Bulletin of Aichi Univ. of Education, 57 (Natural Sciences), pp. 3 - 8, March, 2008

A proof of self-similarity of cut and project sets in 1-dimension

Eriko NODA* and Yukihiro HASHIMOTO

Department of Mathematics, Aichi University of Education, Kariya, 448-8542, Japan

1. Introduction

The Fibonacci substitution $\sigma(0)=01$ and $\sigma(1)=0$ generates an infinite sequence

$$0100101001001010010100\ldots$$

which is a fixed point of σ, while this sequence is also obtained as a cut and project set along a line with the slope $\gamma=(\sqrt{5}-1)/2$. In the paper [1], the coinsidence of these sequences stated above is proved. We give a generalization of such a result (Theorem 5.1), via proving that a part of the cut and project set has a *rigidity* property (Proposition 4.1).

2. Cut and project sets in 1-dimension

Generally, a cut and project set in $\mathbf{R}^n$ is given as follows. The integer lattice points in $\mathbf{R}^n$ are given by $\mathbf{p}=\sum_{j=1}^{n} p_j\mathbf{e}_j\ (p_j\in\mathbf{Z})$ where $\left\{\mathbf{e}_j={}^t\left(0,\ldots,\overset{j}{1},\ldots,0\right)\right\}$ is a canonical orthonormal basis. For $T=(t_{jk})\in O(n)$, we put $\mathbf{e}'_j=\sum_{k=1}^{n} t_{kj}\mathbf{e}_k$. For a natural number $0<m<n$, the *parallel space* $\mathbf{E}^{\parallel}$ (resp. the *perp space* $\mathbf{E}^{\perp}$) is a linear span of $\{\mathbf{e}'_1,\ldots,\mathbf{e}'_m\}$ (resp. $\{\mathbf{e}'_{m+1},\ldots,\mathbf{e}'_n\}$). $\pi^{\parallel}$ (resp. $\pi^{\perp}$) denotes the orthogonal projection $\pi^{\parallel}:\mathbf{R}^n\to\mathbf{E}^{\parallel}$ (resp. $\pi^{\perp}:\mathbf{R}^n\to\mathbf{E}^{\perp}$). Then the $\mathbf{E}^{\parallel}$-component $\mathbf{e}_j^{\parallel}$ and $\mathbf{E}^{\perp}$-component $\mathbf{e}_j^{\perp}$ of a canonical unit vector $\mathbf{e}_j$ are given by

$$\mathbf{e}_j^{\parallel}=\pi^{\parallel}(\mathbf{e}_j)=\sum_{k=1}^{m} t_{kj}\mathbf{e}'_k,\qquad \mathbf{e}_j^{\perp}=\pi^{\perp}(\mathbf{e}_j)=\sum_{k=m+1}^{n} t_{kj}\mathbf{e}'_k,$$

and the $\mathbf{E}^{\parallel}$-component $\mathbf{e}_j^{\parallel}$ and $\mathbf{E}^{\perp}$-component $\mathbf{e}_j^{\perp}$ of a lattice point $\mathbf{p}$ are given by

$$\mathbf{p}^{\parallel}=\pi^{\parallel}(\mathbf{p})=\sum_{j=1}^{n} p_j\mathbf{e}_j^{\parallel},\qquad \mathbf{p}^{\perp}=\pi^{\perp}(\mathbf{p})=\sum_{j=1}^{n} p_j\mathbf{e}_j^{\perp}$$

Take a unital hyper-cube Q in $\mathbf{R}^n$, $Q=\{(q_1,\ldots,q_n)\mid 0\le q_j<1\}$, and consider the orthogonal projection W of Q into $\mathbf{E}^{\perp}$, that is $W=\{\pi^{\perp}(\mathbf{q})\mid \mathbf{q}\in Q\}$, called the *window*. Given a vector $\mathbf{t}\in\mathbf{E}^{\perp}$ called a *shift*, the *cut and project set* $CP_m^n(T,\mathbf{t})$ is defined by

$$CP_m^n(T,\mathbf{t})=\{\pi^{\parallel}(\mathbf{p})\mid \pi^{\perp}(\mathbf{p})\in W+\mathbf{t},\ \mathbf{p}\in\mathbf{Z}^n\}\subset\mathbf{E}^{\parallel}$$

In the case of $n=2$ and $m=1$, we take

(2.1)
$$T=\frac{1}{\sqrt{\gamma^2+1}}\begin{pmatrix}1 & -\gamma\\ \gamma & 1\end{pmatrix}.$$

where $\gamma\ge 0$. The 1-dimensional subspaces $\mathbf{E}^{\parallel}$ and $\mathbf{E}^{\perp}$ are spaned by

(2.2)
$$\mathbf{e}'_1=\frac{1}{\sqrt{\gamma^2+1}}\begin{pmatrix}1\\ \gamma\end{pmatrix}\quad\text{and}\quad \mathbf{e}'_2=\frac{1}{\sqrt{\gamma^2+1}}\begin{pmatrix}-\gamma\\ 1\end{pmatrix}$$

respectively. The canonical unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ are decomposed as

*undergraduate student

$$\mathbf{e}_1 = \frac{1}{\sqrt{\gamma^2+1}}\mathbf{e}_1' - \frac{\gamma}{\sqrt{\gamma^2+1}}\mathbf{e}_2' = \mathbf{e}_1^{\parallel} + \mathbf{e}_1^{\perp}, \quad \mathbf{e}_2 = \frac{\gamma}{\sqrt{\gamma^2+1}}\mathbf{e}_1' + \frac{1}{\sqrt{\gamma^2+1}}\mathbf{e}_2' = \mathbf{e}_2^{\parallel} + \mathbf{e}_2^{\perp}$$

and then a lattice point $\mathbf{p}$ is decomposed as

$$\begin{aligned}\mathbf{p} &= p_1\mathbf{e}_1 + p_2\mathbf{e}_2 = (p_1\mathbf{e}_1^{\parallel} + p_2\mathbf{e}_2^{\parallel}) + (p_1\mathbf{e}_1^{\perp} + p_2\mathbf{e}_2^{\perp}) \\ &= \left(\frac{1}{\sqrt{\gamma^2+1}}p_1 + \frac{\gamma}{\sqrt{\gamma^2+1}}p_2\right)\mathbf{e}_1' + \left(-\frac{\gamma}{\sqrt{\gamma^2+1}}p_1 + \frac{1}{\sqrt{\gamma^2+1}}p_2\right)\mathbf{e}_2' = \mathbf{p}^{\parallel} + \mathbf{p}^{\perp},\end{aligned}$$

while the window set is given by

$$\begin{aligned}W &= \{x_1\mathbf{e}_1^{\perp} + x_2\mathbf{e}_2^{\perp} + \mathbf{t} \mid 0 \le x_j < 1\} = \left\{\left(-\frac{\gamma x_1}{\sqrt{\gamma^2+1}} + \frac{x_2}{\sqrt{\gamma^2+1}} + t\right)\mathbf{e}_2' \middle| 0 \le x_j < 1\right\} \\ &= \left\{s\,\mathbf{e}_2' \middle| -\frac{\gamma}{\sqrt{\gamma^2+1}} + t < s < \frac{1}{\sqrt{\gamma^2+1}} + t\right\},\end{aligned}$$

where $\mathbf{t} = t\,\mathbf{e}_2'$ is a shift. In this situation, the cut and project set $CP_1^2(T, t)$ is defined by

$$\left\{\left(\frac{p_1+\gamma p_2}{\sqrt{\gamma^2+1}}\right)\mathbf{e}_1' \middle| (p_1, p_2) \in D(\gamma, t) \cap \mathbf{Z}^2\right\}, \tag{2.3}$$

where $D(\gamma, t) = \{(x, y) \in \mathbf{R}^2 | -\gamma + t\sqrt{\gamma^2+1} < -\gamma x + y < 1 + t\sqrt{\gamma^2+1}\}$. As $\gamma + 1 > 1$, there exists at least one $p_2 \in \mathbf{Z}$ such that $(n, p_2) \in CP_1^2(T, t)$ for any $n \in \mathbf{Z}$.

Proposition 2. 1. *For* $(p_1, p_2) \in D(\gamma, t) \cap \mathbf{Z}^2$, *when* $-\gamma p_1 + p_2 \neq t\sqrt{\gamma^2+1}$, *one of the followings holds:*

i) $(p_1, p_2+1) \in D(\gamma, t)$ *and* $(p_1+1, p_2) \notin D(\gamma, t)$
ii) $(p_1+1, p_2) \in D(\gamma, t)$ *and* $(p_1, p_2+1) \notin D(\gamma, t)$

When $-\gamma p_1 + p_2 = t\sqrt{\gamma^2+1}$, *it holds that*

$$(p_1, p_2+1) \notin D(\gamma, t) \text{ and } (p_1+1, p_2) \notin D(\gamma, t)$$

Proof. We see that (p_1, p_2), $(p_1, p_2+1) \in D(\gamma, t)$ shows $-\gamma + t\sqrt{\gamma^2+1} < -\gamma p_1 + p_2 < t\sqrt{\gamma^2+1}$, and (p_1, p_2), $(p_1+1, p_2) \in D(\gamma, t)$ shows $t\sqrt{\gamma^2+1} < -\gamma p_1 + p_2 < 1 + t\sqrt{\gamma^2+1}$, which are exclusive to each other if $-\gamma p_1 + p_2 \neq t\sqrt{\gamma^2+1}$, and neither of them hold if $-\gamma p_1 + p_2 = t\sqrt{\gamma^2+1}$.

A lattice point $(p_1, p_2) \in \mathbf{Z}^2$ is called *exceptional* if $-\gamma(p_1 - 1) + p_2 = t\sqrt{\gamma^2+1}$ holds.

Throughout this paper, we fix the shift

$$t = \frac{\gamma - 1}{\sqrt{\gamma^2+1}}. \tag{2.4}$$

Then the corresponding cut and project set $CP_\gamma = CP_1^2(\gamma, t)$ is given by

$$\{p_1 + \gamma p_2 | -1 < -\gamma p_1 + p_2 < \gamma, \ (p_1, p_2) \in \mathbf{Z}^2\},$$

and the set of exceptional points CP_γ^e is given by

$$\{p_1 + \gamma p_2 | -\gamma p_1 + p_2 = -1, \ (p_1, p_2) \in \mathbf{Z}^2\},$$

denoting $CP_\gamma^+ = CP_\gamma \cup CP_\gamma^e$. The strip $D = D(\gamma, t)$ is given by

$$\{(x, y) \in \mathbf{R}^2 | -1 < -\gamma x + y < \gamma\} \tag{2.5}$$

Note that there exists a unique exceptional point whenever γ is irrational: $CP_\gamma^e = \{(0, 1)\}$. The FIGURE 1 shows a cut and project set CP_γ for $\gamma = \frac{\sqrt{5}-1}{2}$ with a unique exceptional point $(0, -1)$.

For $x \in \mathbf{R}$, $[x]$ denotes the largest integer smaller than or equal to x, and we put $\{x\} = x - [x]$.

Proposition 2. 2. *For an irrational positive number* γ, CP_γ^+ *is described by*

$$CP_\gamma^+ = \{(n, [n\gamma]) | n \in \mathbf{Z}\} \cup \{([m/\gamma], m) | m \in \mathbf{Z}\}$$

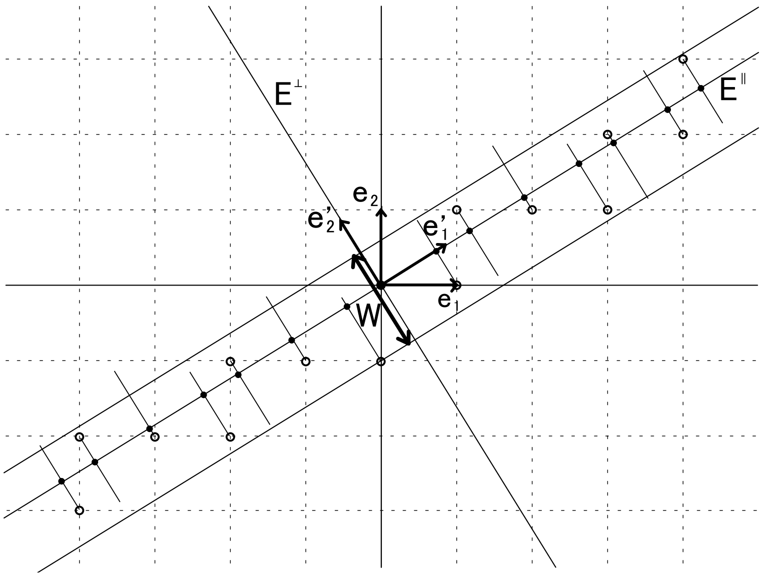

Figure 1. The cut and project set with a shift $\frac{\gamma-1}{\sqrt{\gamma^2+1}}$

and $\{(n, [n\gamma])|n\in\mathbf{Z}\}\cap\{([m/\gamma], m)|m\in\mathbf{Z}\}=\{(0,0)\}$.

Proof. The inequality $n\gamma-1<[n\gamma]\leq[(n+1)\gamma]<(n+1)\gamma$, shows that $(n, m)\in CP_\gamma^+$ if and only if $m=[n\gamma]$ or $[(n+1)\gamma]$. When $[(n+1)\gamma]=[n\gamma]+1$ holds, the inequarity $n\gamma<[n\gamma]+1=[(n+1)\gamma]<(n+1)\gamma$ shows $n<m/\gamma<n+1$, where $m=[(n+1)\gamma]$, and then $[m/\gamma]=n$. Thus any element (n, m) of CP_γ^+ is given by $(n, [n\gamma])$ or $([m/\gamma], m)$. As γ is irrational, $[n\gamma]=n\gamma$ holds only for $n=0$, meaning the rest.

3. A substitution associated with a slope γ

In this section, we consider conditions that a matrix $A\in M(2,\mathbf{Z})$ has eigenvectors $\mathbf{e}_1'$ and $\mathbf{e}_2'$ defined by (2.2). Denoting the eigenvalues, λ_1 and λ_2 corresponding to $\mathbf{e}_1'$ and $\mathbf{e}_2'$ respectively, A is described by

$$A=T\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}{}^tT,$$

where T is given by (2.1). As A is symmetric, we put

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\qquad a, b, c\in\mathbf{Z}, b\neq 0 \tag{3.1}$$

It follows from the identity

$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}={}^tTAT$$

that γ is the positive solution of the equation

$$\gamma^2+\left(\frac{a-c}{b}\right)\gamma-1=0, \tag{3.2}$$

while the eigenvalues, λ_1 and λ_2 of A satisfy the equation

$$(\lambda-a)(\lambda-c)=b^2. \tag{3.3}$$

Moreover, the identity $A\left(\mathbf{e}_1'\ \mathbf{e}_2'\right)=\left(\lambda_1\mathbf{e}_1'\ \lambda_2\mathbf{e}_2'\right)$ shows

$$a+b\gamma=\lambda_1,\qquad b+c\gamma=\lambda_1\gamma, \tag{3.4}$$

$$b-a\gamma=-\lambda_2\gamma,\qquad c-b\gamma=\lambda_2. \tag{3.5}$$

Lemma 3. 1. *For any positive integers a, b, c, we have*

$$\lambda_1 > \max\{a, b, c\}, \quad \textit{and hence} \quad |\lambda_2| < |\Delta| \cdot \min\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}. \tag{3.6}$$

If $\Delta = \det A > 0$,

$$\lambda_1 < a + c, \qquad \lambda_2 > 0, \qquad \frac{b}{a} < \gamma < \frac{c}{b}, \tag{3.7}$$

and for the strip D *defined by* (2.5), *we see*

$$AD = \{(x, y) \in \mathbf{R}^2 | -\lambda_2 < -\gamma x + y < \gamma\lambda_2\}.$$

If $\Delta < 0$,

$$\lambda_1 > a + c, \qquad \lambda_2 < 0, \qquad \frac{c}{b} < \gamma < \frac{b}{a} \tag{3.8}$$

and

$$AD = \{(x, y) \in \mathbf{R}_2 | \gamma\lambda_2 < -\gamma x + y < -\lambda_2\}.$$

Proof. As $a, b, c > 0$, we see $\lambda_1 = a + b\gamma = c + b/\gamma$ by (3.4), showing $\lambda_1 > \max\{a, c, b\gamma, b/\gamma\}$. Since $\max\{\gamma, 1/\gamma\} > 1$, we have $\lambda_1 > b$. The equation (3.3) implies $\lambda_1\lambda_2 = \Delta$, and hence (3.6). Put $f(x) = (x - a)(x - c) - b^2$, then we have $f(a + c) > 0$ if and only if $\Delta > 0$, meaning $\lambda_1 < a + c$. As $\lambda_1 + \lambda_2 = a + c$ by (3.3), we have $\lambda_2 > 0$. It follows from (3.5) that $c - b\gamma = a - b/\gamma = \lambda_2 > 0$, hence(3.7). Noticing that the boundaries of D are described by $-\mathbf{e}_1^{\perp} + \mathbf{R}\mathbf{e}_1'$ and $-\mathbf{e}_2^{\perp} + \mathbf{R}\mathbf{e}_1'$, the boundaries of AD are mapped to $-\lambda_2\mathbf{e}_1^{\perp} + \mathbf{R}\mathbf{e}_1'$ and $-\lambda_2\mathbf{e}_2^{\perp} + \mathbf{R}\mathbf{e}_1'$, hence the assertion. The proof for the case of $\Delta < 0$ is similar.

Lemma 3. 2. *For positive integers a, b, c, consider* $A \in GL(2, \mathbf{Z})$ *defined by* (3.1) *of which eigenvectors are* $\mathbf{e}_1'$ *and* $\mathbf{e}_2'$. *If* $|\Delta| \leq \min\{a, b, c\}$, *we have* $AD \subset D$ *and* $(a, b), (b, c) \in D$.

Proof. We see that the condition $AD \subset D$ is eqivalent to $-1 < a\gamma - b < \gamma$ and $-1 < b\gamma - c < \gamma$, while the condition (a, b), $(b, c) \in D$ is $-1 < -a\gamma + b < \gamma$ and $-1 < -b\gamma + c < \gamma$. Combining these inequarities, we have $|b - a\gamma| < \min(1, \gamma)$ and $|c - b\gamma| < \min(1, \gamma)$. Since $|\Delta| \leq \min\{a, b, c\}$, we have $|c - b\gamma| = |a - b/\gamma| = |\lambda_2| < 1$ by Lemma (3.1). Also we see $|b - a\gamma| = |\lambda_2\gamma| < |\Delta| \cdot \min\{1/a, 1/b, 1/c\} \cdot \max\{b/a, c/b\} < 1$ and $|b - c/\gamma| = |\lambda_2/\gamma| < |\Delta| \cdot \min\{1/a, 1/b, 1/c\} \cdot \max\{a/b, b/c\} < 1$. Hence the assertion.

Note that $\Delta = 0$ implies $\gamma \in \mathbf{Q}$ since the determinant of (3.2) is $(a + c)^2 + 4\Delta$.

For the irrational γ, we define an infinite word $x_\gamma = x_0x_1\cdots$ of an alphabet $\{0, 1\}$ by

$$x_n = \begin{cases} 0, & \text{if}(n, p_2), \quad (n+1, p_2) \in CP_\gamma^+, \\ 1, & \text{if}(n, p_2), \quad (n, p_2 + 1) \in CP_\gamma^+, \end{cases} \quad \text{for } n \geq 0. \tag{3.9}$$

We see that x_γ is well defined by Proposition 2.1 and the definition of CP_γ^+. For $\mathbf{v} = (v_1, v_2) \in \mathbf{R}^2$, $|\mathbf{v}|_1$ stands for $v_1 + v_2$. It follows from the definition of CP_γ^+ that there exists a unique $\mathbf{p} \in CP_\gamma^+$ with $|\mathbf{p}|_1 = n$ for any $n \in \mathbf{Z}$. For $\mathbf{p}, \mathbf{q} \in CP_\gamma^+$ with $|\mathbf{p}|_1 < |\mathbf{q}|_1$, we define

$$C[\mathbf{p}, \mathbf{q}] = \{\mathbf{n} \in CP_\gamma^+ | |\mathbf{p}|_1 \leq |\mathbf{n}|_1 \leq |\mathbf{q}|_1\}.$$

Suppose that positive integers a, b, c defining the matrix $A \in GL(2, \mathbf{Z})$ by (3.1) satisfy $\Delta \leq \min\{a, b, c\}$. Then we call subsets $C_0 = C[\mathbf{0}, A\mathbf{e}_1]$ and $C_1 = C[\mathbf{0}, A\mathbf{e}_2]$ of CP_γ^+ the *basic sets*. The *basic words* W_0 and W_1 are prefix factors of x_γ, given by $W_i = x_0 \cdots x_k$ with $k = |A\mathbf{e}_{i+1}|_1$, $i = 0, 1$. The *substitution* σ_γ *associated with* γ is a word homomorphism defined by $\sigma_\gamma(i) = W_i\ (i = 0, 1)$. Note that $C_i \subseteq C_j$ and thus W_i is a subword of W_j where $i = 0, j = 1$ if $|\mathbf{e}_1|_1 \leq |\mathbf{e}_2|_1$, and $i = 1, j = 0$ if $|\mathbf{e}_1|_1 \geq |\mathbf{e}_2|_1$.

4. Rigidity of basic words

Proposition 4. 1. *Consider a matrix* $A \in GL(2, \mathbf{Z})$ *defined by* (3.1) *with positive integers a, b, c. If* $|\Delta| = 1$, *it holds for any* $(x, y) \in AD$ *that*

(4.1) $$C_i+(x,y)\subset D, \quad i=0,1$$

Proof. Note that $|\Delta|=1$ implies $(a,b)=(b,c)=1$ and $|\Delta|\le\min\{a,b,c\}$. Since the boundaries of D are parallel to $\mathbf{e}_1'$, by translating $C_i+(x,y)$ along $\mathbf{R}\mathbf{e}_1'$, it is sufficient to prove (4.1) for $C_i+(0,\delta)$, where $-\lambda_2<\delta<\lambda_2\gamma$ if $\Delta>0$, and $\lambda_2\gamma<\delta<-\lambda_2$ if $\Delta<0$.

We show (4.1) in the case of $\Delta=1$. Owing to Proposition 2.2, the proof is devided into two cases: for $(n,[n\gamma])$ with $-\lambda_2<\delta\le 0$ and for $([m/\gamma],m)$ with $0\le\delta<\lambda_2\gamma$. To show the former, it is sufficient to prove $[n\gamma]-\lambda_2\ge n\gamma-1$ for $0\le n\le\max\{a,b\}$ which is equivalent to show

(4.2) $$\{n\gamma\}+\lambda_2\le 1, \quad 0\le\forall n\le\max\{a,b\}.$$

With the help of (3.5) and (3.6), we have for $0\le n\le\max\{a,b\}$,

$$n\frac{c}{b}-n\gamma\le\max\{a,b\}\left(\frac{c}{b}-\gamma\right)=\max\left\{1,\frac{a}{b}\right\}\cdot\lambda_2<\max\left\{1,\frac{a}{b}\right\}\cdot\min\left\{\frac{1}{a},\frac{1}{b}\right\}\le\frac{1}{b}.$$

We see $1/b\le\{nc/b\}\le 1-1/b$ whenever $nc/b\notin\mathbf{Z}$. Thus $[nc/b]\le nc/b-1/b<n\gamma$ and hence $[nc/b]\le[n\gamma]$. By (3.7), we see $\{n\gamma\}\le\{nc/b\}$. Then we have $[n\gamma]=[nc/b]$ for $0\le n\le\max\{a,b\}$, $nc/b\notin\mathbf{Z}$. By the definition $\{x\}=[x]-x$, we come to $\{n\gamma\}+\lambda_2<\{nc/b\}+\lambda_2<1-1/b+1/b=1$. For the case $nc/b\in\mathbf{Z}$, we put $n=kb\le\max\{a,b\}$. By (3.6), we see $k\lambda_2<\max\{a,b\}\cdot\min\{1/a,1/b\}<1$. Using (3.5), we have $kc=kb\gamma+k\lambda_2$, and then $kc-1<kb\gamma<kc$, which implies $[kb\gamma]=kc-1$. Hence $\{n\gamma\}+\lambda_2=kb\gamma-(kc-1)+\lambda_2\le kc-(kc-1)=1$ and thus (4.2).

To show the latter, it is sufficient to show $m+\lambda_2\gamma\le([m/\gamma]+1)\gamma$ for $0\le m\le\max\{b,c\}$, which is equivalent to show

(4.3) $$\{m/\gamma\}+\lambda_2\le 1, \quad 0\le\forall m\le\max\{b,c\}$$

Using (3.5), (3.6) and (3.7), a similar argument shows (4.3). We have proved (4.1) in the case of $\Delta=1$.

The proof for $\Delta=-1$ is obtained by showing $\{n\gamma\}-\lambda_2\gamma\le 1$ for $(n,[n\gamma])\in C_0\cup C_1$, and $\{m/\gamma\}-\lambda_2/\gamma\le 1$ for $([m/\gamma],m)\in C_0\cup C_1$. A similar argument leads to the inequalities. Thus Proposition 4.1 is proved.

5. Selfsimilarity of x_γ and remarks

Theorem 5. 1. *Let a, b, c be positive integers with $|ac-b^2|=1$. For a positive irrational number γ given as a positive solution of the quadratic equation (3.2), the infinite word x_γ defined by (3.9) is a fixed point of the substitution σ_γ.*

Proof. For $n\in\mathbf{N}$, we take a unique lattice point $\mathbf{p}_n\in CP_\gamma^+$ with $|\mathbf{p}_n|_1=n$. Then by definition (3.9), we have $\mathbf{p}_n+\mathbf{e}_1\in CP_\gamma^+$ whenever $x_n=0$ and $\mathbf{p}_n+\mathbf{e}_2\in CP_\gamma^+$ whenever $x_n=1$. Suppose that $x_n=0$. As $A\,\mathbf{p}_n\in AD$, it follows from Proposition 4.1 that $C_0+A\,\mathbf{p}_n\subset D$, while $C_0+A\,\mathbf{p}_n\subset\mathbf{Z}^2$, implying that $C_0+A\,\mathbf{p}_n\subset CP_\gamma^+$. Owing to Proposition 2.1, we see that $C_0+A\,\mathbf{p}_n$ coinsides with $C\left[A\,\mathbf{p}_n, A\,(\mathbf{p}_n+\mathbf{e}_1)\right]$, which implies that $x_{|A\,\mathbf{p}_n|_1}\cdots x_{|A\,\mathbf{p}_n+\mathbf{e}_1|_1}=W_0=\sigma_\gamma(0)=\sigma_\gamma(x_n)$. A similar argument in the case of $x_n=1$ brings $C_1+A\,\mathbf{p}_n=C\left[A\,\mathbf{p}_n, A\,(\mathbf{p}_n+\mathbf{e}_2)\right]$ and $x_{|A\,\mathbf{p}_n|_1}\cdots x_{|A\,(\mathbf{p}_n+\mathbf{e}_2)|_1}=W_1=\sigma_\gamma(1)=\sigma_\gamma(x_n)$. Summing up, for any $n\in\mathbf{N}$ there exists unique succesive points $\mathbf{p}_n$, $\mathbf{p}_{n+1}\in CP_\gamma^+$, i.e., $|\mathbf{p}_n|_1=n$ and $|\mathbf{p}_{n+1}|_1=n+1$, such that $x_{|A\,\mathbf{p}_n|_1}\cdots x_{|A\,\mathbf{p}_{n+1}|_1}=\sigma_\gamma(x_n)$. We have shown

$$\begin{aligned}\sigma_\gamma(x_\gamma)&=\sigma_\gamma(x_0)\sigma_\gamma(x_1)\sigma_\gamma(x_2)\cdots\\&=x_{|A\,\mathbf{p}_0|_1}\cdots x_{|A\,\mathbf{p}_1|_1}\cdots x_{|A\,\mathbf{p}_2|_1}\cdots x_{|A\,\mathbf{p}_3|_1}\cdots\\&=x_0x_1x_2\cdots=x_\gamma,\end{aligned}$$

that is, x_γ is a fixed point of σ_γ.

Remark On one hand, since we take the projection $\pi^\perp$ being orthogonal to $\mathbf{E}^\parallel$, we have to restrict ourselves to take a symmetric matrix (3.1), which will be an inessential restriction. Our argument should work for nonsymmetric ones. On the other hand, the condition $|\Delta|=1$ is essential to our discussion. In this case, we are to treat a positive irrational solution of a quadratic equation

$$\gamma^2+\frac{q}{p}\gamma-1=0$$

where the integers p and q are prime to each other. The continued fractional expansion of γ gives succesive good approximating fractions b/a and d/c, satisfying $|ad-bc|=1\,(a,b,c,d\in\mathbf{N})$. Then the matrix

$$A=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

plays a role in our discussion instead of (3.1).

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