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A fractal set associated with the Collatz problem

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1. Introduction

In 1930's, Lothar Collatz had great interest in representation of integer functions by directed graphs. He proposed the following, known as the Hasse-Collatz-Syracuse problem:

Problem 1. 1. *For a natural number $n \in \mathrm{N}$, let us consider the function*

$$f(n) = \begin{cases} 3n+1, & \text{if } n \equiv 1 \pmod 2, \\ n/2, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Then for each n, there exists a finite k such that $f^k(n) = \underbrace{f \circ f \circ \cdots \circ f(n)}_{k\text{-times}} = 1$

Our interest in the dynamics of the Collatz procedure. In this note, we observe a graph of the function f. With the help of the *binary embedding of natural numbers into* $[0, 1]$, we obtain a Cantor set $\mathfrak{K}$ associated with the Collatz procedure with Hausdorff dimension one (Theorem 2.3, 3.1 and Proposition 4.1). The set $\mathfrak{K}$ is generated by an iterated functional system, which satisfies strongly separation condition.

2. Binary embedding of natural numbers and a graph of Collatz procedure

Definition 2. 1. *Let $n = a_k \cdot 2^k + a_{k-1} \cdot 2^{k-1} + \cdots + a_0$ be a binary expansion of a natural number n. The binary embedding β of n is given by*

$$\beta(n) = \frac{a_0}{2} + \frac{a_1}{2^2} + \cdots + \frac{a_k}{2^{k+1}}.$$

By definition, $\beta : \mathrm{N} \to [0, 1)$ is one-to-one and N is densely embedded into $[0, 1]$.

Let N_{od} be the set of odd natural numbers $\{2n-1 \mid n \in \mathrm{N}\}$. Note that N_{od} is densely embedded into $[1/2, 1)$ by β.

Definition 2. 2. *The reduction of the Collatz procedure f is the map $H : \mathrm{N}_{od} \to \mathrm{N}_{od}$ such that for $n \in \mathrm{N}_{od}$,*

$$H(n) = k,$$

where $k \in \mathrm{N}_{od}$ and $3n + 1 = k \cdot 2^{\nu}$ for some $\nu \in \mathrm{N}$.

The exponent ν is given by 2-adic valuation v_2 of $3n+1$, $v_2(3n+1) = -\nu$.

Let us consider the graph $\mathfrak{C} = \{(\beta(n), \beta \circ H(n)) \mid n \in \mathrm{N}_{od}\}$ (See figure 1). By definitions of β and H, $\mathfrak{C} \subset [1/2, 1]^2$. One see that $\mathfrak{C}$ has a natural decomposition. Consider the subsets of $\mathfrak{C}$,

$$\mathfrak{C}_0 = \mathfrak{C} \cap \left[\frac{1}{2}, \frac{5}{8}\right] \times \left[\frac{1}{2}, 1\right],$$

$$\mathfrak{C}_1 = \mathfrak{C} \cap \left[\frac{3}{4}, 1\right] \times \left[\frac{1}{2}, 1\right].$$

Then the graph $\mathfrak{C}$ is decomposed as follows.

Theorem 2. 3.

$$\mathfrak{C} = \bigsqcup_{k=0}^{\infty} \hat{A}^k (\mathfrak{C}_0 \sqcup \mathfrak{C}_1),$$

where $A : \mathrm{R}^2 \to \mathrm{R}^2$ is given by

$$\hat{A} : (x, y) \to \left(\frac{1}{4}x + \frac{1}{2}, y\right).$$

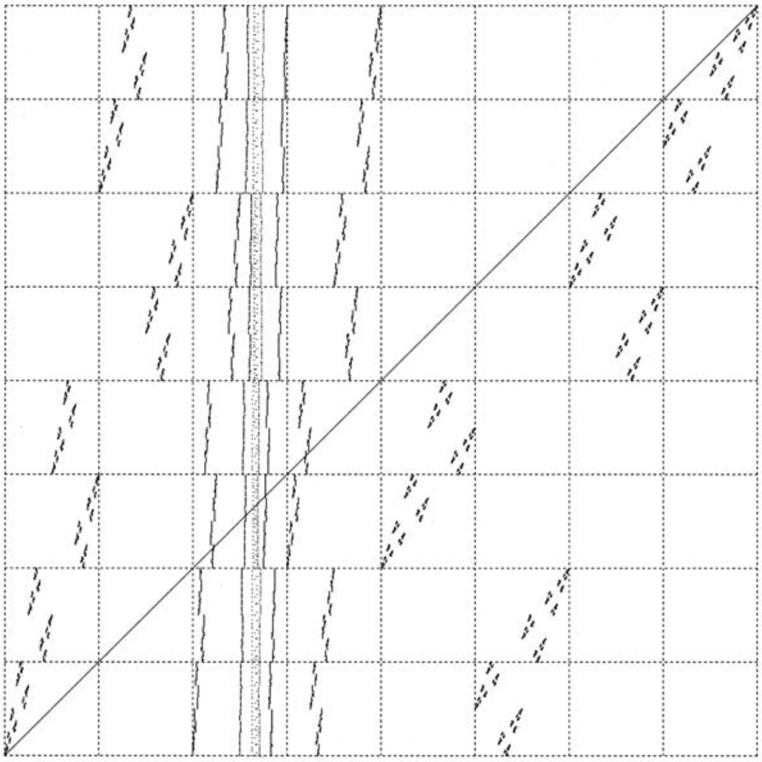

Figure 1. The graph of H

To show Theorem 2. 3, we need

Lemma 2. 4. *For a, k,* $n \in \mathrm{N}$ *such that* $a < 2^k$,

$$\beta(2^k n + a) = \frac{\beta(n)}{2^k} + \beta(a)$$

Particularly,

$$\beta(4n+1) = \frac{\beta(n)}{4} + \frac{1}{2}.$$

Proof. Let $n = n_l \cdot 2^l + n_{l-1} \cdot 2^{l-1} + \cdots + n_0$ and $a = a_h \cdot 2^h + a_{h-1} \cdot 2^{h-1} + \cdots + n_0\ (h < k)$ be binary expansions respectively. Then we have

$$\beta(2^k n + a) = \beta\left(\sum_{i=0}^{l} n_i \cdot 2^{i+k} + \sum_{j=0}^{h} a_j \cdot 2^j\right)$$
$$= \sum_{i=0}^{l} n_i \cdot 2^{-i-k-1} + \sum_{j=0}^{h} a_j \cdot 2^{-j-1} = \frac{\beta(n)}{2^k} + \beta(a).$$

Lemma 2. 5. *For* $n \in \mathrm{N}_{od}$, $H(4n+1) = H(n)$.

Proof. Since $3(4n+1)+1 = 2^2(3n+1)$, it follows from the definition of H.

Lemma 2. 6.

$$\mathrm{N}_{od} = \bigsqcup_{k=0}^{\infty} A^k\left(\{8k+1 \mid k \in \mathrm{N}_0\} \sqcup \{4k+3 \mid k \in \mathrm{N}_0\}\right),$$

where $A(n) = 4n+1$ *and* $\mathrm{N}_0 = \mathrm{N} \sqcup \{0\}$.

Proof. For $k \in \mathrm{N}_0$, we see that $8k+3 = 4(2k)+3$, $8k+5 = 4(2k+1)+1 = A(2k+1)$ and $8k+7 = 4(2k+1)+3$. $8k+1 \in A(\mathrm{N}_{od})$ means $8k+1 = 4l+1$ for some $l \in \mathrm{N}_{od}$ and then $l = 2k$ contradicts to $l \in \mathrm{N}_{od}$. $4k+3 \in A(\mathrm{N}_{od})$ means $4k+3 = 4l+1$ for some $l \in \mathrm{N}_{od}$, which leads to a contradiction $2l = 2k+1$. Finally, infinite descent method shows that for any $k \in \mathrm{N}_0$, there exists $h \in \{8l+1,\ 4l+3 \mid l \in \mathrm{N}_0\}$ and $\nu \in \mathrm{N}$ such that $8k+5 = A^{\nu}(h)$ since $0 < n < A(n)$ generally.

Lemma 2. 7. *For* $n \in \mathrm{N}_{od}$, $(\beta \circ A(n), \beta \circ H \circ A(n)) = \hat{A}(\beta(n), \beta \circ H(n))$.

Proof. By Lemma 2. 4 and 2. 5, we have for $n \in \mathrm{N}_{od}$,

$$\hat{A}(\beta(n), \beta\circ H(n)) = \left(\frac{\beta(n)}{4}+\frac{1}{2}, \beta\circ H(n)\right)$$
$$= (\beta(4n+1), \beta\circ H(n)) = (\beta(4n+1), \beta\circ H(4n+1)).$$

Proof. of Theorem 2.3. Lamma 2.7 shows that $\hat{A}^k(\mathfrak{C}_0 \sqcup \mathfrak{C}_1) \subset \mathfrak{C}$ for each $k \in \mathrm{N}_0$, that is $\bigsqcup_{k=0}^{\infty} \hat{A}^k(\mathfrak{C}_0 \sqcup \mathfrak{C}_1) \subset \mathfrak{C}$.

We show the converse. For $k \in \mathrm{N}_0$ we see $\frac{1}{2} \le \beta(8k+1) = \frac{\beta(k)}{8} + \frac{1}{2} < \frac{5}{8}$ and $\frac{3}{4} \le \beta(4k+3) = \frac{\beta(k)}{4} + \frac{3}{4} < 1$ by Lemma 2.4, which means

(2.1)
$$\mathfrak{C}_0 = \{(\beta(8k+1), \beta\circ H(8k+1)) \mid k \in \mathrm{N}_0\},$$
$$\mathfrak{C}_1 = \{(\beta(4k+3), \beta\circ H(4k+3)) \mid k \in \mathrm{N}_0\}.$$

By Lemma 2.6 we find $h \in \{8l+1, 4l+3 \mid l \in \mathrm{N}_0\}$ and $\nu \in \mathrm{N}_0$ such that $n = A^{\nu}(h)$ for each $n \in \mathrm{N}_{od}$. Then Lemma 2.7 shows

$$(\beta(n), \beta\circ H(n)) = (\beta(A^{\nu}(h)), \beta\circ H(A^{\nu}(h))) = \hat{A}^{\nu}(\beta(h), \beta\circ H(h)).$$

It follows that $\mathfrak{C} \subset \bigsqcup_{k=0}^{\infty} \hat{A}^k(\mathfrak{C}_0 \sqcup \mathfrak{C}_1)$, hence the assertion.

Note that $H(8k+1) = (3(8k+1)+1)/4 = 6k+1$ and $H(4k+3) = (3(4k+3)+1)/2 = 6k+5$, hence

(2.2)
$$\mathfrak{C}_0 = \{(\beta(8k+1), \beta(6k+1) \mid k \in \mathrm{N}_0\},$$
$$\mathfrak{C}_1 = \{(\beta(4k+3), \beta(6k+5) \mid k \in \mathrm{N}_0\}.$$

3. A self-similar set associated with the Collatz procedure

We consider contructing maps $g_1, g_2, g_3 : [0, 1]^2 \to [0, 1]^2$,

$$g_1(x, y) = \frac{1}{2}(x+1, y+1),$$
$$g_2(x, y) = \frac{1}{4}(x+1, y),$$
$$g_3(x, y) = \frac{1}{4}(1-x, 2-y).$$

It is known that any set of contructing maps, that is, any iterated functional system (IFS) has a unique compact set $\mathfrak{K} \subset [0, 1]^2$ satisfying

$$\mathfrak{K} = g_1(\mathfrak{K}) \cup g_2(\mathfrak{K}) \cup g_3(\mathfrak{K})$$

(See figure 2). Since g_i's are affine maps and the IFS $\{g_1, g_2, g_3\}$ satisfies the open set condition, it follows from Hutchin-

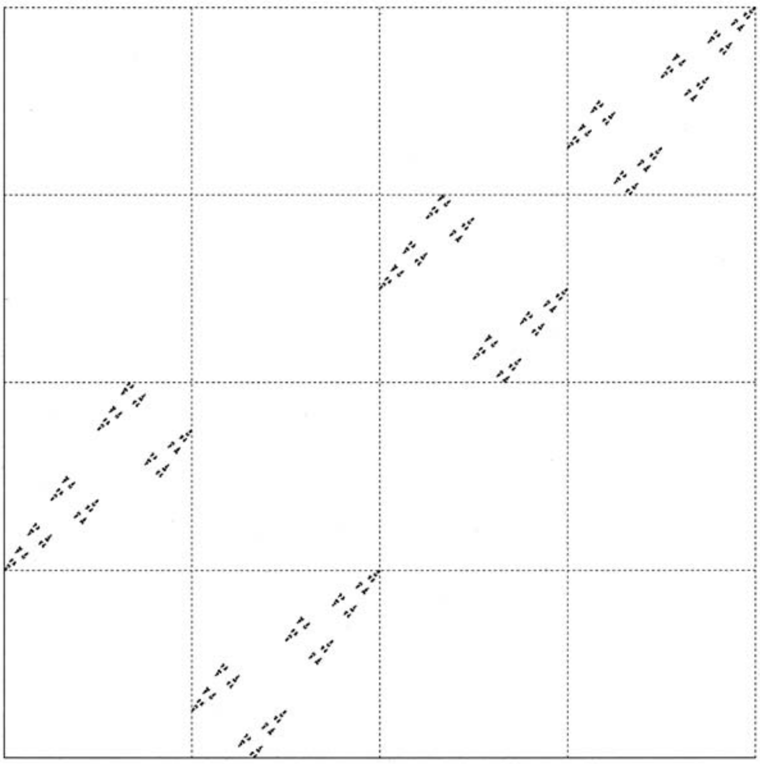

Figure 2. The set $\mathfrak{K}$

son's theorem [1] that the Hausdorff dimension s of $\mathfrak{K}$ is given as a solution of

$$\left(\frac{1}{2}\right)^s+\left(\frac{1}{4}\right)^s+\left(\frac{1}{4}\right)^s=1,$$

that is, $s=1$.

Theorem 3. 1. *The closures* $\overline{\mathfrak{C}_0}$ *and* $\overline{\mathfrak{C}_1}$ *are homeomorphic to* $\mathfrak{K}$.

Proof. We prove $\overline{\mathfrak{C}_1}\approx\mathfrak{K}$. Take the affine map $T_1:\left[\frac{3}{4},\ 1\right]\times\left[\frac{1}{2},\ 1\right]\to[0,\ 1]^2$, $T_1(x,\ y)=(4x-3,\ 2y-1)$ and put $G_i=T_1^{-1}\circ g_i\circ T_1$. We have

$$G_1(x,\ y)=\frac{1}{2}(x+1,\ y+1),$$
$$G_2(x,\ y)=\frac{1}{8}(2x+5,\ 2y+3),$$
$$G_3(x,\ y)=\frac{1}{8}(8-2x,\ 7-2y).$$

We show $\overline{\mathfrak{C}_1}=G_1(\overline{\mathfrak{C}_1})\sqcup G_2(\overline{\mathfrak{C}_1})\sqcup G_3(\overline{\mathfrak{C}_1})$. Let us denote $\mathfrak{C}_{1,1}=\mathfrak{C}_1\cap\left[\frac{7}{8},\ 1\right]\times\left[\frac{3}{4},\ 1\right]$, $\mathfrak{C}_{1,2}=\mathfrak{C}_1\cap\left[\frac{13}{16},\ \frac{7}{8}\right]\times\left[\frac{1}{2},\ \frac{5}{8}\right]$ and $\mathfrak{C}_{1,3}=\mathfrak{C}_1\cap\left[\frac{3}{4},\ \frac{13}{16}\right]\times\left[\frac{5}{8},\ \frac{3}{4}\right]$. Note that for each $(x,\ y)\in\mathfrak{C}_1$ we take $n=4k+3$, $k\in\mathrm{N}_0$ such that $(x,\ y)=(\beta(n),\ \beta\circ H(n))$ by the definition of $\mathfrak{C}_1$.

Since $H(2n+1)=12k+11=2H(n)+1$, $\beta(2n+1)=\beta(8k+7)=\frac{\beta(k)}{8}+\frac{7}{8}\in\left[\frac{7}{8},\ 1\right]$ and $\beta(12k+11)=\beta(4(k+2)+3)=\frac{\beta(k+2)}{4}+\frac{3}{4}\in\left[\frac{3}{4},\ 1\right]$, we see

$$G_1(\beta(n),\ \beta\circ H(n))=\left(\frac{1}{2}(\beta(n)+1),\ \frac{1}{2}(\beta\circ H(n)+1)\right)=(\beta(2n+1),\ \beta(2H(n)+1))$$
$$=(\beta(2n+1),\ \beta\circ H(2n+1))\in\mathfrak{C}_{1,1},$$

using Lemma 2. 4, and hence $G_1(\mathfrak{C}_1)\subset\mathfrak{C}_{1,1}$. Conversary, for any $m\in\mathrm{N}_{od}$, $\beta(m)\in\left[\frac{7}{8},\ 1\right]$ means $\beta(m)=\beta(8k+7)$, that is $m=8k+7=2(4k+3)+1$. Thus we have $G_1(\mathfrak{C}_1)=\mathfrak{C}_{1,1}$. Taking the closures, we have $G_1(\overline{\mathfrak{C}_1})=\overline{\mathfrak{C}_{1,1}}$.

With the help of Lemma 2. 4, we see that for $m=2l+1\in\mathrm{N}_{od}$,

$$\frac{\beta(m)}{4}+\frac{5}{8}=\frac{\beta(2l+1)}{4}+\frac{5}{8}=\frac{\beta(l)}{8}+\frac{3}{4}$$
$$=\frac{\beta(l)}{8}+\beta(3)=\beta(8l+3)=\beta(4m-1),$$

and also $\frac{\beta(m)}{4}+\frac{3}{8}=\beta(4m-3)$ in the similar way. Since $H(4n-1)=24k+17=4H(n)-3$, $\beta(4n-1)=\beta(16k+11)=\frac{\beta(k)}{16}+\frac{13}{16}\in\left[\frac{13}{16},\ \frac{7}{8}\right]$ and $\beta(24k+17)=\beta(8(3k+2)+1)=\frac{\beta(3k+2)}{8}+\frac{1}{2}\in\left[\frac{1}{2},\ \frac{5}{8}\right]$, we have

$$G_2(\beta(n),\ \beta\circ H(n))=\left(\frac{1}{4}\beta(n)+\frac{5}{8},\ \frac{1}{4}\beta\circ H(n)+\frac{3}{8}\right)=(\beta(4n-1),\ \beta(4H(n)-3))$$
$$=(\beta(4n-1),\ \beta\circ H(4n-1))\in\mathfrak{C}_{1,2}$$

hence $G_2(\mathfrak{C}_1)\subset\mathfrak{C}_{1,2}$. An argument similar to the case of G_1 shows the converse $G_2(\mathfrak{C}_1)\supset\mathfrak{C}_{1,2}$, and hence $G_2(\overline{\mathfrak{C}_1})=\overline{\mathfrak{C}_{1,2}}$.

To see $G_3(\overline{\mathfrak{C}_1})=\overline{\mathfrak{C}_{1,3}}$ we need

Lemma 3. 2. *For* $n\in\mathrm{N}$, *we have*

$$1-\beta(n)=\beta\left(3\cdot 2^{\mathrm{ord}(n)}-(n+1)\right),$$

where $\mathrm{ord}(n)=[\log_2 n]$, *the highest degree of the binary expansion of* n.

Proof. Let $n=a_l\cdot 2^l+a_{l-1}\cdot 2^{l-1}+\cdots+a_0$ be a binary expansion where $l=\mathrm{ord}(n)$. Noticing $a_l=1$, we see

$$1-\beta(n)=\sum_{i=0}^{l}2^{-(i+1)}+2^{-(l+1)}-\sum_{i=0}^{l}a_i\cdot 2^{-(i+1)}$$
$$=\sum_{i=0}^{l-1}(1-a_i)\cdot 2^{-(i+1)}+2^{-(l+1)}$$
$$=\beta\left(2^l+\sum_{i=0}^{l-1}(1-a_i)\cdot 2^i\right)=\beta\left(2^l+\sum_{i=0}^{l-1}2^i-n+2^l\right)$$
$$=\beta(2^{l+1}-1-n+2^l)=\beta(3\cdot 2^l-(n+1))$$

Lemma 3. 3. *For* $p_j=2^{j+1}-(4s+1)$, $s\in\mathrm{N}_0$, *we have*

$$(1-\beta(4s),\ 1-\beta(6s))=\lim_{j\to\infty}(\beta(p_j),\ \beta\circ H(p_j))\in\overline{\mathfrak{C}_1}.$$

Proof. Taking $j > \operatorname{ord}(6s) > \operatorname{ord}(4s)$, we see $\operatorname{ord}(2^j + 4s) = j$. It follows from Lemma 3.2 that

$$\begin{aligned} 1-\beta(4s)-2^{j+1} &= 1-\beta(2^j+4s) = \beta(3\cdot 2^j-(2^j+4s+1)) \\ &= \beta(2^{j+1}-(4s+1)) = \beta(p_j). \end{aligned}$$

As $p_j = 2^{j+1}-(4n+1) = 4(2^{j-1}-n-1)+3$, we see $(\beta(p_j), \beta\circ H(p_j)) \in \mathfrak{C}_1$ and $H(p_j) = (3p_j+1)/2 = 3\cdot 2^j-(6n+1)$. For any natural numbers $a > b$, it holds that

$$\begin{aligned} \beta(2^a-2^b) &= \beta(2^{a-1}+2^{a-2}+\cdots+2^b) \\ &= \beta(2^{a-1})+\beta(2^{a-2})+\cdots+\beta(2^b) = \frac{1}{2^b}-\frac{1}{2^a}. \end{aligned}$$

Since $j > l = \operatorname{ord}(6s)$, we have

$$\begin{aligned} \beta\circ H(p_j) &= \beta(3\cdot 2^j-(6s+1)) \\ &= \beta(2^{j+1}+2^j-2^{l+1}+(2^{l+1}-1-6s)) \\ &= \frac{1}{2^{j+2}}+\frac{1}{2^{l+1}}-\frac{1}{2^j}+\beta(2^{l+1}-1-6s). \end{aligned}$$

As $\operatorname{ord}(2^{l+1}-1-6s) = l-1$, the limit $j\to\infty$ brings

$$\begin{aligned} \lim_{j\to\infty}\beta\circ H(p_j) &= \frac{1}{2^{l+1}}+\beta(2^{l+1}-1-6s) = \beta(2^l+2^{l+1}-(6s+1)) \\ &= \beta(3\cdot 2^l-(6s+1)) = 1-\beta(6s). \end{aligned}$$

We have $H(n) = \dfrac{3n+1}{2}$ as $n = 4k+3$. It follows from Lemma 2.4 that

$$\begin{aligned} \frac{7}{8}-\frac{\beta\circ H(n)}{4} &= 1-\left\{\frac{1}{8}+\frac{\beta\circ H(n)}{4}\right\} = 1-\left\{\frac{1}{8}+\beta\left(8\left(\frac{H(n)-1}{2}\right)+4\right)\right\} \\ &= 1-\left\{\frac{1}{4}+\beta(4(H(n)-1))\right\} = 1-\beta(4H(n)-2). \\ &= 1-\beta(6n) \end{aligned}$$

Thus we have

$$G_3(\beta(n), \beta\circ H(n)) = (1-\beta(4n),\ 1-\beta(6n)).$$

Note that $1-\beta(4n) = 1-\beta(4(4k+3)) = 1-\dfrac{\beta(k)}{16}-\dfrac{3}{16}\in\left[\dfrac{3}{4},\dfrac{13}{16}\right]$ and $1-\beta(6n) = 1-\beta(8(k+2)+2) = 1-\dfrac{\beta(k+2)}{8}-\dfrac{1}{4}$ $\in\left[\dfrac{5}{8},\dfrac{3}{4}\right]$. Lemma 3.2 shows that $(1-\beta(4n), 1-\beta(6n)) = (\beta(p), \beta(q))$ where $p = 3\cdot 2^{\operatorname{ord}(4n)}-(4n+1)$ and $q = 3\cdot 2^{\operatorname{ord}(6n)}-(6n+1)$, while $H(p)\neq q$.

Applying Lemma 3.3 for $p_j = 2^{j+1}-(4n+1)$, we have

$$(1-\beta(4n), 1-\beta(6n)) = \lim_{j\to\infty}(\beta(p_j), \beta\circ H(p_j)) \in \overline{\mathfrak{C}_{1,3}}.$$

Hence $G_3(\mathfrak{C}_1)\subset\overline{\mathfrak{C}_{1,3}}$, which means $G_3(\overline{\mathfrak{C}_1})\subset\overline{\mathfrak{C}_{1,3}}$.

We show the converse $\overline{\mathfrak{C}_{1,3}}\subset G_3(\overline{\mathfrak{C}_1})$. Note that $\beta(n)\in\left[\dfrac{3}{4},\dfrac{13}{16}\right]$ means $n = 2^4k+3\ (k\in\mathrm{N}_0)$, then $H(n) = (3n+1)/2 = 2^3\cdot 3k+5$. For $n = 2^4k+3\ (k\in\mathrm{N}_0)$, we have

$$\begin{aligned} \beta\circ H(n)+\frac{1}{8} &= \frac{\beta(3k)}{2^3}+\beta(5)+\frac{1}{8} = \frac{\beta(3k)}{2^3}+\beta(3) = \beta(2^3\cdot 3k+3) \\ &= \beta(H(n)-2) = \beta\left(\frac{3}{2}(n-1)\right), \end{aligned}$$

and also

$$\begin{aligned} \operatorname{ord}\left(\frac{3(n-1)}{2}\right) &= \operatorname{ord}(2^3\cdot 3k+3) = \operatorname{ord}(6k)+2, \\ \operatorname{ord}\left(\frac{3n-1}{8}\right) &= \operatorname{ord}(6k+1) = \operatorname{ord}(6k), \end{aligned}$$

that is, $\operatorname{ord}\left(\dfrac{3(n-1)}{2}\right)-2 = \operatorname{ord}\left(\dfrac{3n-1}{8}\right)$. It follows from Lemma 3.2 that

$$\begin{aligned} G_3^{-1}(\beta(n), \beta\circ H(n)) &= 4\left(1-\beta(n), 1-\beta\circ H(n)+\frac{1}{8}\right) \\ &= 4\left(1-\beta(n), 1-\beta\left(\frac{3}{2}(n-1)\right)\right) \\ &= 4\left(\beta\left(3\cdot 2^{\operatorname{ord}(n)}-(n+1)\right), \beta\left(3\cdot 2^{\operatorname{ord}\left(\frac{3}{2}(n-1)\right)}-\frac{3n-1}{2}\right)\right) \\ &= \left(\beta\left(3\cdot 2^{\operatorname{ord}(n)-2}-\frac{n+1}{4}\right), \beta\left(3\cdot 2^{\operatorname{ord}\left(\frac{3}{2}(n-1)\right)-2}-\frac{3n-1}{8}\right)\right) \end{aligned}$$

$$=\left(\beta\left(3\cdot 2^{\mathrm{ord}(4k)}-(4k+1)\right),\ \beta\left(3\cdot 2^{\mathrm{ord}(6k)}-(6k+1)\right)\right)$$
$$=(1-\beta(4k),\ 1-\beta(6k))$$

Applying Lemma 3.3 for $p_j = 2^{j+1}-\frac{n+1}{4} = 2^{j+1}-(4k+1)$, we have

$$(1-\beta(4k), 1-\beta(6k)) = \lim_{j\to\infty}(\beta(p_j), \beta\circ H(p_j)) \in \overline{\mathfrak{C}_1}.$$

Hence $\mathfrak{C}_{1,3} \subset G_3(\overline{\mathfrak{C}_1})$, which shows $\overline{\mathfrak{C}_{1,3}} \subset G_3(\overline{\mathfrak{C}_1})$.

We have shown that $\mathfrak{C}_1$ is a solution of set valued equation

$$\mathfrak{X} = G_1(\mathfrak{X}) \sqcup G_2(\mathfrak{X}) \sqcup G_3(\mathfrak{X}).$$

It follows from the uniqueness of the invariant set of IFS $\{G_1, G_2, G_3\}$ that $\mathfrak{K} = T_1(\overline{\mathfrak{C}_1})$. Similar argument shows $\mathfrak{K} \approx \overline{\mathfrak{C}_0}$.

4. $\mathfrak{K}$ is a Cantor set

Proposition 4. 1. *$\mathfrak{K}$ is a Cantor set with the Hausdorff dimension one.*

Proof. As $\mathfrak{K} = T_1(\overline{\mathfrak{C}_1}) = \overline{T_1(\mathfrak{C}_1)}$, $T_1(\mathfrak{C}_1)$ is dense in $\mathfrak{K}$. It follows from

$$\lim_{k\to\infty}(\beta(n+2^k), \beta\circ H(n+2^k)) = (\beta(n), \beta\circ H(n))$$

for $n \in \mathrm{N}_{od}$ that any elements in $\mathfrak{C}_1$ are accumulation points, i.e., $\mathfrak{C}_1$ is a perfect set. Thus $\mathfrak{K}$ is so. As the IFS $\{g_1, g_2, g_3\}$ satisfies strong separation condition

$$\mathfrak{K} = g_1(\mathfrak{K}) \sqcup g_2(\mathfrak{K}) \sqcup g_3(\mathfrak{K}) \quad \text{and} \quad g_i(\mathfrak{K}) \cap g_j(\mathfrak{K}) = \emptyset,\ i \neq j,$$

$\mathfrak{K}$ is totally disconnected. We see that the Hausdorff dimension of $\mathfrak{K}$ is one in the previous section. Hence the assertion.

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