Largest regular multigraphs with three distinct eigenvalues

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Abstract

We deal with connected k-regular multigraphs of order n that has only three distinct eigenvalues. In this paper, we study the largest possible number of vertices of such a graph for given k. For $k = 2, 3, 7$, the Moore graphs are largest. For $k \neq 2, 3, 7, 57$, we show an upper bound $n \leq k^2 - k + 1$, with equality if and only if there exists a finite projective plane of order $k - 1$ that admits a polarity.

Key words: Graph spectrum, Moore bound, linear programming bound, projective plane,

1 Introduction

Let G be a connected k-regular multigraph (V, E), which may have a loop. For $u, v \in V$, let $m(u, v)$ be the number of edges between u and v if $u \neq v$, and the number of loops on u if $u = v$. The *adjacency matrix* $\boldsymbol{A}$ of G is defined to be the square matrix indexed by V whose (u, v) entry is $m(u, v)$ if $\{u, v\} \in E$ and 0 otherwise. The eigenvalues of $\boldsymbol{A}$ are called the eigenvalues of G. In this paper, we deal with a k-regular multigraph G with only 3 distinct eigenvalues. Since the degree of the minimal polynomial of $\boldsymbol{A}$ is 3, the diameter of G is at most 2. This implies that the Moore bound $|V| \leq k^2 + 1$ holds for k-regular multigraphs with only 3 distinct eigenvalues. If G attains this bound, G is called a *Moore graph*, which is simple. A Moore graph does not exist except for $(d, k) = (2, 2), (2, 3), (2, 7), (2, 57)$ [2, 5]. The following Moore graphs uniquely exist: the 5-cycle for $k = 2$, the Petersen graph for $k = 3$, and the Hoffman–Singleton graph for $k = 7$ [9]. For $k = 57$, the existence of the Moore graph is still open. The main problem of this

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paper is to improve the Moore bound, and to determine the largest k-regular multigraph with only 3 distinct eigenvalues for given $k \geq 3$.

A k-regular simple graph of order n is called a *strongly regular graph* with parameters (n, k, λ, μ) if there exist integers λ and μ such that any two adjacent vertices have λ common neighbours, and any two non-adjacent vertices have μ common neighbours. If a connected regular simple graph has only 3 distinct eigenvalues, then it is strongly regular. If a connected k-regular simple graph satisfies that any two adjacent vertices have at least λ common neighbours, and any two non-adjacent vertices have at least μ common neighbours, then the order n has the bound $n \leq k+1+k(k-1-\lambda)/\mu$ (see [3]). Strongly regular graphs are characterized as the graphs that attain this bound.

The point-line geometry $(\mathcal{P}, \mathcal{L})$ is called a *finite projective plane* of order q if $|\mathcal{P}| = |\mathcal{L}| = q^2 + q + 1$, there exist $q + 1$ points in each line, and there exist $q + 1$ lines through each point. The *incidence matrix* of $(\mathcal{P}, \mathcal{L})$ is the matrix indexed by $\mathcal{P}$ and $\mathcal{L}$ whose (p, l) entry is 1 if $p \in l$, and 0 otherwise. An isomorphism φ from $(\mathcal{P}, \mathcal{L})$ to the dual plane $(\mathcal{L}, \mathcal{P})$ is a *polarity* if φ is an involution. We say $(\mathcal{P}, \mathcal{L})$ admits polarity if there exists a polarity from $(\mathcal{P}, \mathcal{L})$ to $(\mathcal{L}, \mathcal{P})$. The classical finite projective planes admit a polarity. A finite projective plane $(\mathcal{P}, \mathcal{L})$ admits a polarity if and only if the incidence matrix of $(\mathcal{P}, \mathcal{L})$ can be symmetric. The symmetric incidence matrix of $(\mathcal{P}, \mathcal{L})$ is the adjacency matrix of a $(q - 1)$-regular multigraph with only 3 distinct eigenvalues which has loops. For $k \neq 2, 3, 7, 57$, we show an upper bound $n \leq k^2 - k + 1$ for k-regular multigraphs of order n with only 3 distinct eigenvalues. The equality holds if and only if the adjacency matrix of the graph is the symmetric incidence matrix of a finite projective plane of order $k - 1$ that admits a polarity.

The paper is organized as follows. In Section 2, the linear programming bound [11] is generalized for connected regular multigraphs. We also give a certain improvement of the Moore bound with prescribed distinct eigenvalues. In Section 3, we prove the upper bound $n \leq k^2 - k + 1$ for $k \neq 2, 3, 7, 57$. In Section 4, we show that the existence of a connected k-regular multigraph G of order $k^2 - k + 1$ with only 3 distinct eigenvalues is equivalent to the existence of a finite projective plane $PG(2, k - 1)$ that admits a polarity.

2 Bounds for regular multigraphs

Let G be a multigraph (V, E). For $v_j \in V$ and $e_j \in E$, a sequence $w_p = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{p-1}, e_p, v_p)$ is a *walk* if $e_j = \{v_{j-1}, v_j\}$ for each $j \in \{1, \ldots, p\}$. We shortly write a walk $w_p = (e_1, \ldots, e_p)$. The number p is called the *length* of a walk. A walk w_p is *non-backtracking* if there does not exist $j \in \{1, \ldots, p - 1\}$ such that $e_j = e_{j+1}$, or $p = 1$. A non-backtracking walk w_p is a *cycle* if $v_0 = v_p$ and $v_0, \ldots, v_{p-1}$ are distinct. The minimum length of cycles in G is called the *girth* of G. If G has a loop, then the girth of G is 1. It is well known that the (u, v)-entry of $\boldsymbol{A}^i$ is the number of walks of length i from u to v. A multigraph G is *k-regular* if $\sum_{v \in V} m(u, v)$ is k for each $u \in V$.

Let $F_i^{(k)}$ denote a polynomial of degree i defined by

$$F_0^{(k)}(x) = 1, \qquad F_1^{(k)}(x) = x, \qquad F_2^{(k)}(x) = x^2 - k,$$

and

$$F_i^{(k)}(x) = xF_{i-1}^{(k)}(x) - (k-1)F_{i-2}^{(k)}(x)$$

for $i \geq 3$. Note that $F_i^{(k)}(k) = k(k-1)^{i-1}$ for $i \geq 1$.

Singleton [13] proved the following theorem only for k-regular simple graphs.

Theorem 2.1. *Let G be a connected k-regular multigraph with adjacency matrix $\boldsymbol{A}$. Then the (u, v)-entry of $F_i^{(k)}(\boldsymbol{A})$ is the number of non-backtracking walks of length i from u to v.*

Proof. We use induction on i. Let $b_{uv}^{(i)}$ be the number of non-backtracking walks of length i from u to v. Let $f_{uv}^{(i)}$ be the (u, v)-entry of $F_i^{(k)}(\boldsymbol{A})$. For $i = 1$, the assertion is trivial. For $i = 2$, the (u, v)-entry $a_{uv}^{(2)}$ of $\boldsymbol{A}^2$ is the number of walks of length 2 from u to v. A walk that has backtracking must form (e_i, e_i). The assertion follows from $b_{uv}^{(2)} = a_{uv}^{(2)} - k\delta_{uv}$, where δ is the Kronecker delta.

Suppose $f_{uv}^{(j)} = b_{uv}^{(j)}$ for each $j \in \{1, \dots, i-1\}$. Since $F_i^{(k)}(\boldsymbol{A}) = \boldsymbol{A}F_{i-1}^{(k)}(\boldsymbol{A}) - (k-1)F_{i-2}^{(k)}(\boldsymbol{A})$, we have

$$\begin{aligned} f_{uv}^{(i)} &= \sum_{s \in V} f_{us}^{(1)} f_{sv}^{(i-1)} - (k-1) f_{uv}^{(i-2)} \\ &= \sum_{s \in V} b_{us}^{(1)} b_{sv}^{(i-1)} - (k-1) b_{uv}^{(i-2)}. \end{aligned}$$

The value $\sum_{s \in V} b_{us}^{(1)} b_{sv}^{(i-1)}$ is the number of walks $(e_1, \dots, e_p)$ such that $e_1 = \{u, *\}$, $e_p = \{*, v\}$, and $(e_2, \dots, e_p)$ is non-backtracking. We remove walks that have backtracking, namely the ones satisfying $e_1 = e_2$. For given non-backtracking walk $(e_3, \dots, e_p)$, the number of choices of e_1 is equal to $k-1$ because $e_1 \neq e_3$. Therefore $f_{uv}^{(i)} = b_{uv}^{(i)}$ follows. □

Let $\boldsymbol{I}$ denote the identity matrix. Let $\boldsymbol{J}$ denote the matrix whose entries are all 1. In [11] we proved the following theorem only for k-regular simple graphs.

Theorem 2.2. *Let G be a connected k-regular multigraph of order n with adjacency matrix $\boldsymbol{A}$. Let $\tau_0, \dots, \tau_d$ be the distinct eigenvalues of $\boldsymbol{A}$, where*

$\tau_0 = k$. Let $f(x)$ be the polynomial defined by $f(x) = \sum_{i=0}^{s} f_i F_i^{(k)}(x)$ with a positive integer s and real numbers $f_0, \dots, f_s$ such that $f_0 > 0$, $f_i \geq 0$ for each $i \in \{1, \dots, s\}$. If $f(k) > 0$ and $f(\tau_j) \leq 0$ for each $j \in \{1, \dots, d\}$, then

$$n \leq \frac{f(k)}{f_0}.$$

Proof. Since $\boldsymbol{A}$ is a real symmetric matrix, we have the spectral decomposition $\boldsymbol{A} = \sum_{i=0}^{d} \tau_i \boldsymbol{E}_i$, where $\boldsymbol{E}_0 = (1/n)\boldsymbol{J}$. It follows that

$$\sum_{j=0}^{d} f(\tau_j)\boldsymbol{E}_j = f(\boldsymbol{A}) = \sum_{i=0}^{s} f_i F_i^{(k)}(\boldsymbol{A}). \tag{2.1}$$

Taking the traces in (2.1), we have

$$\begin{aligned} f(k) = \operatorname{tr}(f(k)\boldsymbol{E}_0) &\geq \operatorname{tr}\left(\sum_{j=0}^{d} f(\tau_j)\boldsymbol{E}_j\right) \\ &= \operatorname{tr}\left(\sum_{i=0}^{s} f_i F_i^{(k)}(\boldsymbol{A})\right) \geq \operatorname{tr}(f_0 \boldsymbol{I}) = n f_0, \end{aligned}$$

because $\boldsymbol{E}_j$ is positive semidefinite, and each entry in $F_i^{(k)}(\boldsymbol{A})$ is non-negative by Theorem 2.1. It therefore follows $n \leq f(k)/f_0$. □

Let $k_i = k(k-1)^{i-1}$ and $k_0 = 1$.

Theorem 2.3. *Let G be a connected k-regular multigraph of order n with adjacency matrix $\boldsymbol{A}$. Let $F(x)$ be the polynomial defined by*

$$F(x) = \sum_{i=0}^{s} f_i F_i^{(k)}(x) \tag{2.2}$$

for some real numbers $f_0, \dots, f_s$. If the entries of $F(\boldsymbol{A})$ are all positive, then

$$n \leq \sum_{i \in \{0,\dots,d\} : f_i > 0} k_i. \tag{2.3}$$

Proof. Since each (u, v)-entry of $F(\boldsymbol{A})$ is positive, there exists $i \in \{0, \dots, d\}$ such that $f_i > 0$ and the (u, v)-entry in $F_i^{(k)}(\boldsymbol{A})$ is positive. For each $u \in V$, the number of non-backtracking walks of length i from u is equal to k_i. Thus the number of non-zero entries in $F_i^{(k)}(\boldsymbol{A})$ is at most nk_i. Comparing the numbers of positive entries in the both sides in (2.2), it follows that

$$n^2 \leq \sum_{i \in \{0,\dots,s\} : f_i > 0} n k_i.$$

This implies the theorem. □

Let $H_G(x)$ denote the Hoffman polynomial [7, 8] of a regular multigraph G, which is the polynomial of least degree satisfying $H_G(\boldsymbol{A}) = \boldsymbol{J}$. If the distinct eigenvalues of G are $\tau_0 = k, \tau_1, \ldots, \tau_d$ and the order of G is n, then H_G can be expressed by

$$H_G(x) = n \prod_{i=1}^{d} \frac{x - \tau_i}{k - \tau_i}.$$

Corollary 2.4. *Let G be a k-regular multigraph of order n, with only $d+1$ distinct eigenvalues $\tau_0 = k, \tau_1, \ldots, \tau_d$. Let $F_G(x)$ be the polynomial defined by $F_G(x) = \prod_{i=1}^{d}(x - \tau_i)$. Then, from the expression $F_G(x) = \sum_{i=0}^{d} f_i F_i^{(k)}(x)$, it follows that $n \leq \sum_{i \in \{0,\ldots,d\}: f_i > 0} k_i$.*

Proof. The polynomial $F_G(x)$ can be expressed by $F_G(x) = (\prod_{i=1}^{d}(k - \tau_i)/n) H_G(x)$. Therefore, each entry of $F_G(\boldsymbol{A}) = (\prod_{i=1}^{d}(k - \tau_i)/n)\boldsymbol{J}$ is positive. Applying Theorem 2.3 to $F_G(x)$, we obtain the bound $n \leq \sum_{i \in \{0,\ldots,d\}: f_i > 0} k_i$. □

If each f_i is positive in Corollary 2.4, then the bound (2.3) coincides with the Moore bound.

3 Upper bound for regular multigraphs with three eigenvalues

In this section, we prove an upper bound for k-regular multigraphs with only 3 distinct eigenvalues, which means Theorem 3.5. First we prove several lemmas to prove Theorem 3.5.

Lemma 3.1. *Let G be a connected k-regular multigraph of order n with only 3 distinct eigenvalues k, τ_1, τ_2. If $\tau_1 + \tau_2 \geq 0$, then $n \leq k^2 - k + 1$.*

Proof. The polynomial $F_G(x) = (x - \tau_1)(x - \tau_2)$ can be expressed by

$$F_G(x) = F_2^{(k)}(x) - (\tau_1 + \tau_2) F_1^{(k)}(x) + (k + \tau_1 \tau_2) F_0^{(k)}(x).$$

By $\tau_1 + \tau_2 \geq 0$ and Corollary 2.4, we have $n \leq k_0 + k_2 = k^2 - k + 1$. □

Lemma 3.2. *In a multigraph of maximum degree at most k, if a vertex u is incident with a multiedge then there are at most $k^2 - k$ vertices within distance two of u.*

Proof. Let v be a vertex adjacent to u with a multiedge. Then, it follows that

$$\begin{aligned}
|\{w \in V\colon \partial(u, w) \leq 2\}| &= 1 + |\{w \in V\colon \partial(u, w) = 1\}| + |\{w \in V\colon \partial(u, w) = 2\}| \\
&\leq 1 + (k - 1) + |\{w \in V\colon \partial(u, w) = 2, (v, w) \in E\}| \\
&\quad + |\{w \in V\colon \partial(u, w) = 2, (v, w) \notin E\}| \\
&\leq 1 + (k - 1) + (k - 2) + (k - 1)(k - 2) = k^2 - k,
\end{aligned}$$

where $\partial(u, w)$ is the distance between u and w. □

Let l_v denote the number of loops of $v \in V$.

Lemma 3.3. *Let G be a connected k-regular multigraph of order n with only 3 distinct eigenvalues. If $n > k^2 - k + 1$, then G is simple and strongly regular.*

Proof. It suffices to show that G is simple. Let τ_1, τ_2 be the distinct eigenvalues of G with $\tau_1, \tau_2 \neq k$. By Lemma 3.1, we have $\tau_1 + \tau_2 < 0$. By Lemma 3.2, G has no multiedge. The Hoffman polynomial of G can be expressed by

$$H_G(x) = n\frac{(x - \tau_1)(x - \tau_2)}{(k - \tau_1)(k - \tau_2)}.$$

It therefore follows that

$$n(\boldsymbol{A}^2 - (\tau_1 + \tau_2)\boldsymbol{A} + \tau_1\tau_2\boldsymbol{I}) = (k - \tau_1)(k - \tau_2)\boldsymbol{J}, \tag{3.1}$$

where $\boldsymbol{A}$ is the adjacency matrix of G. Comparing the (v, v)-entry of the both sides in (3.1), we obtain

$$l_v^2 - (\tau_1 + \tau_2 + 1)l_v = \frac{1}{n}(k - \tau_1)(k - \tau_2) - k - \tau_1\tau_2.$$

The value $l_v^2 - (\tau_1 + \tau_2 + 1)l_v$ is constant for each $v \in V$. If $l_v > 0$ for each $v \in V$, then

$$n \leq 1 + (k - 2) + (k - 2)(k - 2) = k^2 - 3k + 3 < k^2 - k + 1,$$

which contradicts our assumption. We may suppose some $v \in V$ satisfies $l_v = 0$. This implies that $l_u^2 - (\tau_1 + \tau_2 + 1)l_u = 0$, namely $l_u = 0$ or $l_u = \tau_1 + \tau_2 + 1$ for each $u \in V$. Since $\tau_1 + \tau_2 < 0$ holds, it follows that $l_u = \tau_1 + \tau_2 + 1 < 1$ and $l_u = 0$ for each $u \in V$. □

Lemma 3.4. *Let G be a connected k-regular multigraph of order n with only 3 distinct eigenvalues. If $n > k^2 - k + 1$ and $k \geq 3$, then there does not exist G except for Moore graphs. If $n > k^2 - k + 1$ and $k = 2$, then G is the cycle graph of order 4 or 5.*

Proof. By Lemma 3.3, G is strongly regular, and let (n, k, λ, μ) be the parameters of G. The assertion clearly holds for $k = 2$. Suppose $k \geq 3$. Let τ_1, τ_2 be the distinct eigenvalues of G with $\tau_1, \tau_2 \neq k$. For connected strongly regular graphs, it follows that $\mu \neq 0$. If $\mu \geq 2$, then

$$n = k + 1 + \frac{k^2 - \lambda k - k}{\mu} \leq \frac{k^2}{2} + \frac{k}{2} + 1 \leq k^2 - k + 1 \tag{3.2}$$

from $k \geq 3$. Thus $\mu = 1$. If $\lambda = 0$, then G is a Moore graph. If $\lambda = 1$, then G gives rise to a projective plane with a polarity containing no absolute points, which is not possible [6]. If $\lambda > 1$, then there exists an integer s such that $k = s(\lambda + 1)$ and $n = 1 + s(\lambda + 1) + s(s - 1)(\lambda + 1)^2$ [6], which gives

$$n = 1 + k + k^2 - s(\lambda + 1)^2 \leq 1 + k + k^2 - 3k < k^2 - k + 1. \qquad \square$$

Theorem 3.5. *Let G be a connected k-regular multigraph of order n with only 3 distinct eigenvalues. Then, one has $n \leq k^2 - k + 1$ for $k \neq 2, 3, 7, 57$.*

Proof. By Lemma 3.4, if $n > k^2 - k + 1$, then G is a Moore graph. There does not exist a Moore graph except for $k \in \{2, 3, 7, 57\}$ [2, 5]. This implies the theorem. □

4 Largest regular multigraphs with three eigenvalues

For $k \neq 2, 3, 7, 57$, we have $n \leq k^2 - k + 1$ by Theorem 3.5. The largest multigraphs are constructed from finite projective planes. Refer to [12] for projective planes. Suppose $q = k - 1$ is a prime power. Let $\mathbb{F}_q$ be the finite field of order q. Let V_q be a 3-dimensional vector space over $\mathbb{F}_q$. Let $\mathcal{P}_q$ (*resp.* $\mathcal{L}_q$) be the set of all 1-dimensional (*resp.* 2-dimensional) subspaces of V_q. Note that $|\mathcal{P}_q| = |\mathcal{L}_q| = q^2 + q + 1 = k^2 - k + 1$. A point $p \in \mathcal{P}_q$ is incident with a line $l \in \mathcal{L}_q$ if $p \subset l$. The point-line geometry $(\mathcal{P}_q, \mathcal{L}_q)$ is called a *classical* finite projective plane. Let Γ_q denote the incidence graph of $(\mathcal{P}_q, \mathcal{L}_q)$. The graph Γ_q is bipartite and its adjacency matrix can be expressed by

$$\begin{pmatrix} O & \boldsymbol{A} \\ \boldsymbol{A}^\top & O \end{pmatrix},$$

where $\boldsymbol{A}$ is the incidence matrix of $(\mathcal{P}_q, \mathcal{L}_q)$. The set of eigenvalues of Γ_q is $\{\pm(q+1), \pm\sqrt{q}\}$. We may suppose $\boldsymbol{A}$ is symmetric by the correspondence $\{(p, l) \in \mathcal{P}_q \times \mathcal{L}_q \colon p \perp l\}$, where we use the usual inner product of V_q. This implies that $\boldsymbol{A}$ is the adjacency matrix of a $(q+1)$-regular graph G_q and has only 3 distinct eigenvalues $\{q + 1, \pm\sqrt{q}\}$. Note that G_q has loops. For any prime power q, the graph G_q is a largest k-regular multigraph attaining the bound from Theorem 3.5.

The following is a necessary condition for a graph to attain the bound from Theorem 3.5.

Lemma 4.1. *Let G be a connected k-regular multigraph of order n with only 3 distinct eigenvalues k, τ_1, τ_2. If $n = k^2 - k + 1$, then G has a loop and no multiedge, $l_v \in \{0, 1\}$ for each $v \in V$, and $\tau_1 + \tau_2 = 0$.*

Proof. By $n = k^2 - k + 1$ and Lemma 3.2, there does not exist a multiedge in G. If there exists $v \in V$ such that $l_v > 1$, then

$$n \leq 1 + (k - 2) + (k - 2)(k - 1) = k^2 - 2k + 1 < k^2 - k + 1.$$

Thus $l_v \leq 1$ for each $v \in V$. As we see in the proof of Lemma 3.3, there exists $v \in V$ such that $l_v = 0$. Moreover $l_u^2 - (\tau_1 + \tau_2 + 1)l_u = 0$, namely $l_u = 0$ or $l_u = \tau_1 + \tau_2 + 1$ for each $u \in V$. If there exists $u \in V$ such that $l_u = \tau_1 + \tau_2 + 1 = 1$, then $\tau_1 + \tau_2 = 0$. Assume $l_u = 0$ for each $u \in V$. Now G is a strongly regular graph with parameters (v, k, λ, μ). If $\mu \geq 2$, then (3.2) holds. The last equality in (3.2) is attained only for $(n, k) = (7, 3)$, which is impossible. Thus $\mu = 1$. By the same argument as the last part in the proof of Lemma 3.4, for any λ there does not exist G of order $k^2 - k + 1$. □

The following is the main theorem in this section.

Theorem 4.2. *The existence of a connected k-regular multigraph G of order $k^2 - k + 1$ with only 3 distinct eigenvalues is equivalent to the existence of a finite projective plane $PG(2, k-1)$ that admits a polarity.*

Proof. If a finite projective plane $PG(2, k-1)$ that admits a polarity exists, then the incidence matrix can be symmetric, and it is the adjacency matrix of a k-regular multigraph of order $k^2 - k + 1$ with only 3 distinct eigenvalues.

Let G be a connected k-regular multigraph of order $k^2 - k + 1$ with only 3 distinct eigenvalues. By Lemma 4.1, the eigenvalues are $k, \pm\tau$, and the bipartite double graph G' of G is simple. Since the eigenvalues of G' are $\pm k, \pm\tau$, the diameter of G' is at most 3. Thus the graph G' attains the bipartite Moore bound $n \leq 2(1 + (k-1) + (k-1)^2) = 2(k^2 - k + 1)$, and the girth of G' is 6. The graph G' is the cage $v(k, 6)$, and G' must be the incidence graph of a finite projective plane $PG(2, k-1)$ (see [3, Section 6.9]). Now the incidence matrix of the projective plane $PG(2, k-1)$ is symmetric, and hence there exists a polarity on it. □

By Theorem 4.2, largest k-regular multigraphs with only 3 distinct eigenvalues are obtained for a prime power $q = k-1$. Open cases of small degrees are $k = 11, 13, 15, 16, 19, 21, 22, 23, \ldots$. For $q \equiv 1, 2 \pmod 4$, if a projective plane of order q exists, then q is the sum of two integral squares [4]. Therefore for $k = 13$ a projective plane of order 14 does not exist. For $k = 11$, there does not exist a finite projective plane of order 10 by a computer search [10]. If $\boldsymbol{A}$ is the adjacency matrix of some k-regular multigraph, then $\boldsymbol{A} + t\boldsymbol{I}$ is that of a $(k+t)$-regular multigraph, and has the same number of distinct eigenvalues as $\boldsymbol{A}$. This construction gives a candidate of the largest graph when a projective plane does not exist.

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