Journal of Mathematical Physics

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Cite as: J. Math. Phys. **60**, 011702 (2019); https://doi.org/10.1063/1.5074114
Submitted: 22 October 2018 . Accepted: 02 January 2019 . Published Online: 25 January 2019

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Journal of Mathematical Physics ARTICLE scitation.org/journal/jmp

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Submitted: 22 October 2018 • Accepted: 2 January 2019 •
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Nobuhiro Asai[1,a)] and Hiroaki Yoshida[2,b)]

AFFILIATIONS

[1] Department of Mathematics, Aichi University of Education, Hirosawa 1, Igaya, Kariya 448-8542, Japan
[2] Department of Information Sciences, Ochanomizu University, 2-1-1, Otsuka, Bunkyo, Tokyo 112-8610, Japan

[a)] **Email:** nasai@auecc.aichi-edu.ac.jp
[b)] **Email:** yoshida@is.ocha.ac.jp

ABSTRACT

The main purpose of this paper is to propose an (α, q)-analogue of the Poisson operators on the Fock space of type B in the sense of Bożejko, Ejsmont, and Hasebe [J. Funct. Anal. **269**, 1769–1795 (2015)] and to find a probability law of this operator. We shall show that the probability law is expressed by the q-Meixner distribution in the sense of Definition 3.2. Our results contain not only symmetric distributions as in Bożejko-Ejsmont-Hasebe but also the non-symmetric ones such as free Poisson, q and q^2-deformations of Poisson, Pascal, Gamma, and Meixner distributions.

I. INTRODUCTION

Based on a general procedure in Ref. 10, Bożejko *et al.*[7] considered a deformation of the (algebraic) full Fock space with two parameters $\alpha, q \in (-1, 1)$, namely, the (α, q)-Fock space (or the Fock space of type B) $\mathcal{F}_{\alpha,q}(\mathscr{H})$ over a complex Hilbert space $\mathscr{H}$. The deformation with $\alpha = 0$ is equivalent to the q-deformation by Bożejko and Speicher[9] and Bożejko *et al.*[8] In Ref. 7, a crucial point is to replace the Coxeter group of type A, that is, symmetric group $\mathfrak{S}_n$ for the q-Fock space by the Coxeter group of type B, $B(n) := \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$ in (2.1), to construct $\mathcal{F}_{\alpha,q}(\mathscr{H})$ equipped with the $(\alpha.q)$-inner product $\langle\cdot,\cdot\rangle_{\alpha,q}$. This replacement provides us to define more general creation $B^{\dagger}_{\alpha,q}(x)$ and annihilation $B_{\alpha,q}(x)$ operators acting on $\mathcal{F}_{\alpha,q}(\mathscr{H})$ and to compute a probability distribution $\nu_{\alpha,q}$ on $\mathbb{R}$ of the (α, q)-Gaussian operator (the Gaussian operator of type B), $B^{\dagger}_{\alpha,q}(x) + B_{\alpha,q}(x)$, $x \in \mathscr{H}$, with respect to the vacuum state. In fact, $\nu_{\alpha,q}$ is identified with the orthogonality and symmetric probability measure on $\mathbb{R}$ associated with the (α, q)-orthogonal polynomials $\{P_n^{(\alpha,q)}(t)\}$ given by the recurrence relation in (3.1). We should note that $\{\nu_{\alpha,q}\}_{\alpha,q\in(-1,1)}$ contains important examples, the laws of the free Gaussian ($\alpha = q = 0$), symmetric free Meixner ($q = 0$), q-Gaussian ($\alpha = 0$), and q^2-Gaussian ($\alpha = q$).

On the other hand, in Ref. 18, the q-Poisson operator (the Poisson operator of type A) is introduced as the sum of q-creation $b_q^{\dagger}$, q-annihilation b_q, and q-number $b_q^{\dagger}b_q$ operators and its probability law is identified with the q-Poisson distribution for $q \in [0, 1)$. However, an (α, q)-counterpart of the Poisson operator is not considered to the best of our knowledge. Hence, it is a natural question to consider how to define its (α, q)-analogue.

The organization of this paper is as follows: In Sec. II, we shall give a quick review on the Fock space of type B from Ref. 7. In Sec. III, we shall recall the (α, q)-Gaussian operator and propose the (α, q)-Poisson operator. In Sec. IV, after relationships between q-Meixner operator $\mathbf{X}_q$ of Ref. 21 and our (α, q^2)-Poisson operator are explained, we shall introduce a weighted $(-q, q^2)$-Poisson operator $\mathbf{Y}_{-q,q^2}$. Our approach is based on the q-Meixner class of orthogonal polynomials in the sense of Definition 3.2 to discuss the probability laws of all field operators. We shall show in Theorem 4.3 that the probability law of $\mathbf{Y}_{-q,q^2}$ is equal to that of the scaled Meixner operator $\mathbf{Y}_q = \frac{\mathbf{X}_q}{1+q}$ with respect to appropriate vacuum states. Moreover, it will be seen that one can treat rich examples of non-symmetric probability distributions such as

- free Poisson and q-Poisson,
- q^2-Poisson, q^2-Pascal, q^2-Gamma, and q^2-Meixner,

within the framework of the Fock space of type B. This is a significant development in this line of research.

II. PRELIMINARIES ON THE FOCK SPACE OF TYPE B

Let $\mathrm{B}(n)$ be the set of bijections σ of the $2n$ points $\{\pm 1, \pm 2, \ldots, \pm n\}$ with $\sigma(-k) = -\sigma(k)$. Equipped with the composition operation as a product, $\mathrm{B}(n)$ becomes what is called a Coxeter group of type B. It is generated by $\pi_0 := (1, -1)$ and $\pi_i := (i, i+1)$, $1 \le i \le n-1$, which satisfy the generalized braid relations

$$\begin{cases} \pi_i^2 = e, & 0 \le i \le n-1, \\ (\pi_0\pi_{n-1})^4 = (\pi_i\pi_{i+1})^3 = e, & 1 \le i \le n-1, \\ (\pi_i\pi_j)^2 = e, & |i-j| \ge 2,\ 0 \le i,j \le n-1. \end{cases} \tag{2.1}$$

An element $\sigma \in \mathrm{B}(n)$ expresses an irreducible form

$$\sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \le i_1, \ldots, i_k \le n-1,$$

and in this case

$$\ell_1(\sigma) := \text{the number of } \pi_0 \text{ in } \sigma,$$
$$\ell_2(\sigma) := \text{the number of } \pi_i,\ 1 \le i \le n-1, \text{ in } \sigma$$

are well defined.

Let $\mathscr{H}$ be a complex Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, where the inner product is linear on the right and conjugate linear on the left. For a given self-adjoint involution $x \mapsto \overline{x}$ for $x \in \mathscr{H}$, an action of $\mathrm{B}(n)$ on $\mathscr{H}^{\otimes n}$ is defined by

$$\begin{cases} \pi_0(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes x_2 \otimes \cdots \otimes \overline{x}_n, & n \ge 1, \\ \pi_i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes x_i \otimes x_{i+2} \otimes \cdots \otimes x_n, & n \ge 2,\ 1 \le i \le n-1. \end{cases}$$

Throughout this paper, we assume that the involution $\overline{x}$ of $x \in \mathscr{H}$ is defined in such a way that $\langle x, \overline{x}\rangle \in \mathbb{R}$ holds and $\langle x, \overline{x}\rangle = 0$ is equivalent to $x = 0$.

Let $\mathcal{F}_{\mathrm{fin}}(\mathscr{H})$ denote the algebraic full Fock space over $\mathscr{H}$

$$\mathcal{F}_{\mathrm{fin}}(\mathscr{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}^{\otimes n},$$

where Ω denotes the vacuum vector. We note that the elements of $\mathcal{F}_{\mathrm{fin}}(\mathscr{H})$ are expressed as finite linear combinations of the elementary vectors $x_1 \otimes \cdots \otimes x_n \in \mathscr{H}^{\otimes n}$. We equip $\mathcal{F}_{\mathrm{fin}}(\mathscr{H})$ with the inner product

$$\langle x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_n\rangle_{0,0} := \delta_{m,n} \prod_{k=1}^{n} \langle x_k, y_k\rangle, \quad x_k, y_k \in \mathscr{H}.$$

For $\alpha, q \in (-1, 1)$, define the symmetrization operator of type B on $\mathscr{H}^{\otimes n}$ as

$$\mathrm{P}_{\alpha,q}^{(n)} = \sum_{\sigma \in \mathrm{B}(n)} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \sigma, \quad n \ge 1,$$
$$\mathrm{P}_{0,q}^{(n)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\ell_2(\sigma)} \sigma, \quad n \ge 1,$$
$$\mathrm{P}_{\alpha,q}^{(0)} = \mathrm{I}_{\mathscr{H}^{\otimes 0}},\ \mathrm{P}_{0,0}^{(n)} = \mathrm{I}_{\mathscr{H}^{\otimes n}},$$

where we put $0^0 = 1$ and $\mathscr{H}^{\otimes 0} = \mathbb{C}\Omega$ by convention and

$$\mathrm{P}_{\alpha,q} = \bigoplus_{n=0}^{\infty} \mathrm{P}_{\alpha,q}^{(n)}$$

be the symmetrization operator of type B on $\mathcal{F}_{\mathrm{fin}}(\mathscr{H})$. Since $\mathrm{P}_{\alpha,q}^{(n)}$ is known to be strictly positive,

$$\langle x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_n \rangle_{\alpha,q} := \langle x_1 \otimes \cdots \otimes x_m, \mathrm{P}_{\alpha,q}(y_1 \otimes \cdots \otimes y_n) \rangle_{0,0}$$

becomes an inner product and $\langle \cdot, \cdot \rangle_{\alpha,q}$ is called the (α, q)-inner product with the convention $0^0 = 1$ and $y_{-k} = \overline{y_k}$, $k = 1, 2, \ldots, n$.

Definition 2.1. (1) For $\alpha, q \in (-1, 1)$, the (algebraic) full Fock space $\mathcal{F}_{\mathrm{fin}}(\mathscr{H})$ with respect to $\langle \cdot,\cdot \rangle_{\alpha,q}$ is called the (α, q)-Fock space (the Fock space of type B) denoted by $\mathcal{F}_{\alpha,q}(\mathscr{H})$. In this paper, we do not take completion. In particular, $\mathcal{F}_{0,q}(\mathscr{H})$ is nothing but the q-Fock space (the Fock space of type A) $\mathcal{F}_q(\mathscr{H})$ equipped with the q-inner product $\langle \cdot,\cdot \rangle_q := \langle \cdot,\cdot \rangle_{0,q}$ of Bożejko and Speicher.[9]

(2) Let $\mathrm{B}^{\dagger}_{\alpha,q}(x)$ be defined as the usual left creation operator

$$\mathrm{B}^{\dagger}_{\alpha,q}(x)\Omega = x,$$
$$\mathrm{B}^{\dagger}_{\alpha,q}(x)(x_1 \otimes \cdots \otimes x_n) = x \otimes x_1 \otimes \cdots \otimes x_n, \quad n \geq 1,$$

and $\mathrm{B}_{\alpha,q}(x)$ be its adjoint with respect to $\langle \cdot,\cdot \rangle_{\alpha,q}$, that is, $\mathrm{B}_{\alpha,q} = (\mathrm{B}^{\dagger}_{\alpha,q})^*$. $\mathrm{B}^{\dagger}_{\alpha,q}$ and $\mathrm{B}_{\alpha,q}$ are called the (α, q)-creation and (α, q)-annihilation operators, respectively.

The next proposition is direct consequences of the definition.

Proposition 2.2. (1) The (α, q)-annihilation operator $\mathrm{B}_{\alpha,q}$ acts on the elementary vectors as follows:

$$\mathrm{B}_{\alpha,q}(x)\Omega = 0, \quad \mathrm{B}_{\alpha,q}(x)x_1 = \langle x, x_1 \rangle\, \Omega,$$
$$\mathrm{B}_{\alpha,q}(x)(x_1 \otimes \cdots \otimes x_n) = \mathrm{L} + \mathrm{R},$$

where

$$\mathrm{L} = \sum_{k=1}^{n} q^{k-1} \langle x, x_k \rangle\, x_1 \otimes \cdots \otimes \check{x}_k \otimes \cdots \otimes x_n,$$
$$\mathrm{R} = \alpha q^{n-1} \sum_{k=1}^{n} q^{k-1} \langle x, \overline{x}_{n-(k-1)} \rangle\, x_1 \otimes \cdots \otimes \check{x}_{n-(k-1)} \otimes \cdots \otimes x_n,$$

for $n \geq 2$ where $\check{x}_k$ means that x_k should be deleted from the tensor product.

(2) The (α, q)-creation and the (α, q)-annihilation operators satisfy the commutation relation

$$\mathrm{B}_{\alpha,q}(x)\mathrm{B}^{\dagger}_{\alpha,q}(y) - q\mathrm{B}^{\dagger}_{\alpha,q}(y)\mathrm{B}_{\alpha,q}(x) = \langle x, y \rangle \mathrm{I} + \alpha \langle x, \overline{y} \rangle q^{2\mathrm{N}}, \quad x, y \in \mathscr{H}.$$

The readers can refer to Ref. 7 for details. It is easy to see that the operators $\mathrm{B}^{\dagger}_{0,q}$ and $\mathrm{B}_{0,q}$ are the same as the q-creation operator $b^{\dagger}_q(x)$ and q-annihilation operator $b_q(x)$, respectively, with respect to the inner product $\langle \cdot,\cdot \rangle_q$, that is, $b_q = (b^{\dagger}_q)^*$ (see Ref. 9).

Corollary 2.3. (1) The q-annihilation operator $b_q(x)$ acts on the elementary vectors as follows:

$$b_q(x)\Omega = 0, \quad b_q(x)x_1 = \langle x, x_1 \rangle\, \Omega,$$
$$b_q(x)(x_1 \otimes \cdots \otimes x_n) = \sum_{k=1}^{n} q^{k-1} \langle x, x_k \rangle\, x_1 \otimes \cdots \otimes \check{x}_k \otimes \cdots \otimes x_n, \quad n \geq 2,$$

where $\check{x}_k$ means that x_k should be deleted from the tensor product.

(2) The q-creation and the q-annihilation operators satisfy the q-commutation relation (q-CCR)

$$b_q(x)b^{\dagger}_q(y) - q\, b^{\dagger}_q(y)b_q(x) = \langle x, y \rangle \mathbf{1}, \qquad x, y \in \mathscr{H}.$$

III. (α, q)-OPERATORS AND PROBABILITY DISTRIBUTIONS

Let us recall standard notations from q-calculus, which can be found in Refs. 15 and 17, for example. Let $[n]_q!$ be the q-factorial as $[n]_q! := [1]_q[2]_q \cdots [n]_q$ for $n \geq 1$, where $[n]_q$ denotes the q-number, $[n]_q := 1 + q + \cdots + q^{n-1}$ for $n \geq 1$. The q-shifted factorials are defined by

$$(a;q)_0 := 1, \quad (a;q)_k := \prod_{\ell=1}^{k} (1 - aq^{\ell-1}), \; k = 1, 2, \ldots, \infty.$$

Remark 3.1. The q-shifted factorials are a natural extension of the Pochhammer symbol $(\cdot)_n$ because one can see that $\lim_{q\to 1}[k]_q = k$ implies

$$\lim_{q\to 1}\frac{(q^k;q)_n}{(1-q)^n} = (k)_n,$$

where $(k)_0 := 1$, $(k)_n := k(k+1)\cdots(k+n-1)$, $n \geq 1$.

A. (α, q)-Gaussian operator on $\mathcal{F}_{\alpha,q}(\mathscr{H})$

For $\alpha, q \in (-1, 1)$, let $\nu_{\alpha,q}$ be the orthogonalizing probability measure of the sequence of monic polynomials $\{P_n^{\alpha,q}(t)\}$ defined by the recurrence relation

$$\begin{cases} P_0^{\alpha,q}(t) = 1,\ P_1^{\alpha,q}(t) = t, \\ tP_n^{\alpha,q}(t) = P_{n+1}^{\alpha,q}(t) + (1+\alpha q^{n-1})[n]_q P_{n-1}^{\alpha,q}(t), \quad n \geq 1. \end{cases} \tag{3.1}$$

The measure $\nu_{\alpha,q}$ is symmetric, and its explicit expression can be found in Refs. 5, 7, and, 17. In Ref. 7, the (α, q)-Gaussian operator (the Gaussian operator of type B) on $\mathcal{F}_{\alpha,q}(\mathscr{H})$

$$\mathbf{G}_{\alpha,q}(x) := B^{\dagger}_{\alpha,q}(x) + B_{\alpha,q}(x), \quad x \in \mathscr{H},$$

is introduced and its spectral measure with respect to the vacuum state $\langle\Omega, \cdot\Omega\rangle_{\alpha,q}$ is identified with the symmetric probability measure $\nu_{\alpha\langle x,\bar{x}\rangle,q}$ on $\mathbb{R}$ for $\alpha\langle x, \bar{x}\rangle, q \in (-1, 1)$. When $\alpha = 0$, one can see that the q-Gaussian operator $\mathbf{G}_{0,q}(x)$ (the Gaussian operator of type A) on $\mathcal{F}_q(\mathscr{H})$,

$$\mathbf{G}_{0,q}(x) := b^{\dagger}_q(x) + b_q(x), \quad x \in \mathscr{H},$$

is recovered and its spectral measure with respect to the vacuum state $\langle\Omega, \cdot\Omega\rangle_q$ is the q-Gaussian measure, which is the orthogonalizing measure of the q-Hermite polynomials (see Refs. 8 and 9).

Definition 3.2. For given constants $q, \kappa_1, \kappa_2, \gamma, \delta$ with $0 \leq q < 1$, $\kappa_2 > 0$, $\delta \geq 0$, let $\mathbf{m}_q$ denote the probability measure $\mu(q; \kappa_1, \kappa_2, \gamma, \delta)$ on $\mathbb{R}$ such that the sequence of monic polynomials $\{Q_n^{(q)}(t)\}$ given by the recurrence relation

$$\begin{cases} Q_0^{(q)}(t) = 1, \quad Q_1^{(q)}(t) = t - \kappa_1, \\ tQ_n^{(q)}(t) = Q_{n+1}^{(q)}(t) + \big(\kappa_2 + \delta[n-1]_q\big)[n]_q Q_{n-1}^{(q)}(t) + \big(\kappa_1 + \gamma[n]_q\big)Q_n^{(q)}(t), \quad n \geq 1. \end{cases}, \tag{3.2}$$

is orthogonal with respect to the $L^2(\mathbf{m}_q)$-inner product. We shall refer the measure $\mathbf{m}_q$ as the q-Meixner distribution. For the q-Meixner class, see Refs. 4, 11, and 20 and the references cited therein. For the free Meixner class $q = 0$, see Refs. 3, 6 and 19.

Definition 3.3. For $s \in \mathbb{R}$, we define the translation T_s of a probability measure μ by $T_s\mu(\cdot) = \mu(\cdot - s)$. For $\lambda \in \mathbb{R}, \lambda \neq 0$, we define the dilation D_λ of μ by $D_\lambda\mu(\cdot) = \mu(\cdot/\lambda)$.

Remark 3.4. The existence of probability measure $\mathbf{m}_q := \mu(q; \kappa_1, \kappa_2, \gamma, \delta)$ is guaranteed by Favard's theorem, for example, in Refs. 12 and 16.

Remark 3.5. (1) The equality $1 + \alpha q^{n-1} = 1 + \alpha - \alpha(1-q)[n-1]_q$ holds. Hence, $\{P_n^{\alpha,q}(t)\}$ for $\alpha \in (-1, 0]$ can be considered as a special case of $\{Q_n^{(q)}(t)\}$ in the sense of Definition 3.2. Hence the measure $\nu_{\alpha,q}$ for $\{P_n^{\alpha,q}(t)\}$ coincides with $\mu(q; 0, 1+\alpha, 0, -\alpha(1-q))$ for $\alpha \in (-1, 0]$.

(2) In particular, by $(1+q^n)[n]_q = [2n]_q = (1+q)[n]_{q^2}$,

$$B_{q,q}(x) = (1+q)B_{0,q^2}(x) = (1+q)b_{q^2}(x), \quad x \in \mathscr{H},$$

holds. Therefore, $\nu_{q,q}$ is equal to the q^2-Gaussian measure with variance $1 + q$ (see Sec. IV).

Remark 3.6. (1) It is known[12] that the classical Meixner class of orthogonal polynomials and distributions ($q = 1$) can be classified into five types by parameters

$$\begin{cases} \theta := \dfrac{\gamma}{\sqrt{\kappa_2}}, \quad \tau := \dfrac{\delta}{\kappa_2}, \\ D := \theta^2 - 4\tau. \end{cases}$$

A q-analogue of the classical case is discussed in Ref. 4 and characterized as well as the $q = 1$ case by the same parameters (Refs. 11 and 20). More precisely, the q-Meixner distribution is classified into five types as follows:

(i) q-Gaussian: $\tau = 0, \theta = 0$.
(ii) q-Poisson: $\tau = 0, \theta \neq 0$.
(iii) q-Pascal: $\tau > 0, D > 0$.
(iv) q-Gamma: $\tau > 0, D = 0$.
(v) q-Meixner: $D < 0$.

(2) Monic polynomials $\{Q_n^{(q)}(t)\}$ can be obtained by the affine transformation of Al-Salam-Chihara polynomials,[2] but we accept Eq. (3.2), because it is rather convenient to examine the five types of distributions from the probabilistic viewpoint as well as the classical case ($q = 1$).

B. (α, q)-Poisson operator on $\mathcal{F}_{\alpha,q}(\mathscr{H})$

In Ref. 18, the q-Poisson operator (the Poisson operator of type A) is examined as the sum of $b_q^\dagger, b_q$ and $b_q^\dagger b_q$ and its distribution is identified with the q-Poisson distribution with $\delta = 0$ ($\tau = 0$) of the Meixner's classification. However, an (α, q)-counterpart of Poisson is not considered to the best of our knowledge, and hence it is a natural question to consider how to define its (α, q)-analogue.

Let us first examine a self-adjoint operator $\mathbf{P}_{\alpha,q}(x)$ defined by the form

$$\mathbf{P}_{\alpha,q}(x) := B_{\alpha,q}^\dagger(x) + B_{\alpha,q}(x) + c_1 N_q(x) + c_2\mathbf{1},$$

where $N_q := b_q^\dagger b_q$ and $c_1 \geq 0, c_2 \in \mathbb{R}$, and compute the probability distribution of this operator with respect to the vacuum state $\langle \Omega, \cdot\Omega\rangle_{\alpha,q}$. In this paper, the operator $\mathbf{P}_{\alpha,q}(x)$ is called the (α, q)-Poisson operator (the Poisson operator of type B). By Remark 3.5 (2), in particular, we have

$$\mathbf{P}_{q,q}(x) := b_{q^2}^\dagger(x) + (1+q)b_{q^2}(x) + c_1(1+q)N_{q^2}(x) + c_2\mathbf{1}.$$

Moreover, we note that $\mathbf{P}_{0,q}(x)$ is the same as the q-Poisson operator. It is trivial to see that the Poisson operator with $c_1 = 0$ is equal to the Gaussian operator with mean c_2.

Theorem 3.7. *Suppose $\alpha, q \in (-1, 1)$ and $x \in \mathscr{H}$ with $\|x\| = 1$. Let $\rho_{\alpha,q,x}$ be the probability distribution of $\mathbf{P}_{\alpha,q}(x)$ with respect to the vacuum state $\langle \Omega, \cdot\Omega\rangle_{\alpha,q}$.*

(1) *If $q \in (-1, 1)$ and $-1 < \alpha\langle x, \overline{x}\rangle \leq 0$, then $\rho_{\alpha,q,x}$ is*

$$\mu(q; c_2, 1 + \alpha\langle x, \overline{x}\rangle, c_1, -\alpha(1-q)\langle x, \overline{x}\rangle).$$

(2) *If $c_1 = 0$, $q \in (-1, 1)$ and $-1 < \alpha\langle x, \overline{x}\rangle < 1$, then $\rho_{\alpha,q,x}$ is equal to $T_{c_2}\nu_{\alpha\langle x,\overline{x}\rangle,q}$, where it is the probability distribution of $\mathbf{G}_{\alpha,q}(x) - c_2\mathbf{1}$.*

Proof. The map $\Phi : \operatorname{span}\{x^{\otimes n} \mid x \in \mathscr{H}, n \geq 0\} \to L^2(\mathbf{m}_q)$ given by $\Phi(x^{\otimes n}) = Q_n(t)$ is an isometry; in fact, we have

$$\|x^{\otimes n}\|_{\alpha,q}^2 = \|Q_n(t)\|_{L^2}^2 = (-\alpha\langle x, \overline{x}\rangle; q)_n [n]_q!, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

In addition, one can see

$$\begin{aligned}
&\mathbf{P}_{\alpha,q}(x)x^{\otimes n}\\
&\quad = B_{\alpha,q}^\dagger(x)x^{\otimes n} + B_{\alpha,q}(x)x^{\otimes n} + c_1 N_q(x)x^{\otimes n} + c_2\mathbf{1}x^{\otimes n}\\
&\quad = x^{\otimes n+1} + (1 + \alpha\langle x, \overline{x}\rangle q^{n-1})[n]_q x^{\otimes(n-1)} + (c_1[n]_q + c_2)x^{\otimes n}\\
&\quad = x^{\otimes n+1} + (1 + \alpha\langle x, \overline{x}\rangle - \alpha(1-q)\langle x, \overline{x}\rangle[n-1]_q)[n]_q x^{\otimes(n-1)} + (c_1[n]_q + c_2)x^{\otimes n}.
\end{aligned}$$

Hence, one can get inductively $\Phi(\mathbf{P}_{\alpha,q}(x)^n\Omega) = t^n$ and

$$\langle \Omega, \mathbf{P}_{\alpha,q}(x)^n\Omega\rangle_{\alpha,q} = \int t^n d\mu(q; c_2, 1 + \alpha\langle x, \overline{x}\rangle, c_1, -\alpha(1-q)\langle x, \overline{x}\rangle).$$

Since $\mu(q; c_2, 1 + \alpha\langle x, \overline{x}\rangle, c_1, -\alpha(1-q)\langle x, \overline{x}\rangle)$ has a compact support, it can be determined uniquely by the moment sequences. Therefore, $\rho_{\alpha,q,x} = \mu(q; c_2, 1 + \alpha\langle x, \overline{x}\rangle, c_1, -\alpha(1-q)\langle x, \overline{x}\rangle)$ if $-1 < \alpha\langle x, \overline{x}\rangle \leq 0$ is satisfied. It is easy to see the case $c_1 = 0$. □

IV. RELATIONSHIP BETWEEN (α, q^2)-POISSON AND q-MEIXNER OPERATORS

A. q-Meixner operator on $\mathcal{F}_q(\mathscr{H})$

Due to the replacement of α by $\alpha/\langle x, \overline{x}\rangle$ for $x \neq 0$ in the Proof of Theorem 3.7, it is enough to consider one-mode operators to obtain a probability distribution of a field operator on the Fock space of type B with respect to the vacuum state. This is justified in general by the idea of one-mode interacting Fock spaces by Accardi and Bożejko[1] (see also Ref. 16). Therefore, we shall restrict our consideration to the one-mode case from now on so that we simply denote all operators $B^\dagger_{\alpha,q}(x), B_{\alpha,q}(x), b^\dagger_q(x), b_q(x), N_q(x)$ by $B^\dagger_{\alpha,q}, B_{\alpha,q}, b^\dagger_q, b_q, N_q$, respectively.

First, we would like to recall a self-adjoint operator $\mathbf{X}_q(c_3, c_4)$ on $\mathcal{F}_q(\mathscr{H})$ given by

$$\mathbf{X}_q(c_3, c_4) = (b^\dagger_q)^2 + (b_q)^2 + c_3 b^\dagger_q b_q + c_4\mathbf{1}, \quad c_3 \geq 0, c_4 \in \mathbb{R},$$

and the probability distribution of this operator denoted by $\mu_{\mathbf{x}_q}$ with respect to the vacuum state $\langle\Omega, \cdot\Omega\rangle_q$.[20] In this paper, $\mathbf{X}_q$, $(b^\dagger_q)^2$ and $(b_q)^2$ are called the q-Meixner operator, the double q-creation and annihilation operators acting on $\mathcal{F}_q(\mathscr{H})$, respectively.

Remark 4.1. The operators, $b^\dagger_1, b_1, b^\dagger_1 b^\dagger_1, (b^\dagger_1)^2, (b_1)^2, \mathbf{1}$, are generators of the centrally extended Schrödinger Lie algebra $\mathcal{S}_1$, which is decomposed as the semi-direct product of the Heisenberg-Weyl algebra and $sl(2)$ (see Refs. 14 and 21). We will not mention this point in this paper.

It is our main concern in this section to clarify the relationship between probability distributions of $\mathbf{X}_q$ and $\mathbf{P}_{\alpha,q}$ in a sense. For this purpose, we shall take two steps as follows:

Step 1: Let us begin to see a fundamental identity to connect the operator $\mathbf{X}_q$ with (α, q^2)-operator. One can get

$$\begin{aligned}(1+\alpha q^{2(n-1)})[n]_{q^2} &= \left(1+\alpha-\alpha(1-q^2)[n-1]_{q^2}\right)[n]_{q^2}\\ &= \left\{1+\alpha-\alpha(1-q^2)\frac{1-(q^2)^{n-1}}{1-q^2}\right\}\frac{1-(q^2)^n}{1-q^2}\\ &= \frac{1}{1+q}\left\{1+\alpha-\frac{\alpha(1-q)}{q}\left(-1+[2n-1]_q\right)\right\}[2n]_q\\ &= \frac{1}{q(1+q)}\left(\alpha+q-\alpha(1-q)[2n-1]_q\right)[2n]_q,\end{aligned}$$

and hence $\alpha = -q$ implies

$$\begin{aligned}[2n-1]_q[2n]_q &= \frac{1+q}{1-q}(1-q^{2n-1})[n]_{q^2}\\ &= (1+q+q(1+q)^2[n-1]_{q^2})[n]_{q^2}.\end{aligned} \tag{4.1}$$

On the other hand, due to the definition of $b^\dagger_q$ and b_q, we have, for $x \in \mathscr{H}$,

$$\begin{cases}(b^\dagger_q)^2\, x^{\otimes 2n} = x^{\otimes 2(n+1)}, & n \geq 0,\\ (b_q)^2\, x^{\otimes 2n} = [2n]_q\,[2n-1]_q\, x^{\otimes 2(n-1)}, & n \geq 1,\\ b^\dagger_q b_q\, x^{\otimes 2n} = [2n]_q\, x^{\otimes 2n}, & n \geq 1.\end{cases} \tag{4.2}$$

Now let us state the following result, where Theorem 4.2 (1) is recalled from Ref. 20.

Theorem 4.2. *The probability distribution $\mu_{\mathbf{x}_q}$ of the operator $\mathbf{X}_q(c_3, c_4)$ with respect to the vacuum state $\langle\Omega, \cdot\Omega\rangle_q$ is given as follows:*

(1) *If $c_3 > 0$, then*

$$\mu_{\mathbf{x}_q} = \mu(q^2; c_4, 1+q, c_3(1+q), q(1+q)^2)$$

for $q \in [0, 1)$.

(2) *If $c_3 = 0$, then $\mu_{\mathbf{x}_q} = T_{c_4}\nu_{-q,q^2}$ for $q \in (-1, 1)$.*

Proof. It can be shown that the map $\Phi_1 : \{x^{\otimes 2n} \mid x \in \mathscr{H}, \|x\| = 1, n \geq 0\} \to L^2(\mathbf{m}_q)$ given by $\Phi_1(x^{\otimes 2n}) = Q_n(t)$ is an isometry. Moreover, due to equalities (4.1) and (4.2), we get

$$\begin{aligned}
&\mathbf{X}_q x^{\otimes 2n}\\
&= (b_q^\dagger)^2 x^{\otimes 2n} + (b_q)^2 x^{\otimes 2n} + c_3 b_q^\dagger b_q x^{\otimes 2n} + c_4 \mathbf{1} x^{\otimes 2n}\\
&= x^{\otimes 2(n+1)} + [2n]_q[2n-1]_q x^{\otimes 2(n-1)} + (c_4 + c_3[2n]_q) x^{\otimes 2n}\\
&= x^{\otimes 2(n+1)} + (1+q+q(1+q)^2[n-1]_{q^2})[n]_{q^2} x^{\otimes 2(n-1)} + (c_4 + c_3(1+q)[n]_{q^2}) x^{\otimes 2n}.
\end{aligned}$$

Hence, one can get inductively $\Phi_1(\mathbf{X}_q^n\Omega) = t^n$ and

$$\langle \Omega, {\mathbf{X}_q}^n \Omega\rangle_{\alpha,q} = \int t^n d\mu_{\mathbf{x}_q}.$$

Since $\mu_{\mathbf{x}_q}$ has a compact support, it can be determined uniquely by the moment sequences. Hence our first claim (1) is derived. It is easy to see our second claim (2). □

Step 2: For $0 \le q < 1$, consider a scaled operator $\mathbf{Y}_q$ of $\mathbf{X}_q$,

$$\mathbf{Y}_q := \frac{1}{1+q}\mathbf{X}_q, \tag{4.3}$$

and a weighted Poisson operator $\mathbf{Y}_{-q,q^2}$ defined by

$$\mathbf{Y}_{-q,q^2} := \frac{1}{1+q}\left\{ \mathrm{B}^\dagger_{-q,q^2} + \frac{1+q}{1-q}\mathrm{B}_{-q,q^2} + c_1(1+q)\mathrm{N}_{q^2} + c_2\mathbf{1}\right\}. \tag{4.4}$$

We remark here that if $c_1 = c_3 = 0$, the condition on q can be relaxed to $q \in (-1, 1)$.

Since $\mathbf{Y}_{-q,q^2}$ is not self-adjoint with respect to $\langle\cdot,\cdot\rangle_{-q,q^2}$ due to the second term in RHS of (4.4), which is a counterpart of $(b_q)^2$ in (4.3), we need to modify (α, q)-creation and annihilation operators by adding a weight $\beta > 0$ as follows:

Let $\mathrm{B}^\dagger_{\beta,\alpha,q}(x)$ be the β-weighted (α, q)-creation defined as the (α, q)-creation operator, and $\mathrm{B}^\dagger_{\alpha,q}(x)$ and $\mathrm{B}_{\beta,\alpha,q}(x)$ be the β-weighted (α, q)-annihilation operator given by

$$\mathrm{B}_{\beta,\alpha,q}(x) := \beta \mathrm{B}_{\alpha,q}(x), \quad \beta > 0.$$

The above two β-weighted operators are adjoint each other with respect to the β-weighted (α, q)-inner product

$$\langle x_1\otimes\cdots\otimes x_m, y_1\otimes\cdots\otimes y_n\rangle_{\beta,\alpha,q} := \delta_{m,n}\beta^n\langle x_1\otimes\cdots\otimes x_m, y_1\otimes\cdots\otimes y_n\rangle_{\alpha,q}.$$

By setting $\beta = \frac{1+q}{1-q}$, the operator $\mathbf{Y}_{-q,q^2}$ can be expressed as

$$\mathbf{Y}_{-q,q^2} = \frac{1}{1+q}\left\{\mathrm{B}^\dagger_{\beta,-q,q^2} + \mathrm{B}_{\beta,-q,q^2} + c_1(1+q)\mathrm{N}_{q^2} + c_2\mathbf{1}\right\},$$

and hence $\mathbf{Y}_{-q,q^2}$ is the self-adjoint operator with respect to the inner product $\langle\cdot,\cdot\rangle_{\beta,-q,q^2}$.

Then we can clarify the relationship between probability distributions of $\mathbf{Y}_q$ and $\mathbf{Y}_{-q,q^2}$ with respect to the vacuum state.

Theorem 4.3. *Suppose $c_1 = c_3$ and $c_2 = c_4$. Then the probability law of $\mathbf{Y}_q$ with respect to $\langle\Omega,\cdot\Omega\rangle_q$ is equal to that of $\mathbf{Y}_{-q,q^2}$ with respect to $\langle\Omega,\cdot\Omega\rangle_{\beta,-q,q^2}$ with $\beta = \frac{1+q}{1-q}$. In fact, the probability distribution $\rho_{\mathbf{Y}}$ of these operators is given as follows:*

(1) *If $c_1 > 0$, then $\rho_{\mathbf{Y}}$ is*

$$\mathrm{D}_a\mu_{\mathbf{x}_q} = \mu\left(q^2; \frac{c_2}{1+q}, \frac{1}{1+q}, c_1, q\right), \quad a = \frac{1}{1+q},$$

for $q \in [0, 1)$.

(2) *If $c_1 = 0$, then $\rho_{\mathbf{Y}}$ is $\mathrm{D}_a\mu_{\mathbf{x}_q} = \mathrm{D}_a\mathrm{T}_{c_2}\nu_{-q,q^2}$ for $q \in (-1, 1)$.*

Proof. We shall follow the same procedure as in the Proof of Theorem 4.2.

The map $\Phi_2 : \{y^{\otimes n} \mid y \in \mathscr{H}, \|y\| = 1, n \ge 0\} \to L^2(\mathbf{m}_q)$ given by $\Phi_2(y^{\otimes n}) = Q_n(t)$ is an isometry and

$$\begin{aligned}
&\mathbf{Y}_{-q,q^2} y^{\otimes n}\\
&= \frac{1}{1+q}\left\{\mathrm{B}^\dagger_{-q,q^2} y^{\otimes n} + \frac{1+q}{1-q}\mathrm{B}_{-q,q^2} y^{\otimes n} + c_1(1+q)\mathrm{N}_{q^2} y^{\otimes n} + c_2 \mathbf{1y}^{\otimes n}\right\}\\
&= \frac{1}{1+q}\left\{y^{\otimes(n+1)} + \frac{1+q}{1-q}(1-qq^{2(n-1)})[n]_{q^2} y^{\otimes(n-1)} + (c_2 + c_1(1+q)[n]_{q^2}) y^{\otimes n}\right\}\\
&= \frac{1}{1+q}\left\{y^{\otimes(n+1)} + [2n]_q[2n-1]_q y^{\otimes(n-1)} + (c_2 + c_1[2n]_q) y^{\otimes n}\right\}.
\end{aligned}$$

Hence, one can get inductively $\Phi_2(\mathbf{Y}^n_{-q,q^2}\Omega) = t^n$ and

$$\langle \Omega, \mathbf{Y}^n_{-q,q^2}\Omega\rangle_{\alpha,q} = \int t^n d(D_a \mu_{\mathbf{x}_q}), \quad a = \frac{1}{1+q},$$

with the help of (4.1). Since $D_a\mu_{\mathbf{x}_q}$ has a compact support, it can be determined uniquely by the moment sequences.

On the other hand, by Theorem 4.2, the probability distribution of $\mathbf{Y}_q$ is equal to $D_a\mu_{\mathbf{x}_q}$. Therefore, we have obtained our claim. □

Remark 4.4. In Theorems 4.2 and 4.3, the classification parameters θ and τ under $c_1 = c_3$ for the q-Meixner class are given by

$$\begin{cases} \theta = c_1\sqrt{1+q}, \\ \tau = q(1+q) \geq 0, \\ D = (1+q)(c_1^2 - 4q). \end{cases} \tag{4.5}$$

Note that the condition $\tau \geq 0$ in (4.5) implies $0 \leq q < 1$.

(I) If $q = 0$ ($\tau = 0$) and
   (1) $c_1 = c_3 = 0$ ($\theta = 0$), then $\mu_{\mathbf{x}_0} = T_{c_2}\nu_{0,0}$ and $D_a\mu_{\mathbf{x}_0} = D_aT_{c_2}\nu_{0,0}$ are the free Gaussian. Of course, this is a special case of $\nu_{\alpha,q}$ discussed in Ref. 7
   (2) $c_1 = c_3 \neq 0$ ($\theta \neq 0$), then $\mu_{\mathbf{x}_0}$ and $D_a\mu_{\mathbf{x}_0}$ are the free Poisson.
(II) If $0 < q < 1$ ($\tau > 0$) and
   (3) $c_1 = c_3 > 2\sqrt{q}$ ($D > 0$), then $\mu_{\mathbf{x}_q}$ and $D_a\mu_{\mathbf{x}_q}$ are the q^2-Pascal.
   (4) $c_1 = c_3 = 2\sqrt{q}$ ($D = 0$), then $\mu_{\mathbf{x}_q}$ and $D_a\mu_{\mathbf{x}_q}$ are the q^2-Gamma.
   (5) $0 \neq c_1 = c_3 < 2\sqrt{q}$ ($D < 0$), then $\mu_{\mathbf{x}_q}$ and $D_a\mu_{\mathbf{x}_q}$ are the q^2-Meixner.

We have shown by introducing the (α, q^2)-Poisson and the q-Meixner operators that non-symmetric probability distributions such as (2)–(5) can be treated within the framework of the Fock space of type B. In Ref. 7, non-symmetric cases are not treated.

As a final remark, we shall quickly mention about the q^2-Gaussian and q^2-Poisson distributions. Due to the classification in Remark 4.4, neither $\mathbf{X}_q$ nor $\mathbf{Y}_{-q,q^2}$ produces the q^2-Gaussian and q^2-Poisson laws. On the other hand, the $\mathbf{Y}_{q,q}$-operator given by

$$\mathbf{Y}_{q,q} := a\mathbf{P}_{q,q}, \ a = \frac{1}{1+q},$$

has these probability laws for $q \in (-1, 1)$. It is the self-adjoint operator with respect to the inner product $\langle\cdot,\cdot\rangle_{1+q,0,q^2}$. It is easy to see that if $c_1 = 0$, then $\mathbf{Y}_{q,q} = a(\mathbf{G}_{q,q} + c_2\mathbf{1})$. Hence we have the following proposition:

Proposition 4.5. *For $q \in (-1, 1)$, the probability law of $\mathbf{Y}_{q,q}$ with respect to $\langle\Omega, \cdot\Omega\rangle_{1+q,0,q^2}$ is as follows:*

(1) $c_1 = 0 \Rightarrow$ *the q^2-Gaussian,* $D_aT_{c_2}\nu_{0,q^2}$.
(2) $c_1 > 0 \Rightarrow$ *the q^2-Poisson,* $\mu\left(q^2; \frac{c_2}{1+q}, \frac{1}{1+q}, c_1, 0\right)$.

Note added in proof: We would like to thank an anonymous referee(s) informing us a relevant paper Ref. 13, where the Poisson distribution of type B is different from ours in this paper.

ACKNOWLEDGMENTS

This work was supported by Grant-in-Aid for Scientific Research (C) (JSPS No. 23540131 to N. Asai and JSPS No. 26400112 to H. Yoshida).

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