Bulletin of Aichi Univ. of Education, 57 (Natural Sciences), pp. 1－2, March, 2008

A Simple Proof of a Characterization Theorem of the Sphere

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S. M. Ulam posed a problem which states: "If a body rests in equilibrium in every direction on a flat horizontal surface, must it be a sphere?" See [1] for detail. In 1974, L.Montejano [2] solved it as follows.

Theorem. *If a body $D \subset \mathbb{R}^3$ rests in equilibrium in every direction, then its convex hull $C(D)$ must be a sphere.* (*If a planar body $D \subset \mathbb{R}^2$ rests in equilibrium in every direction, then $C(D)$ must be a circle.*)

In this note, we give a simple proof of the 2-dimensional case of the above theorem. The reader can prove the 3-dimensional case in the same way.

Let $D \subset \mathbb{R}^2$ be a connected closed region. Then, by the definition, the centroid G of D is an interior point of the convex hull $C(D)$ of D. For every unit vector $\vec{u}$, there is a line l orthogonal to $\vec{u}$ such that $l \cap D \neq \emptyset$ and

$$\overrightarrow{PQ} \cdot \vec{u} \leqq 0 \text{ for every } P \in l,\ Q \in D. \tag{1}$$

Figure 1 shows the situation. We call l the *supporting line* of D with respect to the direction $\vec{u}$. Let P be the foot of the perpendicular from G to l. We say that D *rests in equilibrium in a direction* $\vec{u}$ if $P \in C(D)$. Figure 2 shows a planar body D which does not rest in equilibrium in the downward direction.

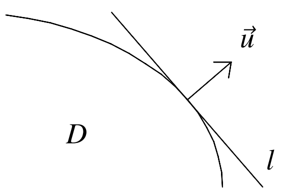

Figure 1

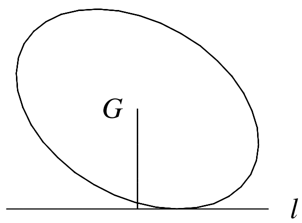

Figure 2

Proof of Theorem in 2-Dimensional Case. Since G is an interior point of the convex set $C(D)$, every half line from G meets the boundary $\partial C(D)$ at exactly one point. So we can represent the closed curve $\partial C(D)$ by the polar equation $r = r(\theta)$ with G as the origin. Take two arguments θ_0, θ_1, and the corresponding points $P_0, P_1 \in \partial C(D)$, that is,

$$\overrightarrow{GP_i} = r(\theta_i)\vec{u}_i, \quad \vec{u}_i = (\cos\theta_i, \sin\theta_i), \quad i = 0,1. \tag{2}$$

Let l_0 be the supporting line of D with respect to the direction $\vec{u}_0$. Since D rests in equilibrium in the direction $\vec{u}_0$, we obtain that $P_0 \in l_0$. Applying (1) to P_0, P_1, we obtain that

$$\overrightarrow{P_0P_1} \cdot \vec{u}_0 = r(\theta_1)\cos h - r(\theta_0) \leqq 0, \tag{3}$$

where $h = \theta_1 - \theta_0$. Changing the role of P_0 and P_1, we obtain that

$$r(\theta_0)\cos h - r(\theta_1) \leqq 0 \tag{4}$$

Combining (3) and (4), we obtain that

$$-r(\theta_0)(1-\cos h) \leqq r(\theta_1) - r(\theta_0) \leqq r(\theta_1)(1-\cos h). \tag{5}$$

Dividing it by h, and tending h→0, we obtain that $r'(\theta) = 0$ for every θ. Hence $r(\theta)$ must be a constant.

References

[1] H. T. Croft, K. J. Falconer and R. K. Guy, Unsolved Problems in Geometry, Springer-Verlag, New York, 1991.

[2] L. Montejano, On a problem of Ulam concerning a characterization of the sphere, Stud. Appl. Math. **53** (1974), 243. 248.

(Received September 18, 2007)