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A characterization of tonality via the rigidity of chords

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1. Tonality and maximal evenness ansatz: a review

Based on the Pythagorean tuning and the maximal evenness ansatz, we have been trying to build a combinatorial model of tonal music theory [3][4][5][6]. The concept of maximal evenness has been firstly introduced into mathematical music theory by Clough and Douthett[1], where they have realized maximally even collections by mechanical sequences associated with fractions, which is described by J-functions (Definition 2.4). In [3] and [4], we show that any maximally even set has a descriptions as a rotational dynamics (Theorem 2.6, quoted below), which corresponds to an abstraction of the Pythagorean tuning, hence we have found a good reason for the maximal evenness ansatz in tonal music theory. In [5], we have investigated possibilities to describe tonality from a viewpoint of the maximal evenness ansatz. From a combinatorial viewpoint, maximal evenness imposes a kind of limitation on the choice of musical notes in a piece, hence it influences chord progressions. As a result, it brings a smoothness of voice leadings among maximally even chords[6].

To step deeply into tonality, we have to investigate the relation between scales and chords. The tonality of a piece is provided by melodies and chord progressions, dominated by the system of organizing notes named the scale. A chord in a scale, however, belongs to more than one scale generally, thus the number of scales the chord belongs, which we call *rigidity*, seems to relate tonality of the piece. We give the upper bound of the rigidity for the fixed number of notes that form a chord (Theorem 3.2). We see that the suspended chords like $Sus2$ or $Sus4$ are characterized as maximum rigidity chords (Example 2.14). We also characterize tritones in maximally even scales from the rigidity of chords viewpoint (Theorem 3.3), which will bring us a clue to build a combinatorial model on consonance/dissonance, the underground concept of tonality.

2. Maximal evenness of diatonic system

2.1. **General setting of scales.** Since tones with basic frequencies f and $2^n f, n \in \mathbf{Z}$ have a 'similar' quality for human ears, these tones are called *octave equivalent* to each other in music theory. Keeping this psychological fact in mind, we introduce the followings. For a tuple $A = (a_i)$, $|A|$ denotes the set $\{a_i\}$ consists of entries of A. If a tuple $B = (b_j)$ satisfies $|B| \subset |A|$ and that the inclusion $\iota : |B| \hookrightarrow |A|$ is increasing on indexes, that is, for $b_i, b_j \in B$ with $i < j$, $i' < j'$ holds for $a_{i'} = \iota(b_i)$ and $a_{j'} = \iota(b_j)$, we call B is *compatible* with A and write $B \sqsubset A$.

Definition 2.1 (General chromatic scale)**.** A *general chromatic scale* Ch_c is a tuple $(f_0, f_1, \dots, f_{c-1})$ of tones with basic frequencies $f_0 < f_1 < \cdots < f_{c-1}$ satisfying $f_{c-1} < 2f_0$. A *general semitone encoding* θ associated with the chromatic scale Ch_c is a map $|Ch_c| \ni f_k \mapsto k \in \mathbf{Z}$. The octave equivalence leads a periodic extension $\overline{Ch_c}$ of the chromatic scale Ch_c such that $\overline{Ch_c} = (2^n f_0, 2^n f_1, \dots, 2^n f_{c-1})_{n \in \mathbf{Z}}$, and the semitone encoding θ is also extended to a bijection $\theta : |\overline{Ch_c}| \ni 2^n f_k \mapsto cn + k \in \mathbf{Z}$. We call an element of $\overline{Ch_c}$ a *note*.

Definition 2.2 (General scales)**.** Given a general chromatic scale Ch_c, a *scale* is a tuple $S = (t_0, t_1, \dots, t_{d-1})$ compatible with Ch_c. Thus entries of S are arranged in ascending order $\theta(t_0) < \theta(t_1) < \cdots < \theta(t_{d-1})$ with $\theta(t_{d-1}) - \theta(t_0) < c$. For a natural number h, $h \cdot S$ stands for an extension of scale S,

$$h \cdot S = (2^n t_0, \dots, 2^n t_{d-1})_{n=0,\dots,h-1}.$$

$\overline{S}$ denotes an infinite extension of S, $\overline{S} = (2^n t_0, \dots, 2^n t_{d-1})_{n\in\mathbf{Z}}$. The *general wholetone encoding* η_S associated with a scale S is a map $|\overline{S}| \ni 2^n t_k \mapsto dn + k \in \mathbf{Z}$.

Definition 2.3 (General chords)**.** A *chord* $X \sqsubset h\cdot S$ is a tuple of notes $X = (x_0, \dots, x_{n-1})$ compatible with $h \cdot S$. We call $oct(X) = h$ the *octave range* of the chord X. If $\theta(x_p) \not\equiv \theta(x_q) \pmod c$ holds for any entries x_p and x_q, we call X *prime*.

Since the octave equivalence of a and $b \in \overline{Ch_c}$ means $\theta(a) \equiv \theta(b) \pmod c$, the chromatic scale Ch_c is identified with $\mathbf{Z}/c\mathbf{Z}$. When we focus on a scale S of d notes, we identify S with $\mathbf{Z}/d\mathbf{Z}$. Thus the extension $\overline{Ch_c}$ and $\overline{S}$ can be seen as the covering space $\mathbf{Z}$ of $\mathbf{Z}/c\mathbf{Z}$ and $\mathbf{Z}$ of $\mathbf{Z}/d\mathbf{Z}$ respectively. We also identify an extension $h \cdot S$ with $\mathbf{Z}/hd\mathbf{Z}$. Throughout these identification, we also reuse θ and η_S as the semitone and wholetone encoding respectively. We often identify a tuple $X = (x_1, \dots, x_n) \sqsubset Ch_c$ with its image $\theta(X) = (\theta(x_1), \dots, \theta(x_n))$, like $X = (\theta(x_1), \dots, \theta(x_n))$ for short. For a note $a \in \overline{Ch_c}$ and $n \in \mathbf{Z}$, $a^n \in \overline{Ch_c}$ stands for the octave equivalent note such that $\theta(a^n) = \theta(a) + nc$.

$$
\begin{tikzcd}
\mathbf{Z} \arrow[d, two heads, "proj."'] & \overline{S} \arrow[l, tail, two heads, "\eta_S"'] \arrow[r, hook] \arrow[d, two heads] & \overline{Ch_c} \arrow[r, tail, two heads, "\theta"] \arrow[d, two heads] & \mathbf{Z} \arrow[d, two heads, "proj."] \\
\mathbf{Z}/hd\mathbf{Z} \arrow[d, two heads, "proj."'] & h\cdot S \arrow[l, tail, two heads, "\eta_S"'] \arrow[r, hook] \arrow[d, two heads] & h\cdot Ch_c \arrow[r, tail, two heads, "\theta"] \arrow[d, two heads, "oct.eq."'] & \mathbf{Z}/hc\mathbf{Z} \arrow[d, two heads, "proj."] \\
\mathbf{Z}/d\mathbf{Z} & S \arrow[l, tail, two heads, "\eta_S"'] \arrow[r, hook] & Ch_c \arrow[r, tail, two heads, "\theta"] & \mathbf{Z}/c\mathbf{Z}
\end{tikzcd}
$$

2.2. ***J*-function, diatonic scale and diatonic chord.** The J-function is introduced by Clough and Douthett[1], which works well to extract features of tonality (see for details, e.g.,[4][5][6]).

Definition 2.4. For $c, d, m \in \mathbf{Z}$ with $c > d$ and $d \neq 0$, the *J-function* on $\mathbf{Z}$ is defined as

$$J^m_{c,d}(k) = \left\lfloor \frac{ck+m}{d} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. We note $\mathcal{J}^m_{c,d}$ as the tuple $(J^m_{c,d}(k))_{k=0,\dots,d-1}$ and $|\mathcal{J}^m_{c,d}|$ as the set $\{t \in \mathcal{J}^m_{c,d}\}$ of entries of $\mathcal{J}^m_{c,d}$.

We just quote the following and omit the proof.

Proposition 2.5 ([5] Proposition 2.7)**.** *Let c, d be non-zero integers prime to each other. Then for any $x \in \mathbf{R}$,*

$$\sum_{k\in\mathbf{Z}/d\mathbf{Z}} |J^{x+1}_{c,d}(k) - J^x_{c,d}(k)| = 1 \tag{2.1}$$

holds. Thus the Hamming distance of $\mathcal{J}^{x+1}_{c,d}$ and $\mathcal{J}^x_{c,d}$ is 1.

The tuple $\mathcal{J}^m_{c,d} \sqsubset Ch_c$ has a special feature called *Myhill's property*, which is an embodiment of 'maximal evenness'. When we treat $\mathcal{J}^m_{c,d} \sqsubset Ch_c$ as a scale, we call it a (general) *diatonic* scale. The following gives a dynamical description of maximally even sets, which is a crucial tool for our observations of tonality via the maximal evenness ansatz.

Theorem 2.6 (Dynamical characterization of Myhill set[4] Theorem 3.4)**.** *Consider a subset $S \subset \mathbf{Z}/c\mathbf{Z}$ with $\#S = d$ prime to c, and take a translation*

$$T : \mathbf{Z}/c\mathbf{Z} \ni x \mapsto x + d^{-1} \in \mathbf{Z}/c\mathbf{Z},$$

where d^{-1} is the multiplicative inverse of $d \in (\mathbf{Z}/c\mathbf{Z})^\times$. Then S is maximally even if and only if S is a collection of successive d images of some element $g \in \mathbf{Z}/c\mathbf{Z}$ by T, namely

$$S = \{g, T(g), T^2(g), \dots, T^{d-1}(g)\} = \{g, g + d^{-1}, g + 2d^{-1}, \dots, g + 1 - d^{-1}\}.$$

Example 2.7. Classical European music or Western popular music even today is established on 12-tone scale, so-called the *chromatic scale*,

$$Ch_{12} = (C, C^{\#}, D, D^{\#}, E, F, F^{\#}, G, G^{\#}, A, A^{\#}, B),$$

where frequencies of $A, A^{\#}, \ldots$ are determined appropriately in music theory (e.g., $A = 440\text{Hz}, A^{\#} = 466.16\text{Hz}, \ldots$). Hereafter we adopt the semitone encoding $\theta : |Ch_{12}| \to \mathbf{Z}/12\mathbf{Z}$,

$$\theta(C) = 0, \theta(C^{\#}) = 1, \theta(D) = 2, \theta(D^{\#}) = 3, \theta(E) = 4, \theta(F) = 5,$$
$$\theta(F^{\#}) = 6, \theta(G) = 7, \theta(G^{\#}) = 8, \theta(A) = 9, \theta(A^{\#}) = 10, \theta(B) = 11.$$

Tonal music pieces are composed over *diatonic scales* $\mathcal{J}^m_{12,7}, m = 0, \ldots, 11$ which are maximally even in Ch_{12}. For instance the C-major diatonic scale $\mathcal{C} = CDEFGAB$ is expressed as a tuple $(0, 2, 4, 5, 7, 9, 11) = (J^5_{12,7}(k))_{k=0,\ldots,6} = \mathcal{J}^5_{12,7}$. We also note that $|\mathcal{J}^5_{12,7}| = \{0, 2, 4, 5, 7, 9, 11\}$ has a dynamical expression $\{5 + 7k \pmod 7 \mid k = 0, \ldots, 6\}$, since $7^{-1} \equiv 7 \pmod{12}$.

Example 2.8. Diatonic chords of three notes stacked in 'thirds', called *triads*, play crucial roles in tonal music. In the case of C-major scale, adopting a wholetone encoding η_C as

$$\eta_C(C) = 0, \eta_C(D) = 1, \eta_C(E) = 2, \eta_C(F) = 3, \eta_C(G) = 4, \eta_C(A) = 5, \eta_C(B) = 6,$$

the triads are also expressed by the J-function as follows:

$$CEG = \mathcal{J}^0_{7,3},\ DFA = \mathcal{J}^3_{7,3},\ EGB = \mathcal{J}^6_{7,3},\ FAC^1 = \mathcal{J}^9_{7,3},\ GBD^1 = \mathcal{J}^{12}_{7,3},\ AC^1E^1 = \mathcal{J}^{15}_{7,3},\ BD^1F^1 = \mathcal{J}^{18}_{7,3}.$$

Moreover, the 7th chords that consist of four diatonic notes are expressed as

$$CEGB = \mathcal{J}^3_{7,4},\ DFAC^1 = \mathcal{J}^7_{7,4},\ EGBD^1 = \mathcal{J}^{11}_{7,4},\ FAC^1E^1 = \mathcal{J}^{15}_{7,4},$$
$$GBD^1F^1 = \mathcal{J}^{19}_{7,4},\ AC^1E^1G^1 = \mathcal{J}^{23}_{7,3},\ BD^1F^1A^1 = \mathcal{J}^{27}_{7,3}.$$

So, using an abbreviation

$$\mathcal{J}^{m,n}_{c,d,e} = J^m_{c,d}(\mathcal{J}^n_{d,e}) = (J^m_{c,d} \circ J^n_{d,e}(k))_{k=0,\ldots,e-1}$$

for $c > d > e > 0$, the triads or 7-th chords have semitone encoding expressions, such as

$$CEG = \mathcal{J}^{5,0}_{12,7,3},\ CEGB = \mathcal{J}^{5,3}_{12,7,4},$$

meaning that the diatonic triads or 7th chords are understood as 'multi-order maximally even sets', pointed out by Douthett[2].

Definition 2.9. Consider a chord $T = (t_{i_1}, \ldots, t_{i_n})$ compatible with the extended scale $\overline{S} = (\ldots, t_{-1}, t_0, t_1, \ldots)$ of a scale S of d notes. The *translation* $T + l$ of T in $\overline{S}$ is a chord given by $T + l = (t_{i_1+l}, \ldots, t_{i_n+l})$ for any $l \in \mathbf{Z}$. The *first inversion* $T(1)$ of T is a chord given by $T(1) = (t_{i_2}, \ldots, t_{i_n}, t_{i_1+hd})$, where $h = oct(T)$. Inductively, the *p-th inversion* $T(p)$ of T is given as the first inversion of $T(p-1)$.

Proposition 2.10. *Given a scale S of d notes, consider a maximally-even chord $\mathcal{J}^m_{hd,e}$ compatible with $h \cdot S$. Then the translation $\mathcal{J}^m_{hd,e} + l$ for any integer l is a maximally-even chord*

$$\mathcal{J}^m_{hd,e} + l = \mathcal{J}^{m+le}_{hd,e}, \tag{2.2}$$

and the p-th inversion $\mathcal{J}^m_{hd,e}(p)$ for any integer p is a maximally-even chord

$$\mathcal{J}^m_{hd,e}(p) = \mathcal{J}^{m+phd}_{hd,e}. \tag{2.3}$$

This proposition shows that translation and inversion of maximally-even chords are also expressed in terms of J-functions, but we omit the proof (see [6]).

2.3. **Expansion of scales (revisited).** When $h > 1$ is prime to d, $\mathbf{Z}/d\mathbf{Z} \ni k \mapsto hk \in \mathbf{Z}/d\mathbf{Z}$ is isomorphic. Thus a tuple

$$X = \left(J^m_{c,d}(hk)\right)_{k=0,\dots,d-1}$$

gives a maximally even subset $|X|$ in $Ch_c = \mathbf{Z}/c\mathbf{Z}$ while its entries are out of sequence on the semitone encoding. As is pointed out in [5], the equality (on $\mathbf{Z}$),

$$\begin{tikzcd} \mathbf{Z}/d\mathbf{Z} \arrow[r, hook, "J^m_{hc,d}"] \arrow[d, "\times h"', "\approx"] \arrow[dr, phantom, "\circlearrowright"] & \mathbf{Z}/hc\mathbf{Z} \arrow[d, "proj."] \\ \mathbf{Z}/d\mathbf{Z} \arrow[r, hook, "J^m_{c,d}"'] & \mathbf{Z}/c\mathbf{Z} \end{tikzcd}$$

$$J^m_{c,d}(hk) = \left\lfloor \frac{c(hk)+m}{d} \right\rfloor = \left\lfloor \frac{hc\cdot k+m}{d} \right\rfloor = J^m_{hc,d}(k)$$

shows that X is maximally even in $h \cdot Ch_c = \mathbf{Z}/hc\mathbf{Z}$. In other words, a maximally even tuple X in $h \cdot Ch_c$ is folded up in Ch_c.

Definition 2.11 (Expansion of scales)**.** Let $S = (t_0, \dots, t_{d-1})$ be a scale compatible with Ch_c. For a natural number h prime to d, we define an *h-expansion* $S^h = (s_0, \dots, s_{d-1}) \sqsubset h \cdot S$ of S by

$$\theta(s_i) = \theta(t_j) + nc \text{ where } hi = nc + j. \tag{2.4}$$

We note that, when we consider η_S as a function $\overline{S} \to \mathbf{Z}$, the condition (2.4) is equivalent to

$$\eta_S(s_i) = \eta_S(t_j) + nd = j + nd \text{ where } hi = nd + j,$$

that is, $\eta_S(s_i) = hi$ while $\eta_{S^h}(s_i) = i$ by definition. For a scale $S = (t_0, \dots, t_{d-1})$, $\pi_{h\cdot S} : h \cdot S \to S$ stands for a projection corresponding to a natural projection $\mathbf{Z}/hd\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$. Whenever h is prime to d, there exists its multiplicative inverse $h^{-1} \in (\mathbf{Z}/d\mathbf{Z})^\times$. Then the multiplication by $h^{-1} : \mathbf{Z}/d\mathbf{Z} \ni k \mapsto h^{-1}k \in \mathbf{Z}/hd\mathbf{Z}$ is injective, which corresponds to the map $(\cdot)^h : S \ni t_j \mapsto s_i \in h \cdot S$ defined by (2.4). Thus we can define the expansion of a chord $X \sqsubset S$ as the image of the map $(\cdot)^h$, denoted by X^h. We note that X^h is compatible with S^h by definition.

Example 2.12. Consider the C major scale $\mathcal{C} = CDEFGAB$ in Ch_{12}, then its expansions $\mathcal{C}^h$ are

$$\mathcal{C}^2 = CEGBD^1F^1A^1 \sqsubset 2\cdot\mathcal{C}, \quad \mathcal{C}^3 = CFBE^1A^1D^2G^2 \sqsubset 3\cdot\mathcal{C},$$
$$\mathcal{C}^4 = CGD^1A^1E^2B^2F^3 \sqsubset 4\cdot\mathcal{C}, \quad \mathcal{C}^5 = CAF^1D^2B^2G^3E^4 \sqsubset 5\cdot\mathcal{C}$$

and $\mathcal{C}^6 = CBA^1G^2F^3E^4D^5 \sqsubset 6\cdot\mathcal{C}$. We have $\eta_{\mathcal{C}}(\mathcal{C}^2) = (0,2,4,6,8,10,12)$ while $\eta_{\mathcal{C}^2}(\mathcal{C}^2) = (0,1,2,3,4,5,6)$. 2-expansion $\mathcal{C}^2$ represents so called the 'stacking third' process, which are mainly used by pieces in classical music. 3-expansion $\mathcal{C}^3$ shows a stacking 4th process, which was the concept that Claude Debussy used preferably in his pieces. 4-expansion $\mathcal{C}^4$ is nothing but the 'stacking perfect 5-th' process and used by the Pythagorean tuning. We note that all these expansions are maximally even since $\mathcal{C}^h = \mathcal{J}^5_{12h,7} \sqsubset h\cdot\mathcal{C}$.

Consider a chord $DFA \sqsubset \mathcal{C}$ of which the whole tone encoding is $\eta_{\mathcal{C}}(DFA) = (1,3,5) = \mathcal{J}^3_{7,3}$. As $2^{-1} \equiv 4 \pmod 7$, we have $\eta_{\mathcal{C}^2}((DFA)^2) = (4,12,20) \equiv (4,5,6) = \eta_{\mathcal{C}^2}(D^1F^1A^1) \pmod 7$, that is, $(DFA)^2 = D^1F^1A^1$, which is not maximally even. Conversely, consider a chord $DEF \sqsubset \mathcal{C}$ with $\eta_{\mathcal{C}}(DEF) = (1,2,3)$. As $3^{-1} \equiv 5 \pmod 7$, we have $\eta_{\mathcal{C}^3}((DEF)^3) = (5,10,15) \equiv (5,3,1) = \eta_{\mathcal{C}^3}((D^2E^1F)$, hence $(DEF)^3 = D^2E^1F$ which is maximally even in $\mathcal{C}^3$.

The following states that any chord of which entries form an arithmetic sequence in the sense of wholetone encoding, compatible with a scale S, becomes a maximally even chord compatible with S^h for some octave range h.

Theorem 2.13. *Let S be a scale of d notes and take natural numbers b and t which are prime to d. Identifying S with $(0,1,2,\dots,d-1)$ via a wholetone encoding η_S, consider a chord X compatible with S of which entries form an arithmetic sequence,*

$$\eta_S(|X|) = \{a + kb \pmod d \mid k = 0,\dots,t-1\}.$$

Then, its expansion $X^h \sqsubset S^h$ is maximally even in S^h, where $0 < h < d$ satisfies $h \equiv bt \pmod d$.

Proof. Putting $S^h = (y_0, \dots, y_{d-1})$, we have seen $\eta_S(y_i) = hi = h \cdot \eta_{S^h}(y_i)$. It comes from the equation

$$\eta_S(y_i) \equiv a + kb \pmod d$$

that

$$\eta_{S^h}(y_i) \equiv h^{-1}(a + bk) \equiv h^{-1}a + t^{-1}k \pmod d \tag{2.5}$$

where h^{-1} is the multiplicative inverse of $h = bt \in (\mathbf{Z}/d\mathbf{Z})^\times$. Let $X^h \sqsubset S^h$ be the h-expansion of X. It comes from Theorem 2.6 that a chord X of $t \in (\mathbf{Z}/d\mathbf{Z})^\times$ notes compatible with S is maximally even if and only if $|X|$ forms an arithmetic sequence,

$$\eta_S(|X|) = \{g + t^{-1}k \pmod d \mid k = 0, \dots, t-1\}.$$

By (2.5) we see

$$\eta_{S^h}(|X^h|) = \{h^{-1}a + t^{-1}k \pmod d \mid k = 0, \dots, t-1\},$$

hence X^h is a maximally even chord compatible with S^h. □

Example 2.14 (Maximal evenness of suspended chords)**.** In music theory, a chord CFG is called a 'suspended 4th' chord denoted by Csus4. Historically, the name 'suspended' comes from voice leadings: for instance, when a chord progression $CFA \to CEG$ occurs, a deformation $CFA \to CFG \to CEG$ may be used in order to get more mild change of voices. In this case, the note F in CFG is suspended from CFA.

Here we give another interpretation to the suspended chords from the maximal evenness ansatz. The whole tone encoding of CFG on C major scale $\mathcal{C}$ is given by $\eta_{\mathcal{C}}(CFG) = (0, 3, 4)$, however, if we take 2nd inversion GC^1F^1, we obtain an arithmetic sequence $\eta_{\mathcal{C}}(GC^1F^1) = (4, 7, 10)$. Thus we can apply Theorem 2.13 for $d = 7, t = 3, b = 4$. As $h \equiv bt \equiv 2 \pmod 7$ and $h^{-1} \equiv 4 \pmod 7$ we have

$$\eta_{\mathcal{C}^2}((CFG)^2) = (0, 12, 16) \equiv (0, 5, 2) = \eta_{\mathcal{C}^2}(CF^1G) \pmod 7.$$

You see $\{0, 2, 5\} = |\mathcal{J}^1_{7,3}|$, so even though CFG is not maximally even in $\mathcal{C}$, its 2-expansion $(CFG)^2 = CF^1G$ is maximally even in $\mathcal{C}^2$. Also we note that GC^1F^1 is not maximally even in $2\cdot\mathcal{C}$: as $\eta_{\mathcal{C}}(GC^1F^1) = (2, 7, 12)$, the distances between adjacent notes G, C^1 and F^1 in $2 \cdot \mathcal{C} = \mathbf{Z}/2\cdot 7\mathbf{Z}$ are $3, 5$ and 3, which means $(2, 7, 12)$ is not balanced (see [9]).

Similar argument can be done for so called 'suspended 2nd' Csus2 such as CDG in $\mathcal{C}$: its first inversion DGC^1 gives an arithmetic sequence $\eta_{\mathcal{C}}(DGC^1) = (1, 4, 7)$, hence taking octave range $h \equiv bt = 9 \equiv 2 \pmod 7$, 2-expansion $(CDG)^2 = CD^1G$ is maximally even in $\mathcal{C}^2$, since $\eta_{\mathcal{C}^2}(|CD^1G|) = \{0, 4, 2\} = |\mathcal{J}^0_{7,3}|$.

3. Rigidity of chords and tonality

3.1. **Rigidity of chords.** The triad CEG is regarded as one of the primary triads in not only C-major scale but also G-major and F-major scales: $CEG \sqsubset \mathcal{J}^4_{12,7}, \mathcal{J}^5_{12,7}, \mathcal{J}^6_{12,7}$, thus CEG is invariant for the change of 'key' $\mathcal{J}^{5\pm1}_{12,7}$.

Definition 3.1 (Rigidity of chords)**.** Given a general chromatic scale Ch_c and a set of extended scales $\mathcal{S} = \{\overline{S}_k\}$ in Ch_c. For a chord X in some $\overline{S} \in \mathcal{S}$, we define the *rigidity* $R_{\mathcal{S}}(X)$ of X as the number of scales with which X is compatible,

$$R_{\mathcal{S}}(X) = \#\{\overline{S} \in \mathcal{S} \mid X \sqsubset \overline{S}\}.$$

$C_{\mathcal{S}}(X)$ stands for the set of scales in $\mathcal{S}$ compatible with X, $C_{\mathcal{S}}(X) = \{\overline{S} \in \mathcal{S} \mid X \sqsubset \overline{S}\}$.

For usual chromatic Ch_{12}, we put the set Δ of the extended diatonic scales, $\Delta = \{\overline{\mathcal{J}^m_{12,7}} \mid m = 0, \dots, 11\}$. Then, it can be seen $R_\Delta(CEG) = 3$.

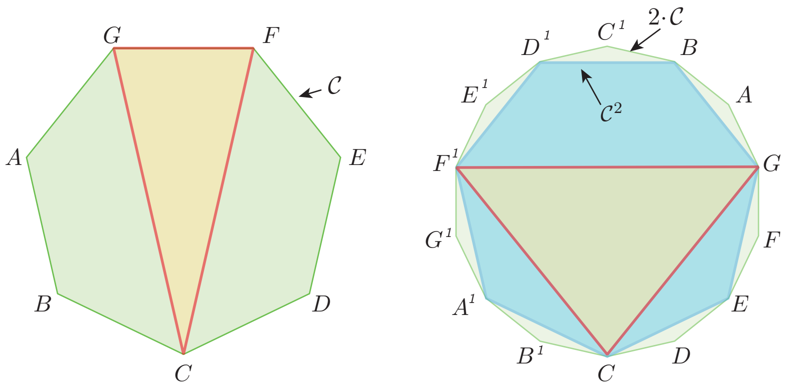

FIGURE 1. The chord CFG in C-major scale $\mathcal{C}$ is not maximally even (left) while its 2-expansion CF^1G in $\mathcal{C}^2$ is maximally even (right). Also we note that CF^1G is not maximally even in $2\cdot\mathcal{C}$.

Theorem 3.2. *Let $c > d$ be natural numbers prime to each other. For a general chromatic scale Ch_c and a set of maximally-even scales $\mathcal{S} = \{\overline{\mathcal{J}^m_{c,d}} \mid m = 0,\dots,c-1\}$ in Ch_c, let X be a prime chord consists of t notes and compatible with some $\overline{\mathcal{J}^m_{c,d}} \in \mathcal{S}$. Then we have*

$$R_{\mathcal{S}}(X) \le d - t + 1. \tag{3.1}$$

There exists a prime chord with maximum rigidity,

$$\max\{R_{\mathcal{S}}(X) \mid X \sqsubset \overline{\mathcal{J}^m_{c,d}} \textit{ for some } \overline{\mathcal{J}^m_{c,d}} \in \mathcal{S} \textit{ and } X \textit{ is a prime chord consists of } t \textit{ notes.}\} = d - t + 1$$

Proof. As X is prime and $X \sqsubset \overline{\mathcal{J}^m_{c,d}}$, we see $X = (J^m_{c,d}(k_1),\dots,J^m_{c,d}(k_t))$ where $J^m_{c,d}(k_i) \not\equiv J^m_{c,d}(k_j) \pmod{c}$, or equivalently $k_i \not\equiv k_j \pmod{d}$ for any $i \ne j$. Taking $m_k = \min\{m' \mid J^{m'}_{c,d}(k) = J^m_{c,d}(k)\}$, we see $J^{m_k-1}_{c,d}(k) = J^{m_k}_{c,d}(k) - 1$, hence $ck + m_k \equiv 0 \pmod{d}$. Thus $J^{m'}_{c,d}(k) = J^m_{c,d}(k)$ holds if and only if $m_k \le m' \le m_k + d - 1$. It comes from Proposition 2.5 that $m_{k_1},\dots,m_{k_t}$ are different each other. Thus

$$\#\left(\bigcap_{i=1}^{t}\{m_{k_i}, m_{k_i}+1,\dots,m_{k_i}+d-1\}\right) \le d - t + 1 \tag{3.2}$$

holds, hence (3.1).

By taking $k_1 = k, k_2 = k + c^-, k_3 = k + 2c^-,\dots,k_t = k + (t-1)c^-$ where $c^- > 0$ is the multiplicative inverse of $c \in (\mathbf{Z}/d\mathbf{Z})^\times$; $cc^- \equiv 1 \pmod{d}$, the chord $X = (J^{m_k}_{c,d}(k_1),\dots,J^{m_k}_{c,d}(k_t))$ is prime, and

$$0 \equiv ck_i + m_{k_i} \equiv ck + m_{k_i} + i - 1 \pmod{d},$$

hence $m_{k_i} \equiv m_k - i + 1 \pmod{d}$. Applying the octave equivalence to $k_i \equiv k + (i-1)c^- \pmod{d}$ if necessary so that $m_{k_i} = m_k - i + 1$ holds for all $i = 1,\dots,t-1$, the equality of (3.2) holds for X. Thus we have $R_{\mathcal{S}}(X) = d - t + 1$. □

3.2. **Rigidity under the diatonic scales Δ and tonality: a characterization of tritone.** Here we come back to the usual chromatic scale Ch_{12}. Figure 2 shows the all diatonic scales as a function $\mathcal{J}^m_{12,7}$ of m. From this, we see the rigidity of primary triads are 3, $R_\Delta(CEG) = R_\Delta(CFA) = R_\Delta(BDG) = 3$, which is smaller than the possible maximum rigidity $7 - 3 + 1 = 5$. When the chord CEG appears in a piece, you will find the possible key of the piece is $C_\Delta(CEG) = \{\mathcal{F} = \mathcal{J}^4_{12,7}, \mathcal{C} = \mathcal{J}^5_{12,7}, \mathcal{G} = \mathcal{J}^6_{12,7}\}$. If the piece contains CEG, CFA and BDG, the possible key is uniquely determined: $C_\Delta(CEG) \cap C_\Delta(CFA) \cap C_\Delta(BDG) = \{\mathcal{C}\}$. Namely, the primary chords CEG, CFA and BDG characterize C major scale $\mathcal{C}$. In this sense, since the diminished chord $BDF = \mathcal{J}^{5,-3}_{12,7,3}$ and the $G7$ chord $GBDF = \mathcal{J}^{5,-9}_{12,7,4}$ is compatible with only $\mathcal{C}$, they determine the key of piece (see Table 1). You see that any chord X compatible with some diatonic scale and which contains F and B belongs to either $\mathcal{C}$ or $\mathcal{F}^\sharp$. We state this phenomenon more generally.

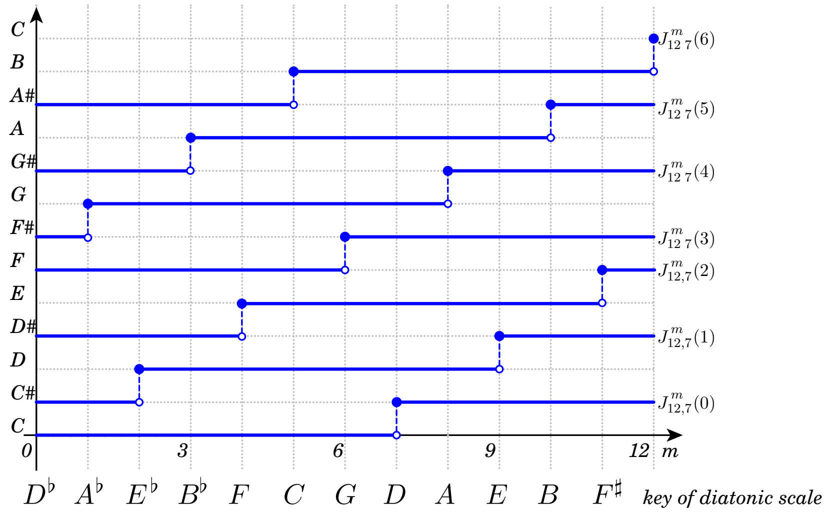

FIGURE 2. A representation of diatonic scales Δ by J-function. By increasing m, the key of diatonic scale is changed, $D^\flat \to A^\flat \to E^\flat \to \ldots$.

Rigidity $R_\Delta(X)$	Diatonic scales close to C: $B^\flat$	F	C	G	D
3	$\times$	$\longleftarrow$	CEG	$\longrightarrow$	$\times$
3	$\longleftarrow$	DFA	$\longrightarrow$	$\times$	$\times$
3	$\times$	$\times$	$\longleftarrow$	EGB	$\longrightarrow$
3	$\longleftarrow$	FAC	$\longrightarrow$	$\times$	$\times$
3	$\times$	$\times$	$\longleftarrow$	GBD	$\longrightarrow$
3	$\times$	$\longleftarrow$	ACE	$\longrightarrow$	$\times$
1	$\times$	$\times$	BDF	$\times$	$\times$

Rigidity $R_\Delta(X)$	Diatonic scales close to C: $B^\flat$	F	C	G	D
2	$\times$	$\times$	$CEGB$	$\longrightarrow$	$\times$
3	$\longleftarrow$	$DFAC$	$\longrightarrow$	$\times$	$\times$
3	$\times$	$\times$	$\longleftarrow$	$EGBD$	$\longrightarrow$
2	$\times$	$FACE$	$\longrightarrow$	$\times$	$\times$
1	$\times$	$\times$	$GBDF$	$\times$	$\times$
3	$\times$	$\longleftarrow$	$ACEG$	$\longrightarrow$	$\times$
1	$\times$	$\times$	$BDFA$	$\times$	$\times$

TABLE 1. The rigidity of triadic chords $\mathcal{J}^{5,n}_{12,7,3}$ (left) and 7th chords $\mathcal{J}^{5,n}_{12,7,4}$ (right) for $n = 0,\ldots,6$ compatible with C-major scale. Their compatibility with diatonic scales close to C-major is shown: $\times$ stands for the incompatibility of the chord with the corresponding scale.

Theorem 3.3. *Let $\mathcal{S}$ be a set of maximally even scales $\mathcal{S} = \{\mathcal{J}^m_{c,d} \mid m = 0,\ldots,c-1\}$ compatible with a chromatic scale Ch_c. Then, for each scale $\mathcal{J}^m_{c,d} \in \mathcal{S}$, there exists a unique chord $X_m \sqsubset \mathcal{J}^m_{c,d}$ with $\#|X_m| = 2$ such that*

$$R_{\mathcal{S}}(X_m) = \#\{(p,q) \in \{0,1,\ldots,d-1\}^2 \mid p+q = 0 \text{ or } c\}. \tag{3.3}$$

Putting $X_m = (x_1,x_2)$, we have $\theta(x_2) - \theta(x_1) \equiv d^{-1} - 1 \pmod c$.

Proof. By Proposition 2.5, there exist unique $0 \le i_m \ne j_m \le d-1$ for each $m = 0,\ldots,c-1$ which satisfy

$$J^m_{c,d}(i_m) - J^{m-1}_{c,d}(i_m) = 1 \quad \text{and} \quad J^{m+1}_{c,d}(j_m) - J^m_{c,d}(j_m) = 1$$

respectively, from which we have

$$ci_m + m = J^m_{c,d}(i_m)d \quad \text{and} \quad cj_m + m + 1 = (J^m_{c,d}(j_m)+1)d, \tag{3.4}$$

hence

$$(J^m_{c,d}(j_m) - J^m_{c,d}(i_m) + 1)d \equiv 1 \pmod c,$$

namely, $J^m_{c,d}(j_m) - J^m_{c,d}(i_m) \equiv d^{-1} - 1 \pmod{c}$. (3.4) also brings for $0 \le p, q \le d-1$,

$$J^{m+p}_{c,d}(i_m) = J^m_{c,d}(i_m) \quad \text{and} \quad J^{m+c-q}_{c,d}(j_m - 1) = J^m_{c,d}(j_m).$$

Then putting $|\theta(X_m)| = |(\theta(x_1), \theta(x_2))| = \{J^m_{c,d}(i_m), J^m_{c,d}(j_m)\}$, we see that $\mathcal{J}^l_{c,d} \in C_{\mathcal{S}}(X_m)$ if and only if $l \equiv m + p \equiv m - q \pmod{c}$ with $(p, q) \in \{0, \ldots, d-1\}^2$. As $0 \le p + q \le 2(d-1) < 2c$, we see $p + q = 0$ or c. Hence the assertion. □

Example 3.4. Let us apply the theorem above for C major scale $\mathcal{C} = \mathcal{J}^5_{12,7}$. As

$$J^6_{12,7}(3) - J^5_{12,7}(3) = J^5_{12,7}(6) - J^4_{12,7}(6) = 1,$$

we have $\theta(X) = \left(J^5_{12,7}(3), J^5_{12,7}(6)\right) = (5, 11)$, that is, $X = FB$. The pair FB is known as the 'tritone' in music theory, which is treated carefully because of its dissonance. In fact, since C (C^2, more explicitly) is the second overtone of F, C is quite harmonious with F while B and C are quite dissonant, as the interval between B and C is one semitone, and one semitone interval causes unpleasantness for human ears (Plomp and Levelt[10]). We also see that the solution to $p + q = 0$ or 12 with $0 \le p, q \le 6$ is $p = q = 0$ and $p = q = 6$, thus we have $\mathcal{C}_\Delta(FB) = \{\mathcal{C}, \mathcal{F}^\sharp\}$ and $R_\Delta(FB) = 2$. In music theory, the relation between $\mathcal{C}$ and $\mathcal{F}^\sharp$ is often used as the *tritone substitution*, a technique of chord progression: for instance, the dominant 7th $G7 = GBDF = \mathcal{J}^{5,5}_{12,7,4}(-2)$ can be changed to $C^\sharp 7 = C^\sharp F G^\sharp B = \mathcal{J}^{11,3}_{12,7,4}$, because both contain the characteristic toritone FB commonly.

Theorem 3.3 brings us a ray of hope for a mathematical model on consonance/dissonance, the basic concept of tonality. By Theorem 2.6, any maximally even set $\mathcal{J}^m_{c,d}$ is generated by a dynamics

$$\mathbf{Z}/c\mathbf{Z} \ni x \to x + d^{-1} \in \mathbf{Z}/c\mathbf{Z},$$

which is just an abstraction of the Pythagorean tuning: adding d^{-1} corresponds to an abstraction of stacking perfect 5th. So, it can be said that two notes x_1 and x_2 are consonant when $|\theta(x_2) - \theta(x_1)| \equiv d^{-1} \pmod{c}$. On the contrary, Plomp and Levelt proposed the dissonant curve model based on psychoacoustic research in [10]. As an abstraction of their model, it can be said that x_1 and x_2 are dissonant when $|\theta(x_2) - \theta(x_1)| \equiv 1 \pmod{c}$. Thus, we can say that a chord $X = (x_1, x_2)$ stated in Theorem 3.3 consists of notes dissonant each other, because $\theta(x_1)$ and $\theta(x_1') = \theta(x_1) + d^{-1}$ are consonant while

$$\theta(x_2) - \theta(x_1') = \theta(x_2) - (\theta(x_1) + d^{-1}) = 1$$

shows that x_2 and x_1' are dissonant, 'hence' x_2 and x_1 are also dissonant. In this sense, Theorem 3.3 gives a combinatorial characterization of the tritone, that is, a 'general' tritone is a chord X consists of two dissonant notes and satisfies (3.3).

3.3. **Does high rigidity mean tonal ambiguity? How about the Tristan chord?** Contrast to primary triads, suspended chords like Csus4 CFG or Csus2 CDG have the maximum rigidity $R_\Delta(CFG) = R_\Delta(CDG) = 5$. The reason for the maximum rigidity is, as is seen in Example 2.14, these suspended chords are generated by 'stacking perfect 5th' up to the octave equivalence, that is parallel to the Pythagorean tuning which generates diatonic scales. Indeed, you will see

$$C_\Delta(CFG) = \{\mathcal{A}^\flat = \mathcal{J}^1_{12,7}, \mathcal{E}^\flat = \mathcal{J}^2_{12,7}, \mathcal{B}^\flat = \mathcal{J}^3_{12,7}, \mathcal{F} = \mathcal{J}^4_{12,7}, \mathcal{C} = \mathcal{J}^5_{12,7}\}$$

and

$$C_\Delta(CDG) = \{\mathcal{E}^\flat = \mathcal{J}^2_{12,7}, \mathcal{B}^\flat = \mathcal{J}^3_{12,7}, \mathcal{F} = \mathcal{J}^4_{12,7}, \mathcal{C} = \mathcal{J}^5_{12,7}, \mathcal{G} = \mathcal{J}^6_{12,7}\}.$$

Thus, it might be said that high rigidity means tonal ambiguity. Such ambiguity has been widely used by composers. Indeed, the ambiguity is useful to change the key of a piece naturally. When a melody has high rigidity, composers can reharmonize it variously. Figure 3 gives such an example of reharmonization for a melody $CDFG$ (but just an arpeggio!), compatible with $C, F, B^\flat$ and $E^\flat$ major scales, namely $C_\Delta(CDFG) = \{\mathcal{C}, F, \mathcal{B}^\flat, \mathcal{E}^\flat\}$. At (i), we put C major tonic chords $CEG = \mathcal{J}^{5,0}_{12,7,3}$ at the first and 5th measures, and characteristic tritones at 2nd, 3rd and 4th measures. At (ii), we put 7th chords, a major 7th $CM7 = CEGB = \mathcal{J}^{5,3}_{12,7,4}$, half diminished 7th $Em7^{\flat 5} = EGB^\flat D = \mathcal{J}^{4,4}_{12,7,4}(1), Am7^{\flat 5} = ACE^\flat G = \mathcal{J}^{3,0}_{12,7,4}$ and $Dm7^{\flat 5} = DFA^\flat C = \mathcal{J}^{2,0}_{12,7,4}(1)$, containing tritones. At (iii), we use the tritone substitution $\mathcal{B}^\flat \to \mathcal{E}$ at measure 3, $D^\sharp m7^{\flat 5} = D^\sharp F^\sharp AC^\sharp = \mathcal{J}^{3,0}_{12,7,4}$, that sounds aggressive. At (iv), we put a diminished chord $E^\flat dim = E^\flat G^\flat AC = \mathcal{J}^{3}_{12,4}$ at measure 3, that sounds milder.

It must be too early to mention the famous Tristan chord, however, the chord raises an objection to our rigidity of chords approach to tonality. The Tristan chord was deliberately introduced in Wagner's opera *Tristan und Isolde* (1859). It can be understood as a half-diminished 7th chord enharmonically, however it has a various kind of possible harmonic functions and voice leadings, because of its tonal ambiguity. $BFAD^1$ is a Tristan chord: it contains the characteristic tritone BF, thus it has low rigidity $R_\Delta(BF) = 2$, meaning that it should has strong tonality from our viewpoint.

In this paper, we pay no attention to harmonic functions of chords or roles of leading tones in a scale, which are fundamental tools to analyze tonal music, at least in classical music. To build a mathematical model for tonal music via the maximal evenness ansatz, we need to change our combinatorial equal treatment of notes in a scale; we may need a a kind of hierarchically organized system of scale notes.

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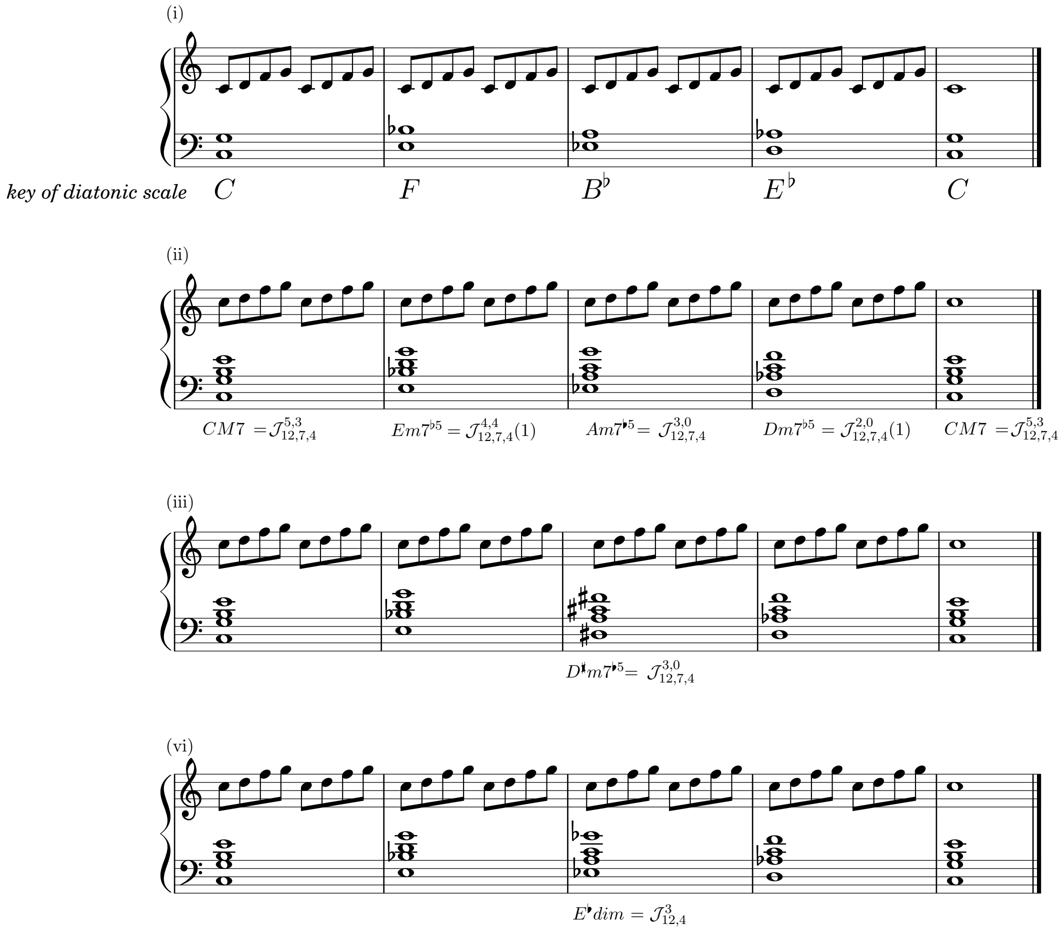

FIGURE 3. Reharmonizations for a high rigidity melody $CDFG$ compatible with $C, F, B^\flat$ and $E^\flat$ major scales. At (i), we put characteristic tritones at 2nd,3rd and 4th measure. At (ii), we use Tristan chords intentionally at 2nd,3rd and 4th measure. At (iii), we apply tritone substitution at 3rd measure as $C_\Delta(E^\flat A) = \{\mathcal{B}^\flat, \mathcal{E}\}$: the right hand plays in $B^\flat$ while the left hand plays in E. As a result, it sounds aggressive, but the top note of the left hand part moves chromatic way: $G \to F^\sharp = G^\flat \to F \to E$. Thus at (iv), we adopt the $E^\flat$ diminished chord as a compromise choice.

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