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Flatness of Simple Ring Extensions over Noetherian Rings

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All rings in this paper are assumed to be commutative with identity.

Let R be a Noetherian integral domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and let π: $R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\phi_\alpha(X)$ be the monic minimal polynomial of α over K with deg $\phi_\alpha(X) = d$ and write

$$\phi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d.$$

Let $I_{[\alpha]} := \cap_{i=1}^{d} (R :_R \eta_i)$. For $f(X) \in K[X]$, let $c(f(X))$ denote the R-submodule generated by the coefficients of $f(X)$. Let $J_{[\alpha]} := I_{[\alpha]} c(\phi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called an *anti-integral element* of degree d over R if Ker $\pi = I_{[\alpha]} \phi_\alpha(X) R[X]$. When α is an anti-integral element of degree d over R, $R[\alpha]$ is called an *anti-integral extension* of R (cf. [1, Introduction]). The element α is called a *super-primitive element* of degree d over R if $J_{[\alpha]} \not\subset p$ for any prime p of depth one. In case that α is an anti-integral element over R, it is known that $A = R[\alpha]$ is R-flat if and only if $J_{[\alpha]} = R$ ([2, Proposition 2.6]).

In this paper, for α is not necessarily anti-integral over R, we shall extend the results obtained under the condition that α is an anti-integral element over R.

We start with the following definition.

Definition. Let $0 \to$ Ker $\pi \to R[X] \to R[\alpha] \to 0$ be an exact sequence. For an integer $t \geqq 0$, we set $\Delta_t = \{f(X) \in \text{Ker}\ \pi \mid \deg f(X) = t\}$. Let $L_t = \{a \in R \mid a = 0$ or a is the leading coefficient of $f(X)$, where $f(X)$ runs over $\Delta_t\}$, which is an ideal of R.

We shall give the following lemma.

Lemma 1. *Let $d = [K(\alpha) : \mathrm{K}]$. Then the following statements hold.*

(1) *If $t < d$, then $\Delta_t = (0)$ and $L_t = (0)$.*

(2) *If $t = d$, then $L_t = I_{[\alpha]}$.*

(3) *If $t \geqq d$, then $L_t \subseteq L_{t+1}$.*

Proof. Since $K(\alpha)$ is an extension field of K with $d = [K(\alpha) : K]$, it follows that

$$\alpha^d + \eta_1 \alpha^{d-1} + \cdots + \eta_d = 0,$$

where $\eta_i \in K$. Clearly, (1) holds. From the definition of $I_{[\alpha]}$, (2) is trivial. Let $f(X) \in \Delta_t$. Then $Xf(X) \in \Delta_{t+1}$. Hence $L_t \subseteq L_{t+1}$.

The following proposition is useful in the sequel.

Proposition 2. *Assume that $t \geqq d$. Then $L_t = L_{t+1} = \cdots = L_n = \cdots$ if and only if* Ker π *is generated by polynomials of degree t at most.*

Proof. (⇒) Let Ker $\pi \ni f(X)$ and let a be the leading coefficient of $f(X)$. Put $n = \deg f(X)$. Suppose that $n \geqq t+1$. Since $a \in L_n = L_t$, there exists a polynomial $g(X) \in \Delta_t$ such that a is the leading coefficient of $g(X)$. Since $f(X) - X^{n-t}g(X) \in$ Ker π and $\deg(f(X) - X^{n-t}g(X)) < n$, we have that $f(X)$ is of degree t at most.

(⇐) Let $f(X) \in$ Ker π and a be the leading coefficient of $f(X)$. Then $f(X) = \Sigma_{i=1}^{s} g_i(X) h_i(X)$, where $\deg g_i(X) \leqq t$, $g_i(X) \in$ Ker π. We may assume that $\deg (g_i(X) h_i(X)) = \deg f(X)$ $(1 \leqq i \leqq m)$ and $\deg (g_i(X) h_i(X)) < \deg f(X)$ $(m+1 \leqq i \leqq s)$. Let b_i and c_i be the leading coefficients of $g_i(X)$ and $h_i(X)$, respectively. Then $a = \Sigma_{i=1}^{m} b_i c_i \in L_t$. Hence $L_n \subseteq L_t$, where $n = \deg f(X)$.

Remark 1. (1) α is an anti-integral element of degree d over R if and only if $L_d = L_{d+1} = \cdots$.

In fact, α is anti-integral if and only if Ker π is generated by some polynomials of degree d.

(2) If α is integral over R, then there exists an integer t $(\geqq d)$ such that $L_t = R$.

Indeed, α has a monic relation over R of degree t. Then $1 \in L_t$ and so $L_t = R$.

The converse of this statement is true. Hence the non-integral locus is given by $V(L_t)$, where $V(L_t) = \{p \in \mathrm{Spec}(R) \mid p \supset L_t\}$.

Definition. Assume that $L_t = L_{t+1} = \cdots = L_n = \cdots$. Then we denote the ideal generated by all coefficients of each polynomial in Δ_t by $L_{[\alpha]}$.

Proposition 3. *Assume that $t \geqq d$ and $L_t = L_{t+1} = \cdots = L_n = \cdots$. Let $p \in$ Spec R and $A = R[\alpha]$. Then $p \supset L_{[\alpha]}$ if and only if $A/pA \cong (R/p)[T]$, where T is an indeterminate over R/p, that is,* Spec A *is a blowing-up at p.*

Proof. (⇒) From Proposition 2, Ker π is generated by polynomials of degree t at most. The coefficients of their polynomials are contained in $L_{[\alpha]}$. Since $L_{[\alpha]} \subset p$, we have that Ker $\pi \subseteq pR[X]$. Therefore $A/pA \cong R[X]/pR[X] \cong (R/p)[T]$.

(⇐) Since Ker $\pi \subseteq pR[X]$, $c(\text{Ker } \pi) = L_{[\alpha]}$. Thus $L_{[\alpha]} \subseteq p$.

In Spec $A \to$ Spec R, the blowing-up locus is given by $V(L_{[\alpha]})$.

Remark 2. Assume that α is an anti-integral element of degree d over R. Then the following statements hold.

(1) $J_{[\alpha]} = L_{[\alpha]}$.

In fact, since Ker $\pi = I_{[\alpha]}\phi_\alpha(X)R[X]$, $\Delta_t = I_{[\alpha]}\phi_\alpha(X) \cdot$ (polynomials of degree t-d). Hence $L_t = I_{[\alpha]}$ and so $L_{[\alpha]} = I_{[\alpha]}c(\phi_\alpha(X)) = J_{[\alpha]}$.

(2) $A = R[\alpha]$ is R-flat if and only if $J_{[\alpha]} = R$ ([2, Proposition 2.6]).
In general, (2) is not necessarily true.

Example. Let $\overline{R}$ be the integral closure of R in K. Assume that $R \subsetneqq \overline{R}$. Take $\alpha \in \overline{R} - R$. Since α is integral over R, we have that $L_{[\alpha]} = R$. But suppose that $R[\alpha]$ is R-flat. Then $R[\alpha]$ is integral over R. Thus $R = R[\alpha]$, a contradiction. Hence $R[\alpha]$ is not R-flat. This example has property that $L_{[\alpha]} = R$ but $R[\alpha]$ is not R-flat.

Proposition 4. *Let $p \in$ Spec R. If $p \not\supset J_{[\alpha]}$, then A_p is R_p-flat.*

Proof. We may assume that (R, p) is a local ring. Then $J_{[\alpha]} = R$. So α is a super-primitive element of degree d over R and so α is an anti-integral element of degree d over R ([2, Theorem 1.12]). From [2, Proposition 2.6], $R[\alpha]$ is R-flat.

Is the converse of this statement true ?

Theorem 5. *If $R[\alpha]$ is R-flat and integral over R, then $I_{[\alpha]}= R$. Hence $R[\alpha]$ is a free R-module of rank d.*

Proof. Since α is an integral element over R, we have that $R[\alpha]$ is a finite R-module. Since $R[\alpha]$ is a flat R-module, $R[\alpha]$ is a locally free R-module of rank d. Thus $R_p[\alpha]$ is a R_p-free module of rank d for $p \in \mathrm{Spec}\ R$. We may assume that (R, m) is a local ring. Put $A = R[\alpha]$ and $A/mA = (R/m)[\bar{\alpha}]$. Since A is a free R-module of rank d, A/mA is a d-dimensional vector space over R/m. Hence $\bar{\alpha}$ has a monic relation over R/m with degree d. Therefore

$$A = (R+R\alpha+\cdots+R\alpha^{d-1})+mA.$$

By Nakayama's lemma, $A = R+R\alpha+\cdots+R\alpha^{d-1}$. This implies that $\alpha^d= a_0+a_1\alpha+\cdots+a_{d-1}\alpha^{d-1}$ for some $a_i \in R$ $(0\leqq i\leqq d-1)$. Therefore $I_{[\alpha]}= R$. And so $J_{[\alpha]}= R$.

Remark 3. If $R[\alpha]$ is integral over R, the non-flat locus is given by $V(I_{[\alpha]})$.

On a general extension $A = R[\alpha]$ of R, the following assertions hold. Assume that $L_t= L_{t+1}=\cdots= L_n=\cdots$.

(1) The non-integral locus of A over R is given by $V(L_t)$.

(2) For Spec $A \to$ Spec R, the blowing-up locus is given by $V(L_{[\alpha]})$.

(3) Suppose that A is integral over R. If $L_t= R$, the non-flat locus A over R is given by $V(I_{[\alpha]})$. If A is not integral over R, the non-flat locus does not know, but in case $p \not\supset J_{[\alpha]}$ we have that A_p/R_p is flat.

References

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