpha - q_k^\beta| = |p_{k+1}^{\alpha-1} - q_{k+1}^{\beta-1}|$, hence

$$D_S(P(\alpha), Q(\beta)) = D_S(P(\alpha-1), Q(\beta-1)) = \cdots = D_S(P, Q(\beta-\alpha)).$$

□

Proposition 2.12. *D_S and ID_S are distances of chords consist of the same number of voices with the same octave range.*

Proof. We just show the triangular inequality. For chords $P = (p_k), Q = (q_k)$ and $R = (r_k)$ of e voices, we have

$$|p_k - q_k| \le |p_k - r_k| + |r_k - q_k| \le \max_{1\le i\le e} |p_i - r_i| + \max_{1\le i\le e} |r_i - q_i| = D_S(P,R) + D_S(R,Q)$$

for any $1 \le k \le e$. Thus we have

$$D_S(P,Q) = \max_{1\le k\le e} |p_k - q_k| \le D_S(P,R) + D_S(R,Q).$$

Suppose $oct(P) = oct(Q) = oct(R)$, then we have for any α, β, γ,

$$ID_S(P,Q) \le D_S(P(\alpha), Q(\beta)) \le D_S(P(\alpha), R(\gamma)) + D_S(R(\gamma), Q(\beta)).$$

By Lemma 2.11, we have

$$ID_S(P,Q) \le D_S(P, R(\gamma-\alpha)) + D_S(R, Q(\beta-\gamma))$$

for any $\alpha, \beta, \gamma \in \mathbf{Z}$. As $\gamma - \alpha$ and $\beta - \gamma$ are taken independently, by taking minimum of each term of right hand side of the inequality, we come to

$$ID_S(P, Q) \leq ID_S(P, R) + ID_S(R, Q).$$

□

2.3. **Smoothness of voice leadings of maximally-even chords: observation.** Consider a typical voice leading $CEG \to CFA \to BDG \to CEG$ of diatonic chords. As diatonic chords are maximally even in the diatonic scale, this progression is expressed by J-functions as

$$\mathcal{J}^0_{7,3} = (0, 2, 4) \to \mathcal{J}^2_{7,3} = (0, 3, 5) \to \mathcal{J}^{-2}_{7,3} = (-1, 1, 4) \to \mathcal{J}^0_{7,3}.$$

Their distances of adjacent chords are

$$D_S(\mathcal{J}^0_{7,3}, \mathcal{J}^2_{7,3}) = 1, D_S(\mathcal{J}^2_{7,3}, \mathcal{J}^{-2}_{7,3}) = 2, D_S(\mathcal{J}^{-2}_{7,3}, \mathcal{J}^0_{7,3}) = 1,$$

where S is the diatonic scale, while one see that their inverted distances are

$$ID_S(\mathcal{J}^0_{7,3}, \mathcal{J}^2_{7,3}) = 1, ID_S(\mathcal{J}^2_{7,3}, \mathcal{J}^{-2}_{7,3}) = 1, ID_S(\mathcal{J}^{-2}_{7,3}, \mathcal{J}^0_{7,3}) = 1.$$

Indeed, by inverting the third chrod BDG to DGB and last chord CEG to EGC, we obtain a new voice leading $CEG \to CFA \to DGB \to EGC$ of which each voice jumps smaller than before. Actually, we will show $ID_S(\mathcal{J}^m_{7,3}, \mathcal{J}^n_{7,3}) \leq 1$ for any integers m, n.

The following proposition is the significant feature of maximally even sets.

Proposition 2.13. *Suppose that hd and e are prime to each other. For any $0 \leq s \leq e-1$, we have*

$$\#\{0 \leq k \leq e-1 \mid J^m_{hd,e}(k) = J^{m+s}_{hd,e}(k)\} = e - s.$$

Thus maximally-even chords of the octave range h in a scale S of d tones fulfill the inequality

$$ID_S(\mathcal{J}^m_{hd,e}, \mathcal{J}^{m+s}_{hd,e}) \leq 1.$$

Proof. Taking $t_k = J^m_{hd,e}(k) = \left\lfloor \dfrac{hdk + m}{e} \right\rfloor$ and $R_k = hdk + m - t_k e \leq e - 1$, we see

$$J^{m+s}_{hd,e}(k) = t_k + \left\lfloor \frac{R_k + s}{e} \right\rfloor = t_k \text{ or } t_k + 1.$$

As e is prime to hd, for each $0 \leq p \leq e-1$ there exists unique $0 \leq k \leq e-1$ fulfilling $R_k = p$. The assertion comes from the fact that $J^{m+s}_{hd,e}(k) = t_k$ if and only if $0 \leq R_k \leq e - s - 1$. □

Lemma 2.14. *Take a scale S of d tones and consider a maximally-even chord $\mathcal{J}^m_{hd,e}$ with the octave range h, where hd and e are prime to each other. Then for any $0 \leq s \leq \lfloor hd/e \rfloor$, we have*

$$ID_S(\mathcal{J}^m_{hd,e}, \mathcal{J}^m_{hd,e} + s) \leq \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor.$$

Proof. Putting $t_k = J^m_{hd,e}(k)$, we see

$$t_{k+1} - t_k \geq \left\lfloor \frac{hd}{e} \right\rfloor, \text{ hence } t_k \leq t_k + s \leq t_{k+1}$$

by Lemma 2.2 and the assumption on s. Then we see

$$D_S(\mathcal{J}^m_{hd,e}, \mathcal{J}^m_{hd,e} + s) = s \text{ and } D_S(\mathcal{J}^m_{hd,e}(1), \mathcal{J}^m_{hd,e} + s) \leq \left\lfloor \frac{hd}{e} \right\rfloor + 1 - s,$$

hence

$$ID_S(\mathcal{J}^m_{hd,e}, \mathcal{J}^m_{hd,e} + s) \leq \min\left\{ s, \left\lfloor \frac{hd}{e} \right\rfloor + 1 - s \right\} \leq \frac{\lfloor hd/e \rfloor + 1}{2}.$$

As ID_S takes natural numbers, we have the assertion. □

Proposition 2.15. *Take a scale S of d tones and consider maximally-even chords $\mathcal{J}^l_{hd,e}$ and $\mathcal{J}^{l+m}_{hd,e}$ with the octave range h, where hd and e are prime to each other. Then for any $l, m \in \mathbf{Z}$, we have*

$$ID_S(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+m}_{hd,e}) \le \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor + 1.$$

Proof. Take p, q, r and s as $m = hdp + q, p = \lfloor m/hd \rfloor$ and $q = es + r, s = \lfloor q/e \rfloor$. Then by Proposition 2.9, we see

$$\mathcal{J}^{l+m}_{hd,e} = \mathcal{J}^{l+q}_{hd,e}(p) = \mathcal{J}^{l+r}_{hd,e}(p) + s.$$

By definition of ID_S and triangular inequality, we have

$$\begin{aligned} ID_S(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+m}_{hd,e}) &= ID_S(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+r}_{hd,e}(p) + s) = ID_S(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+r}_{hd,e} + s) \\ &\le ID_S(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+r}_{hd,e}) + ID_S(\mathcal{J}^{l+r}_{hd,e}, \mathcal{J}^{l+r}_{hd,e} + s) \end{aligned}$$

and applying Proposition 2.13 and Lemma 2.14 since $0 \le r \le e-1$ and $s = \lfloor q/e \rfloor \le \lfloor hd/e \rfloor$,

$$\le 1 + ID_S(\mathcal{J}^{l+r}_{hd,e}, \mathcal{J}^{l+r}_{hd,e} + s) \le \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor + 1.$$

□

However in the case of the diatonic scale S and triadic chords $\mathcal{J}^m_{7,3}$'s, one observe more strict inequality

$$ID_S(\mathcal{J}^l_{7,3}, \mathcal{J}^m_{7,3}) \le 1 = \left\lfloor \frac{\lfloor 7/3 \rfloor + 1}{2} \right\rfloor$$

for all $l, m \in \mathbf{Z}$. This fact depends on the special choice of d and e of the maximally-even chords $\mathcal{J}^m_{d,e}$'s, pointed out by Muramatsu[8] as follows.

Theorem 2.16 (Muramatsu[8])**.** *Take a scale S of d tones and consider maximally-even chords $\mathcal{J}^l_{hd,e}$ and $\mathcal{J}^{l+m}_{hd,e}$ with the octave range h. If $hd \equiv 1 \pmod{e}$, then for any $l, m \in \mathbf{Z}$, we have*

$$ID_S(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+m}_{hd,e}) \le \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor.$$

Proof. As is seen in the proof of Proposition 2.15, there exist p, s and $0 \le r \le e-1$ such that $\mathcal{J}^{l+m}_{hd,e} = \mathcal{J}^{l+r}_{hd,e}(p) + s$ holds. Let us put $s_k = J^l_{hd,e}(k)$ and $t_k = J^{l+r}_{hd,e}(k)$. The assumption $hd \equiv 1 \pmod{e}$ means

$$hd = \left\lfloor \frac{hd}{e} \right\rfloor e + 1 = \left\lfloor \frac{hd}{e} \right\rfloor (e-1) + \left\lfloor \frac{hd}{e} \right\rfloor + 1,$$

then combing Lemma 2.2, we see there exists unique $0 \le i^*, j^* \le e-1$ fulfilling

$$s_{i^*+1} - s_{i^*} = t_{j^*+1} - t_{j^*} = \left\lfloor \frac{hd}{e} \right\rfloor + 1.$$

Since $0 \le r \le e-1$, we see $|s_k - t_k| \le 1$ for all $0 \le k \le e-1$ by Proposition 2.13. Moreover, we may assume, without loss of generality, $t_k = s_k$ holds for $i^*+1 \le k \le j^*$ and $t_k = s_k + 1$ for the rest in $0 \le k \le e-1$ (See Figure 1.). Then we have

$$(t_k + s) - s_k = \begin{cases} s, & \text{if } i^*+1 \le k \le j^*, \\ s+1, & \text{otherwise.} \end{cases} \tag{2.5}$$

Noticing $s_{i^*+1} - t_{i^*} = t_{i^*+1} - t_{i^*} = \lfloor hd/e \rfloor$ and $s_{j^*+1} - t_{j^*} = s_{j^*+1} - s_{j^*} = \lfloor hd/e \rfloor$, we also have

$$s_{k+1} - (t_k + s) = \begin{cases} \left\lfloor \frac{hd}{e} \right\rfloor - s, & \text{if } i^* \le k \le j^*, \\ \left\lfloor \frac{hd}{e} \right\rfloor - s - 1, & \text{otherwise.} \end{cases} \tag{2.6}$$

Then when $s \leq \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor - 1$, we have

$$(t_k + s) - s_k \leq \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor$$

for any k by (2.5), and when $s \geq \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor$,

$$s_{k+1} - (t_k + s) \leq \left\lfloor \frac{hd}{e} \right\rfloor - \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor \leq \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor,$$

where the last inequality comes from the inequality $n \leq 2 \left\lfloor \frac{n+1}{2} \right\rfloor$ that holds for any $n \in \mathbf{Z}$. Therefore we reach

$$\begin{aligned} ID_S(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+m}_{hd,e}) &= ID_S(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+r}_{hd,e} + s) \\ &\leq \min\left\{ D_S\left(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+r}_{hd,e} + s\right), D_S\left(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+r}_{hd,e}(1) + s\right)\right\} \leq \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor. \end{aligned} \tag{2.7}$$

□

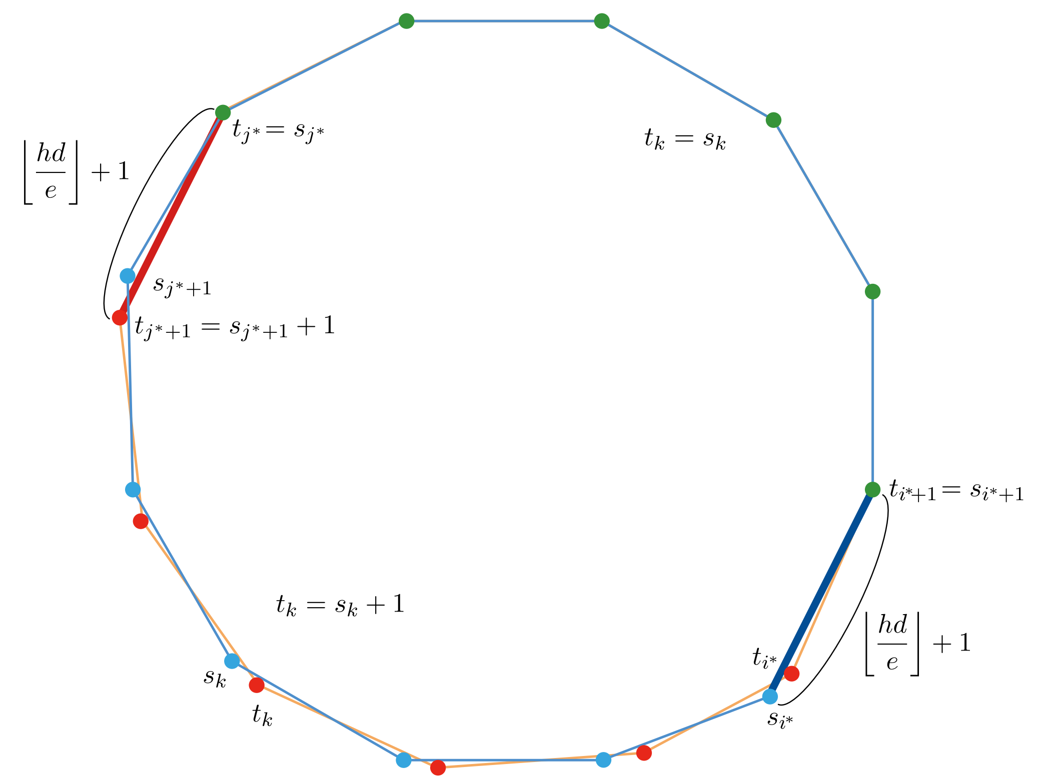

FIGURE 1. The relation between arrangements of s_k's and t_k's in the case of $hd \equiv 1 \pmod{e}$.

Again let us consider triadic chords $\mathcal{J}^m_{7,3}$'s in the diatonic scale S. As $7 \equiv 1 \pmod 3$, Theorem 2.16 shows $ID_S(\mathcal{J}^m_{7,3}, \mathcal{J}^l_{7,3}) \leq 1$ for any $l, m \in \mathbf{Z}$, in other words, any voice leading from a maximally-even triadic chord in the diatonic scale to another proceeds so that each voice moves at most one whole tone, up to the octave equivalence. This fact means that one finds smooth voice leadings corresponding to the chord progressions (in the diatonic system) which one wants.

2.4. **Smoothness of voice leadings derived from Myhill's property.** Theorem 2.16 shows the smoothness of voice leadings under a restricted situation on hd and e of $\mathcal{J}^m_{hd,e}$'s, however, there exists another case of smoothness, e.g., the distance between 7-th chords in a diatonic scale is at most one whole tone. In this subsection, we show that Myhill's property of which maximally-even chords possess derives the smoothness.

Lemma 2.17. *Consider a Myhill sequence (d_i) over $\{0,1\}$. If $\sum_{i=k}^{l}(d_i - d_{i+m}) = 1$ for some k, l and m, then*

$$\sum_{i=k}^{n}(d_i - d_{i+m}) \in \{0,1\}$$

holds for any $n \geq k$.

Proof. By definition, we see $\left|\sum_{i=p}^{q}(d_i - d_{i+m})\right| \leq 1$ for any integers $p < q$. Suppose that there exists an integer $h > 0$ fulfilling $\sum_{i=k}^{h}(d_i - d_{i+m}) = -1$. Then we have $\sum_{i=h+1}^{l}(d_i - d_{i+m}) = 2$ or $\sum_{i=l+1}^{h}(d_i - d_{i+m}) = -2$, hence a contradiction. □

Theorem 2.18. *Take a scale S of d tones and consider maximally-even chords $\mathcal{J}^l_{hd,e}$ and $\mathcal{J}^{l+m}_{hd,e}$ with the octave range h. Suppose that hd is prime to e and $\lfloor hd/e \rfloor$ is odd, then for any $l, m \in \mathbf{Z}$, we have*

$$ID_S\left(\mathcal{J}^l_{hd,e}, \mathcal{J}^{l+m}_{hd,e}\right) \leq \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor. \tag{2.8}$$

Proof. Let us take p, s and $0 \leq r \leq e-1$ such that $\mathcal{J}^{l+m}_{hd,e} = \mathcal{J}^{l+r}_{hd,e}(p) + s$ holds as is seen before, and put $s_k = J^l_{hd,e}(k), t_k = J^{l+r}_{hd,e}(k)$ and their differences $ds_k = s_{k+1} - s_k - \lfloor hd/e \rfloor, dt_k = t_{k+1} - t_k - \lfloor hd/e \rfloor$. We note $ds_k, dt_k \in \{0,1\}$ by Lemma 2.2, and that the both sequences (ds_k) and (dt_k) are mechanical sequences associated with hd/e, which have Myhill's property, hence $\left|\sum_{i=k}^{l}(dt_i - ds_i)\right| \leq 1$ for any integers $k < l$. As $0 \leq r \leq e-1$ and by proposition 2.13, there exists k such as $s_k = t_k$. We may assume $s_0 = t_0$ by applying inversions if needed. We can also assume $\sum_{i=0}^{l}(dt_i - ds_i) \in \{0,1\}$ for any $l > 0$ by Lemma 2.17. Then we see

$$s_k = s_0 + k\left\lfloor \frac{hd}{e} \right\rfloor + \sum_{i=0}^{k-1} ds_i,\ t_k = t_0 + k\left\lfloor \frac{hd}{e} \right\rfloor + \sum_{i=0}^{k-1} dt_i,\ \text{hence } t_k - s_k = \sum_{i=0}^{k-1}(dt_i - ds_i) \in \{0,1\}$$

for any k, meaning that either $t_k = s_k$ or $t_k = s_k + 1$ occurs. As a result, when $s \leq \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor - 1$, we have

$$(t_k + s) - s_k \leq 1 + s \leq \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor$$

for any k. As $\lfloor hd/e \rfloor$ is odd, we see $\left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor = \frac{\lfloor hd/e \rfloor + 1}{2}$. Then when $s \geq \frac{\lfloor hd/e \rfloor + 1}{2}$, we have

$$\begin{aligned} s_{k+1} - (t_k + s) &= s_{k+1} - s_k - (t_k - s_k + s) \\ &\leq \left(\left\lfloor \frac{hd}{e} \right\rfloor + 1\right) - \left(0 + \frac{\lfloor hd/e \rfloor + 1}{2}\right) = \frac{\lfloor hd/e \rfloor + 1}{2} = \left\lfloor \frac{\lfloor hd/e \rfloor + 1}{2} \right\rfloor \end{aligned}$$

for any k, hence the assertion. □

As $\lfloor 7/4 \rfloor = 1$ is odd, any voice leading from a 7-th chord to another which are in the same key is always smooth in the sense of (2.8);

$$ID_S\left(\mathcal{J}^l_{7,4}, \mathcal{J}^m_{7,4}\right) \leq 1$$

for any $l, m \in \mathbf{Z}$. However, when $\lfloor hd/e \rfloor$ is even, the inequality (2.8) is false. Indeed, take $d = 19$ and $e = 7$ for instance, one see

$$ID_S\left(\mathcal{J}^0_{19,7}, \mathcal{J}^1_{19,7}\right) = 2 > \left\lfloor \frac{\lfloor 19/7 \rfloor + 1}{2} \right\rfloor.$$

scale	$ID_{Ch}\left(\mathcal{J}^{5,0}_{12,7,3}, \mathcal{J}^{m,q}_{12,7,3}\right)$	unsmooth chord
$G(m=6), D(m=7), A(m=8)$	≤ 2	none
$F(m=4), B^\flat(m=3), E^\flat(m=2)$	≤ 2	none
$E(m=9)$	≤ 3	$\to D^\sharp F^\sharp A = \mathcal{J}^{9,3}_{12,7,3}$
$B(m=10)$	≤ 3	$\to C^\sharp E A^\sharp = \mathcal{J}^{10,1}_{12,7,3}, C^\sharp F^\sharp A^\sharp = \mathcal{J}^{10,2}_{12,7,3}$
$F^\sharp(m=11)$	≤ 3	$\to C^\sharp E^\sharp A^\sharp = \mathcal{J}^{11,1}_{12,7,3}, C^\sharp F^\sharp A^\sharp = \mathcal{J}^{11,2}_{12,7,3}$
$C^\sharp(m=12) = D^\flat(m=0)$	≤ 3	$\to C^\sharp E^\sharp A^\sharp = \mathcal{J}^{12,1}_{12,7,3}, C^\sharp F^\sharp A^\sharp = \mathcal{J}^{12,2}_{12,7,3}$
$A^\flat(m=1)$	≤ 3	$\to D^\flat F B^\flat = \mathcal{J}^{1,4}_{12,7,3}, D^\flat G B^\flat = \mathcal{J}^{1,5}_{12,7,3}$

TABLE 1. The proximity of diatonic scales and the smoothness of triadic chords. Each voice leading starts from $CEG = \mathcal{J}^{5,0}_{12,7,3}$ in C major scale.

2.5. **Remarks.** Throughout our argument, we fix some scale S, meaning that we only consider the case of voice leadings in a fixed key. The smoothness means that any voice in a chord moves to another at most a wholetone, that is, two halftones away. One may expect that such a smoothness is preserved over the chord progressions among the chords in scales of different keys. Unfortunately, the smoothness breaks down. Actually, consider the progression from CEG, the tonic in C-major scale to $D^\sharp F^\sharp A$, the diminished in E-major scale, which are maximally-even chords $\mathcal{J}^0_{7,3}$ in C-major scale $\mathcal{J}^5_{12,7}$ and $\mathcal{J}^3_{7,3}$ in E-major scale $\mathcal{J}^9_{12,7}$ respectively. One will see the distance in the sense of the chromatic scale is given as (FIGURE 2)

$$ID_{Ch}\left(\mathcal{J}^{5,0}_{12,7,3}, \mathcal{J}^{9,3}_{12,7,3}\right) = 3,$$

meaning that the distance between CEG and $D^\sharp F^\sharp A$ does not become smaller than three semitones away, even though applying inversions. The breaking down of the smoothness depends on the 'proximity' of diatonic scales. TABLE 1 shows the relation between the proximity of scales and the smoothness of triadic chords. We note the upper bound of the distance among triadic chords is given as

$$\begin{aligned} ID_{Ch}\left(\mathcal{J}^{l,p}_{12,7,3}, \mathcal{J}^{m,q}_{12,7,3}\right) &\leq ID_{Ch}\left(\mathcal{J}^{l,p}_{12,7,3}, \mathcal{J}^{m,p}_{12,7,3}\right) + ID_{Ch}\left(\mathcal{J}^{m,p}_{12,7,3}, \mathcal{J}^{m,q}_{12,7,3}\right) \\ &\leq ID_{Ch}\left(\mathcal{J}^{l}_{12,7}, \mathcal{J}^{m}_{12,7}\right) + 2 \leq 3 \text{ (halftones)}. \end{aligned}$$

Similar argument can be discussed for 7-th chords. TABLE 2 shows the smoothness of voice leadings of 7-th chords which start from $CEGB = \mathcal{J}^{5,3}_{12,7,4}$ in C major scale.

REFERENCES

[1] J. Clough and J. Douthett, *Maximal Even Sets,* Journal of Music Theory **35**, pp. 93-173, 1991.

[2] J. Douthett, *Filtered Point-Symmetry and Dynamical Voice-Leading,* in J. Douthett, M. Hyde and C. Smith(eds.), *Music Theory and Mathematics: Chords, Collections, and Transformations, Chapter 4,* University of Rochester Press, pp. 72-106, 2008.

[3] Y. Hashimoto, *A dynamical characterization of Myhill's property,* Bull. of Aichi Univ. of Education, Natural Science **64**, pp. 1-9, 2015. `http://hdl.handle.net/10424/5978`.

[4] Y. Hashimoto, *Maximally evenness ansatz and diatonic system,* Bull. of Aichi Univ. of Education, Natural Science **65**, pp. 7-16, 2017. `http://hdl.handle.net/10424/7039`.

[5] S. Koelsch, *Brain and Music,* John Wiley & Sons Ltd., 2013.

[6] F. Lerdahl and R. Jackendoff, *A Generative Theory of Tonal Music,* MIT Press, 1983.

[7] M. Lothaire, *Algebraic combinatorics on words,* Encyclopedia of mathematics and its applications **90**, Cambridge University Press, 2002.

scale	$ID_{Ch}\left(\mathcal{J}^{5,3}_{12,7,4}, \mathcal{J}^{m,q}_{12,7,4}\right)$	unsmooth chord
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$F(m=4), B^\flat(m=3)$	≤ 2	none
$E(m=9)$	≤ 3	$\to D^\sharp E G^\sharp B = \mathcal{J}^{9,4}_{12,7,4}, D^\sharp F^\sharp G^\sharp B = \mathcal{J}^{9,5}_{12,7,4}$
$B(m=10)$	≤ 3	$\to D^\sharp E G^\sharp B = \mathcal{J}^{10,4}_{12,7,4}, D^\sharp F^\sharp G^\sharp B = \mathcal{J}^{10,5}_{12,7,4}$
$F^\sharp(m=11)$	≤ 3	$\to D^\sharp F G^\sharp B = \mathcal{J}^{11,4}_{12,7,4}, D^\sharp F^\sharp G^\sharp B = \mathcal{J}^{11,5}_{12,7,4}$
$C^\sharp(m=12) = D^\flat(m=0)$	≤ 3	$\to D^\sharp F G^\sharp C = \mathcal{J}^{12,4}_{12,7,4}, D^\sharp F^\sharp G^\sharp C = \mathcal{J}^{12,5}_{12,7,4}$
$A^\flat(m=1)$	≤ 3	$\to C E^\flat F A^\flat = \mathcal{J}^{1,1}_{12,7,4}, C E^\flat G A^\flat = \mathcal{J}^{1,2}_{12,7,4}$
$E^\flat(m=2)$	≤ 3	$\to C E^\flat F A^\flat = \mathcal{J}^{2,1}_{12,7,4}, C E^\flat G A^\flat = \mathcal{J}^{2,2}_{12,7,4}$

TABLE 2. The proximity of diatonic scales and the smoothness of 7-th chords. Each voice leading starts from $CEGB = \mathcal{J}^{5,3}_{12,7,4}$ in C major scale.

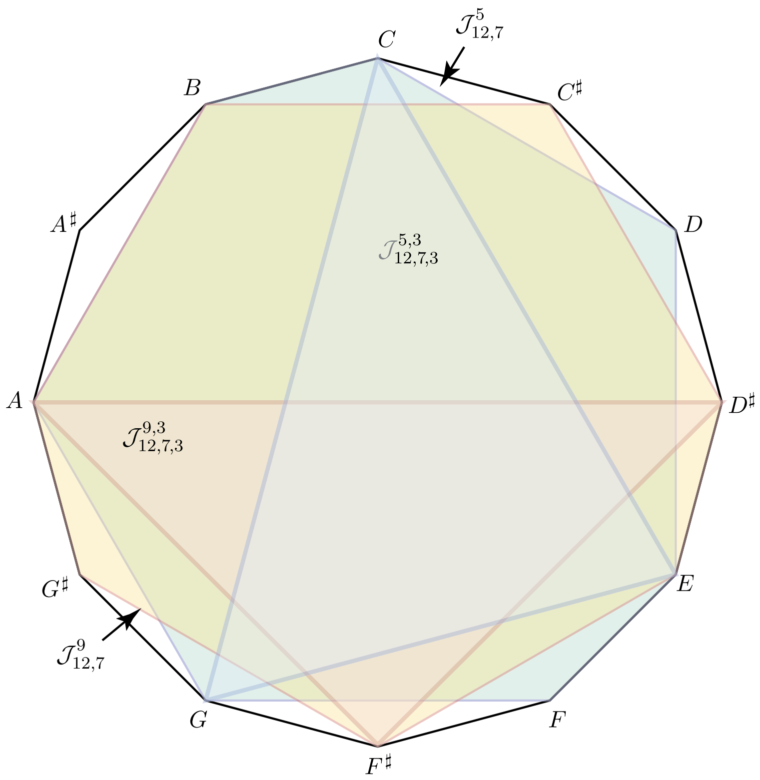

FIGURE 2. A voice leading that breaks down the smoothness.

[8] H. Muramatsu, *The choice of a chord C - A mathematical view of voice leadings (Japanese)*, Graduation thesis, 2017. http://hdl.handle.net/10424/7098.
[9] M. Rohrmeier, *Towards a generative syntax of tonal harmony*, Journal of Mathematics and Music **5**(1), pp. 35-53, 2011.

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