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A Renormalization Approach to Level Statistics on 1-dimensional Rotations

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1. Introduction

In the papers [1] and [2], Bleher observed the nearest-neighbour spacing distribution of energy levels for a 2-dimensional quantum harmonic oscillator with golden mean and generic ratio of frequencies. The essential problem is the determination of the nearest-neighbour spacing distribution generated by the points

$$\{\{n\gamma\} \mid n \in \mathbf{N}\}, \tag{1.1}$$

where $\gamma > 0$ is an irrational number and $\{a\}$ stands for the fractional part of $a > 0$.

In this note, we interpret (1.1) as an orbit of 1-dimensional shift $x \mapsto x + \gamma$ on $\mathbf{R}/\mathbf{Z}$. We see that the nearest-neighbour spacing has a self-similar structure and its behaviour is determined by a kind of a continued fractional expansion of the irrational number γ. In the observation of the shift dynamics, we use a time-scale renormalization. As a result, we describe the nearest-neighbour spacing distribution of the micro ensemble case (see [1]) for any irrational ratio of frequencies.

2. Notations and definitions

Let $\mathcal{I} = \{[0, a] \mid a > 0\}$ be a set of intervals with 0 as the left end. For $A \in \mathcal{I}$, $|A| = a$ whenever $A = [0, a]$. For $A_i = [0, a_i] \in \mathcal{I},\ i = 1, \ldots, n$, we define a *sum* of intervals

$$A_1 \oplus A_2 \oplus \cdots \oplus A_n = \{[0, a_1], [a_1, a_1 + a_2], \ldots, [a_1 + \cdots + a_{n-1}, a_1 + \cdots + a_n]\}.$$

We say $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ gives a *partition* of $B = [0, b] \in \mathcal{I}$ whenever $b = a_1 + \cdots + a_n$, and we denote

$$B \vdash A_1 \oplus A_2 \oplus \cdots \oplus A_n.$$

Conversely, we denote the union of intervals in a partition $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ as

$$[A_1 \oplus A_2 \oplus \cdots \oplus A_n] = [0, a_1 + a_2 + \cdots + a_n].$$

We use an abbreviation for the k-times sum $A \oplus \cdots \oplus A$ of $A \in \mathcal{I}$ as $A^{\oplus k}$. We denote $S + a = \{s + a \mid s \in S\}$ for a subset $S \subset \mathbf{R}$ and $a \in \mathbf{R}$.

3. Renormalization of partitions

Let $\rho_0 : x \mapsto x + \delta_0$ be a shift map on $\mathbf{R}/\mathbf{Z}$, where $0 < \delta_0 < 1$ is an irrational number. We put

$$E_{-1} = [0, 1], \quad \Delta_0 = [0, \delta_0], \quad \sigma_0 = \lfloor |E_{-1}|/\delta_0 \rfloor = \lfloor 1/\delta_0 \rfloor,$$

where $\lfloor a \rfloor$ stands for the integer part of $a > 0$. Since $\epsilon_0 = |E_{-1}| - \sigma_0\delta_0 = 1 - \sigma_0\delta_0$ is positive by definition of δ_0 and σ_0, we put $E_0 = [0, \epsilon_0] \in \mathcal{I}$. Then we see

Lemma 3.1. *The orbit*

$$\rho_0(0) = \delta_0,\ \rho_0^2(0) = 2\delta_0,\ \ldots,\ \rho_0^{\sigma_0}(0) = \sigma_0\delta_0$$

defines the partition of E_{-1} as

$$E_{-1} \vdash \Delta_0^{\oplus\sigma_0} \oplus E_0.$$

□

Consider the iteration $\rho_1 = \rho_0^{\sigma_0+1}$, which is a shift map on $\mathbf{R}/\mathbf{Z}$ described as

$$\rho_1 : x \mapsto x + \delta_1,$$

where $\delta_1 = (\sigma_0+1)\delta_0 - |E_{-1}| = (\sigma_0+1)\delta_0 - 1 = \delta_0 - \epsilon_0$. Putting

$$\Delta_1 = [0, \delta_1], \quad \sigma_1 = \lfloor |E_0|/\delta_1 \rfloor = \lfloor \epsilon_0/\delta_1 \rfloor,$$

we see $\epsilon_1 = |E_0| - \sigma_1\delta_1 = \epsilon_0 - \sigma_1\delta_1 > 0$ and we set $E_1 = [0, \epsilon_1] \in \mathcal{I}$.

Lemma 3.2. *The orbit*

$$\rho_1 \circ \rho_0^{\sigma_0}(0) = \delta_1 + \sigma_0\delta_0,\ \rho_1^2 \circ \rho_0^{\sigma_0}(0) = 2\delta_1 + \sigma_0\delta_0,\ \ldots,\ \rho_1^{\sigma_1} \circ \rho_0^{\sigma_0}(0) = \sigma_1\delta_1 + \sigma_0\delta_0$$

gives the partition of $E_0 + \rho_0^{\sigma_0}(0) = E_0 + \sigma_0\delta_0$ *as*

$$E_0 + \sigma_0\delta_0 \vdash (\Delta_1^{\oplus\sigma_1} \oplus E_1) + \sigma_0\delta_0.$$

For each $0 \le k \le \sigma_0 - 1$, *the orbit*

$$\rho_1 \circ \rho_0^k(0) = \delta_1 + k\delta_0,\ \rho_1^2 \circ \rho_0^k(0) = 2\delta_1 + k\delta_0,\ \ldots,\ \rho_1^{\sigma_1+1} \circ \rho_0^k(0) = (\sigma_1+1)\delta_1 + k\delta_0$$

gives the partition of $\Delta_0 + \rho_0^k(0) = \Delta_0 + k\delta_0$ *as*

$$\Delta_0 + k\delta_0 \vdash (\Delta_1^{\oplus(\sigma_1+1)} \oplus E_1) + k\delta_0.$$

Proof. The first statement is straightforward. As $\delta_0 = \delta_1 + \epsilon_0$, each point $\rho_1 \circ \rho_0^k(0) = \delta_1 + k\delta_0$ $(0 \le k \le \sigma_0 - 1)$ divides the interval $[k\delta_0, (k+1)\delta_0] = \Delta_0 + k\delta_0 = \Delta_0 + \rho_0^k(0)$ as

$$\Delta_0 + k\delta_0 \vdash (\Delta_1 \oplus E_0) + k\delta_0.$$

Hence the second one. □

We continue the process inductively. Being given an irrational number $0 < \delta_0 < 1$, we put $\sigma_0 = \lfloor 1/\delta_0 \rfloor$, $\epsilon_0 = 1 - \sigma_0\delta_0$, $d_0 = \sigma_0$ and $e_0 = 1$. Then we define inductively

$$\delta_{n+1} = \delta_n - \epsilon_n,\ \sigma_{n+1} = \lfloor \epsilon_n/\delta_{n+1} \rfloor,\ \epsilon_{n+1} = \epsilon_n - \sigma_{n+1}\delta_{n+1}, \tag{3.1}$$

$$\begin{pmatrix} d_{n+1} \\ e_{n+1} \end{pmatrix} = \begin{pmatrix} \sigma_{n+1}+1 & \sigma_{n+1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} d_n \\ e_n \end{pmatrix} \tag{3.2}$$

and

$$\Delta_{n+1} = [0, \delta_{n+1}],\ E_{n+1} = [0, \epsilon_{n+1}],\ \rho_{n+1} = \rho_0^{e_{n+1}}. \tag{3.3}$$

Lemma 3.3. *For* $n \ge 0$, *we have*

$$e_{n+1} = (\sigma_n + 1)e_n + \sigma_{n-1}e_{n-1} + \cdots + \sigma_0 e_0.$$

Proof. The recurrence relation (3.2) shows $e_{n+1} = e_n + d_n$ and $d_n = (\sigma_n+1)d_{n-1} + \sigma_n e_{n-1} = \sigma_n e_n + d_{n-1} = \sigma_n e_n + \sigma_{n-1}e_{n-1} + d_{n-2} = \cdots = \sigma_n e_n + \cdots + \sigma_0 e_0$. Hence the assertion. □

Lemma 3.4. *For any natural number* $0 \le N < e_{n+1}$, *we have a unique expression*

$$N = h_n \cdot e_n + h_{n-1} \cdot e_{n-1} + \cdots + h_0 \cdot e_0 \tag{3.4}$$

with the conditions

$$h_k \in \mathbf{Z},\ 0 \le h_k \le \sigma_k + 1 \ \ \text{and} \ \ h_k \cdot e_k + \cdots + h_0 \cdot e_0 < e_{k+1} \ \ \text{for all} \ \ k = 0, 1, \ldots, n. \tag{3.5}$$

Proof. We apply the greedy algorithm to N as follows. It follows from (3.2) that $e_{k+1} = d_k + e_k = (\sigma_k e_k + d_{k-1}) + e_k = (\sigma_k + 1)e_k + d_{k-1}$, hence $(\sigma_k + 1)e_k < e_{k+1} < (\sigma_k + 2)e_k$. We choose n with $e_n \leq N < e_{n+1}$ and put $h_n = \lfloor N/e_n \rfloor$. As $R_{n-1} = N - h_n e_n < e_n$ by definition of h_n, again we put $h_{n-1} = \lfloor R_{n-1}/e_{n-1} \rfloor$ and $R_{n-2} = R_{n-1} - h_{n-1}e_{n-1} < e_{n-1}$. Inductively we define $h_k = \lfloor R_k/e_k \rfloor$ and $R_{k-1} = R_k - h_k e_k < e_k$. We see $h_k = \lfloor R_k/e_k \rfloor \leq \lfloor e_{k+1}/e_k \rfloor = \sigma_k + 1$ and $h_k \cdot e_k + \cdots + h_0 \cdot e_0 < e_{k+1}$ by the construction. Without loss of generality, we assume that N has another expression $N = h'_n \cdot e_n + h'_{n-1} \cdot e_{n-1} + \cdots + h'_0 \cdot e_0$ fulfilling the conditions (3.5) and $h'_n \neq h_n$. If $h'_n \geq h_n + 1$, we have a contradiction

$$\begin{aligned} h'_n e_n &\leq h'_n \cdot e_n + h'_{n-1} \cdot e_{n-1} + \cdots + h'_0 \cdot e_0 \\ &= h_n \cdot e_n + h_{n-1} \cdot e_{n-1} + \cdots + h_0 \cdot e_0 < (h_n + 1)e_n \leq h'_n e_n. \end{aligned}$$

The similar argument shows a contradiction in the case $h'_n + 1 \leq h_n$, hence the uniqueness. □

We note that the expression (3.4) brings the equation $\rho_0^N = \rho_n^{h_n} \circ \rho_{n-1}^{h_{n-1}} \circ \cdots \circ \rho_0^{h_0}$: particularly we have $\rho_0^N(0) = h_n\delta_n + h_{n-1}\delta_{n-1} + \cdots + h_0\delta_0$. If a natural number N has the *greedy expansion* (3.4), we set

$$g(N) = (h_n, h_{n-1}, \ldots, h_0).$$

As $e_1 = \sigma_0 + 1$ by definition, we note $0 \leq h_0 \leq \sigma_0 < e_1$.

Lemma 3.5. *We have*

$$\rho_n^{\sigma_n} \circ \rho_{n-1}^{\sigma_{n-1}} \circ \cdots \circ \rho_0^{\sigma_0}(0) = \sigma_n\delta_n + \sigma_{n-1}\delta_{n-1} + \cdots + \sigma_0\delta_0 = 1 - \epsilon_n$$

and

$$\sum_{k=0}^{\infty} \sigma_k\delta_k = 1, \text{ hence } \sum_{k=n+1}^{\infty} \sigma_k\delta_k = \epsilon_n.$$

Proof. The recurrence relation (3.1) shows the second equation of the first statement. The second statement is shown as $\lim_{n\to\infty} \epsilon_n = 0$. □

Lemma 3.6. *Let $g(N) = (h_n, h_{n-1}, \ldots, h_0)$ be the greedy expansion of a natural number N with $e_n \leq N < e_{n+1}$. Then one of the followings holds:*

(1) $1 \leq h_n \leq \sigma_n$.

(2) $h_n = \sigma_n + 1$ *and* $h_{n-1}e_{n-1} + \cdots + h_0e_0 < e_n - e_{n-1}$.

Conversely, if a greedy expansion $(h_n, \ldots, h_0)$ of N satisfies either the condition (1) or (2), *$e_n \leq N < e_{n+1}$ holds.*

Proof. As $h_{n-1}e_{n-1} + \cdots + h_0e_0 < e_n$ by the condition (3.5), $h_n \geq 1$ is necessary. If $h_n \leq \sigma_n$, we see

$$N = h_ne_n + h_{n-1}e_{n-1} + \cdots + h_0e_0 < \sigma_ne_n + e_n < e_{n+1}$$

by the lemma 3.3 and 3.4. If $h_{n-1}e_{n-1} + \cdots + h_0e_0 < e_n - e_{n-1} = d_{n-1}$ and $h_n = \sigma_n + 1$, also the lemma 3.3 shows

$$\begin{aligned} N &= (\sigma_n + 1)e_n + h_{n-1}e_{n-1} + \cdots + h_0e_0 \\ &< (\sigma_n + 1)e_n + e_n - e_{n-1} \\ &= (\sigma_n + 1)e_n + (\sigma_{n-1} + 1)e_{n-1} + \sigma_{n-1}e_{n-2} + \cdots + \sigma_0e_0 - e_{n-1} = e_{n+1}. \end{aligned}$$

We note that $\#\{N \mid e_n \leq N < e_{n+1} \text{ and } 1 \leq h_n \leq \sigma_n\} = \sigma_ne_n$ and $\#\{N \mid e_n \leq N < e_{n+1} \text{ and } h_{n-1}e_{n-1} + \cdots + h_0e_0 < d_{n-1}\} = d_{n-1}$. Then the total number of N in the case (1) and (2) is $\sigma_ne_n + d_{n-1} =$

$\sigma_n(e_{n-1}+d_{n-1})+d_{n-1}=(\sigma_n+1)d_{n-1}+\sigma_n e_{n-1}=d_n$. Thus either the condition (1) or (2) holds for any natural number N with $e_n \le N < e_{n+1}$. □

Theorem 3.7. *Let*

$$[0,1] \vdash A_0 \oplus \cdots \oplus A_{e_{n+1}-1} \tag{3.6}$$

be the partition defined by the orbit

$$\rho_0(0),\ \rho_0^2(0),\ \ldots,\ \rho_0^{e_{n+1}-1}(0).$$

Let l_k be the natural number such that $[A_0 \oplus \cdots \oplus A_{k-1}] = [0, \rho_0^{l_k}(0)]$ and put $l_0 = 0$. Then we have

$$A_k = \begin{cases} \Delta_n, & \text{if } l_k < e_{n+1}-e_n = d_n, \\ E_n, & \text{otherwise,} \end{cases}$$

and hence we see $\#\{k \mid A_k = \Delta_n\} = d_n$ and $\#\{k \mid A_k = E_n\} = e_n$.

Proof. We prove the theorem by induction on n. It follows from the lemma 3.1 that the theorem holds for $n=0$. Suppose that the theorem holds for n. Let $[0,1] \vdash A_0 \oplus \cdots \oplus A_{e_n-1}$ be the partition defined by points $\rho_0(0),\ \rho_0^2(0),\ \ldots,\ \rho_0^{e_n-1}(0)$. Then, the new $e_{n+1}-e_n = d_n$ points $\rho_0^{e_n}(0), \ldots, \rho_0^{e_{n+1}-1}(0)$ defines the refinement $[0,1] \vdash B_0 \oplus \cdots \oplus B_{e_{n+1}-1}$.

In the case $A_k = E_{n-1}$, it is possible to divide the interval $A_k + \rho_0^{l_k}(0) = [\rho_0^{l_k}(0), \rho_0^{l_{k+1}}(0)]$ by the new points $\rho_n \circ \rho_0^{l_k}(0), \ldots, \rho_n^{\sigma_n} \circ \rho_0^{l_k}(0)$, which define the partition $A_k \vdash \Delta_n^{\oplus \sigma_n} \oplus E_n$ because of $\epsilon_{n-1} = \sigma_n \delta_n + \epsilon_n$. It follows from the lemma 3.6 that the exponential of the point $\rho_n^{h_n} \circ \rho_0^{l_k}(0) = \rho_0^{h_n e_n + l_k}(0)$ $(1 \le h_n \le \sigma_n)$ satisfies $e_n \le h_n e_n + l_k < e_{n+1}$.

In the case $A_k = \Delta_{n-1}$, it is possible to divide the interval $A_k + \rho_0^{l_k}(0) = [\rho_0^{l_k}(0), \rho_0^{l_{k+1}}(0)]$ by the new points $\rho_n \circ \rho_0^{l_k}(0), \ldots, \rho_n^{\sigma_n+1} \circ \rho_0^{l_k}(0)$, which define the partition $A_k \vdash \Delta_n^{\oplus \sigma_n+1} \oplus E_n$ because of $\Delta_{n-1} = [\Delta_n \oplus E_n]$. As $l_k < e_{n+1}-e_n$, again the lemma 3.6 shows that the exponential of the point $\rho_n^{h_n} \circ \rho_0^{l_k}(0) = \rho_0^{h_n e_n + l_k}(0)$ $(1 \le h_n \le \sigma_n + 1)$ satisfies $e_n \le h_n e_n + l_k < e_{n+1}$.

As the numbers of the points dividing the interval of type E_{n-1} and Δ_{n-1} are $\sigma_n e_{n-1}$ and $(\sigma_n + 1)d_{n-1}$ respectively, there appears $\sigma_n e_{n-1} + (\sigma_n+1)d_{n-1} = d_n$ points in the construction above, which means all the points $\rho_0^{e_n}(0), \ldots, \rho_0^{e_{n+1}-1}(0)$ are used to determine the refinement $B_0 \oplus \cdots \oplus B_{e_{n+1}-1}$. By way of the construction, we see $\#\{k \mid B_k = \Delta_n\} = (\sigma_n+1)d_{n-1} + \sigma_n e_{n-1} = d_n$ and $\#\{k \mid B_k = E_n\} = d_{n-1} + e_{n-1} = e_n$. Let m_k be the natural number such that $[B_0 \oplus \cdots \oplus B_{k-1}] = [0, \rho_0^{m_k}(0)]$. If $B_k = \Delta_n$, $[0, \rho_0^{m_{k+1}}(0)] = [B_0 \oplus \cdots \oplus B_k] = [B_0 \oplus \cdots \oplus B_{k-1} \oplus \Delta_n]$ means $\rho_0^{m_{k+1}}(0) = \rho_n \circ \rho_0^{m_k}(0)$ and hence $m_k + e_n = m_{k+1} < e_{n+1}$. Thus $m_k < e_{n+1} - e_n$, which complete the proof. □

Remark. Our discussion works even if $\sigma_n = 0$ occurs for some (infinitely many) n. Thus there is no restriction to the irrational number $0 < \delta_0 < 1$.

4. Continued fractional expansion and level statistics

Given an irrational number $0 < \delta_0 < 1$, the recurrence relation (3.1) give rise to a continued fractional expansion. It follows from the relation $\epsilon_{n-1} = \sigma_n \delta_n + \epsilon_n$ and $\delta_n = \delta_{n+1} + \epsilon_n$ that

$$\frac{\epsilon_{n-1}}{\delta_n} = \sigma_n + \frac{\epsilon_n}{\delta_n} = \sigma_n + \frac{\epsilon_n}{\epsilon_n + \delta_{n+1}} = \sigma_n + \cfrac{1}{1 + \cfrac{1}{\epsilon_n/\delta_{n+1}}},$$

from which we have a continued fractional expansion

$$\frac{1}{\delta_0} = \sigma_0 + \frac{1}{1+}\,\frac{1}{\sigma_1+}\,\frac{1}{1+}\,\frac{1}{\sigma_2+}\cdots\frac{1}{1+}\,\frac{1}{\epsilon_n/\delta_{n+1}}.$$

The number of intervals Δ_n's and E_n's in the partition (3.6) is controlled by σ_n's. Therefore we obtain the nearest-neighbour spacing distribution generated by the 1-dimensional rotation ρ_0.

Proposition 4.1. *Let*

$$\frac{1}{\delta_0} = \sigma_0 + \frac{1}{1+} \frac{1}{\sigma_1+} \frac{1}{1+} \frac{1}{\sigma_2+} \cdots \tag{4.1}$$

be a continued fractional expansion of a real number $0 < \delta_0 < 1$*, and define* e_n*'s,* d_n*'s,* δ_n*'s and* ϵ_n*'s by (3.1) and (3.2). For a natural number* $e_n \leq N < e_{n+1}$*, of which the greedy expansion is* $g(N) = (h_n, h_{n-1}, \ldots, h_0)$*, the orbit* $0, \rho_0(0), \ldots, \rho_0^N(0)$ *generates the nearest-neighbour spacing distribution as follows:*

	number of spacing	
length of spacing	*the case* $M+1 < d_{n-1}$	*the case* $M+1 \geq d_{n-1}$
δ_n	$N - e_n + 1$	$N - e_n + 1$
$\delta_{n-1} - (h_n - 1)\delta_n$	$d_{n-1} - M - 1$	0
$\delta_{n-1} - h_n\delta_n$	$M + 1 + e_{n-1}$	$d_{n-1} + e_n - M - 1$
$\delta_{n-1} - (h_n + 1)\delta_n$	0	$M + 1 - d_{n-1}$

where $M = N - h_n e_n$*.*

Proof. We recall the construction of the partition in the theorem 3.7. As the points $\rho_0(0), \ldots, \rho_0^{e_n-1}(0)$ define the partition

$$[0,1] \vdash A_0 \oplus \cdots \oplus A_{e_n-1}$$

constructed by the intervals Δ_{n-1} and E_{n-1}, the rest points $\rho_0^{e_n}(0), \ldots, \rho_0^N(0)$ generate $N - e_n + 1$ intervals coincide with Δ_n.

Let us consider the case $0 \leq k \leq M$. We note $h_n e_n + k \leq N$. Moreover, if $k < d_{n-1}$, we see that $\rho_0^k(0)$ is the left edge of an interval $A_{j(k)}$ coincide with Δ_{n-1}, and thus the points $\rho_n \circ \rho_0^k(0), \ldots, \rho_n^{h_n} \circ \rho_0^k(0)$ gives the refinement

$$A_{j(k)} \vdash \Delta_n^{\oplus h_n} \oplus [0, \delta_{n-1} - h_n\delta_n].$$

If $d_{n-1} \leq k < e_n$, we see that $\rho_0^k(0)$ is the left edge of an interval $A_{j(k)}$ coincide with E_{n-1}, and thus we have the refinement

$$A_{j(k)} \vdash \Delta_n^{\oplus h_n} \oplus [0, \epsilon_{n-1} - h_n\delta_n].$$

Similar argument holds in the case $M < k < e_n$. We note $(h_n - 1)e_n + k < N < h_n e_n + k$. Thus for each case $k < d_{n-1}$ and $d_{n-1} \leq k < e_n$, the points $\rho_n \circ \rho_0^k(0), \ldots, \rho_n^{h_n-1} \circ \rho_0^k(0)$ derive the refinement

$$A_{j(k)} \vdash \Delta_n^{\oplus(h_n-1)} \oplus [0, \delta_{n-1} - (h_n - 1)\delta_n]$$

and

$$A_{j(k)} \vdash \Delta_n^{\oplus(h_n-1)} \oplus [0, \epsilon_{n-1} - (h_n - 1)\delta_n]$$

respectively. The observation above shows that the numbers of intervals $[0, \delta_{n-1} - h_n\delta_n]$, $[0, \epsilon_{n-1} - h_n\delta_n]$, $[0, \delta_{n-1} - (h_n - 1)\delta_n]$ and $[0, \epsilon_{n-1} - (h_n - 1)\delta_n]$ equal to $\min\{M+1, d_{n-1}\}$, $\max\{M+1-d_{n-1}, 0\}$, $\max\{d_{n-1} - M - 1, 0\}$ and $\min\{e_n - M - 1, e_{n-1}\}$ respectively. Finally we note $\epsilon_{n-1} - (h_n - 1)\delta_n = \delta_{n-1} - h_n\delta_n$ and $\epsilon_{n-1} - h_n\delta_n = \delta_{n-1} - (h_n + 1)\delta_n$ because of $\delta_{n-1} = \epsilon_{n-1} + \delta_n$. □

We apply the proposition to the simple case $\sigma_n = \sigma$ for all n. The continued fractional expansion (4.1) leads to the quadratic relation

$$\sigma\delta^2 = 1 - \sigma\delta, \tag{4.2}$$

which derives $\delta_n = \delta(\sigma\delta^2)^n$ and $\epsilon_n = (\sigma\delta^2)^{n+1}$, because of the recurrence relation (3.1). Meanwhile we see

$$\begin{pmatrix} \sigma+1 & \sigma \\ 1 & 1 \end{pmatrix} = \frac{1}{1+\sigma\delta^2} \begin{pmatrix} 1 & \sigma\delta \\ \delta & -1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \sigma\delta \\ \delta & -1 \end{pmatrix}$$

where $\alpha = (1+\delta)/\delta > 1$ and $\beta = \sigma\delta^2 < 1$, which are the roots of the quadratic equation $\lambda^2 - (\sigma + 2)\lambda + 1 = 0$. With the help of the relation (4.2), we have

$$\begin{pmatrix} d_n \\ e_n \end{pmatrix} = \frac{\alpha^n}{1+\sigma\delta^2} \begin{pmatrix} \frac{1}{\delta} - \sigma^2\delta^3\gamma^n \\ 1 + \sigma\delta^2\gamma^n \end{pmatrix}$$

where $\gamma = \beta/\alpha < 1$. Thus we obtain

$$\text{(4.3)} \qquad \lim_{n\to\infty} \frac{d_n}{e_n} = \frac{1}{\delta} \quad \text{and} \quad \lim_{n\to\infty} \frac{e_{n-1}}{e_n} = \lim_{n\to\infty} \frac{d_{n-1}}{d_n} = \frac{1}{\alpha} = \frac{\delta}{1+\delta} = \sigma\delta^2.$$

Following Bleher's observation [1], we regard a large natural number $e_n \le N < e_{n+1}$ as an internally dividing point $N = xe_{n+1} + (1-x)e_n = e_n + xd_n$ $(0 \le x < 1)$. Then (4.3) shows

$$\lim_{n\to\infty} \frac{N}{e_n} = 1 + \frac{x}{\delta},$$

which means that the coefficient h_n of the highest term in the greedy expansion $g(N) = (h_n, h_{n-1}, \dots, h_0)$ is asymptotically determined by $h_n \sim \lfloor 1 + x/\delta \rfloor = \lfloor x/\delta \rfloor + 1 \le \sigma + 1$ because of the property (3.5) of the greedy expansion. Putting $M = N - h_n e_n$, we have

$$\frac{M}{N} = 1 - h_n \frac{e_n}{N} \sim 1 - h_n \frac{\delta}{\delta + x} = \frac{\delta}{\delta + x}\left(1 + \frac{x}{\delta} - h_n\right) = \frac{\delta}{\delta + x}\left\{\frac{x}{\delta}\right\},$$

$$\frac{e_{n-1}}{N} = \frac{e_n}{N} \cdot \frac{e_{n-1}}{e_n} \sim \frac{\sigma\delta^3}{\delta + x} \quad \text{and} \quad \frac{d_{n-1}}{N} = \frac{e_{n-1}}{N} \cdot \frac{d_{n-1}}{e_{n-1}} \sim \frac{\sigma\delta^2}{\delta + x}$$

as $n \to \infty$. Thus the inequality $M + 1 < d_{n-1}$ asymptotically corresponds to

$$\frac{\delta}{\delta + x}\left\{\frac{x}{\delta}\right\} < \frac{\sigma\delta^2}{\delta + x},$$

which leads to

$$\left\{\frac{x}{\delta}\right\} < \sigma\delta.$$

Summing up, with the help of the proposition 4.1, we have the following.

Proposition 4.2. *Let σ be a natural number and $0 < \delta < 1$ be a root of the quadratic equation*

$$\sigma\delta^2 = 1 - \sigma\delta.$$

Take a real number $0 \le x < 1$ and put $h = \lfloor x/\delta \rfloor + 1$. For $N = e_n + xd_n$, as $n \to \infty$, the orbit $\rho_0(0), \rho_0^2(0), \dots, \rho_0^N(0)$ derives asymptotically the nearest-neighbour spacing distribution as following table:

length of spacing (scaling factor: $\delta(\sigma\delta^2)^n$)	*probability*	
	the case $\left\{\frac{x}{\delta}\right\} < \sigma\delta$	*the case* $\left\{\frac{x}{\delta}\right\} \geq \sigma\delta$
1	$\frac{x}{\delta+x}$	$\frac{x}{\delta+x}$
$\frac{1}{\delta} - h + 2$	$\frac{\delta}{\delta+x}\left(\sigma\delta - \left\{\frac{x}{\delta}\right\}\right)$	0
$\frac{1}{\delta} - h + 1$	$\frac{\delta}{\delta+x}\left(1 - \sigma\delta + \left\{\frac{x}{\delta}\right\}\right)$	$\frac{\delta}{\delta+x}\left(1 + \sigma\delta - \left\{\frac{x}{\delta}\right\}\right)$
$\frac{1}{\delta} - h$	0	$\frac{\delta}{\delta+x}\left(\left\{\frac{x}{\delta}\right\} - \sigma\delta\right)$

where we rescale the length of the spacing by the factor $\delta(\sigma\delta^2)^n$.

□

Bleher [1] has considered the case $\sigma = 1$, that is, the case δ being the golden mean. In the case, we see $d_n = f_{2n+2}$ and $e_n = f_{2n+1}$ where f_n's stand for the Fibonacci numbers, $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$. We see $h = \lfloor x/\delta \rfloor + 1 = 1$ or 2 as $1/2 < \delta < 1$. The restriction $\delta \leq x < 1$ means $\{x/\delta\} = x/\delta - 1$, hence $h = 2$ and $\{x/\delta\} - \delta = (x-1)/\delta < 0$, that is, $\{x/\delta\} < \delta$, using the relation $\delta^2 = 1 - \delta$. Up to the scaling factor, there appears spacings with length $1, 1/\delta$ and δ. The restriction $0 \leq x < \delta$ means $\{x/\delta\} = x/\delta$ and hence $h = 1$. The condition $\{x/\delta\} < \delta$ is equivalent to $x < \delta^2$. We see the length of the spacings are $1, 1/\delta^2, 1/\delta$ and δ in this case. As a result, we obtain the following table, reproducing the micro ensemble case of Bleher's result.

length of spacing (scaling factor: δ^{2n+1})	probability	
	the case $x < \delta^2$	the case $x \geq \delta^2$
1	$\frac{x}{\delta+x}$	$\frac{x}{\delta+x}$
$\frac{1}{\delta^2}$	$\frac{1-\delta-x}{\delta+x}$	0
$\frac{1}{\delta}$	$\frac{\delta^3+x}{\delta+x}$	$\frac{1-x}{\delta+x}$
δ	0	$\frac{\delta-1+x}{\delta+x}$

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