The Bulletin of Aichi Univ. of Education, 43 (Natural Science), pp. 7 ~17, February, 1994

Hitting Point Distribution of Two-Dimensional Random Walk

Masashi SUZUKI

(Department of Mathematics)

(Received September 13, 1993)

ABSTRACT

It is known that each path of the 2-dimensional (standard) random walk starting at arbitrary point on the plane passes the x-axis in the long run with probability 1. The purpose of this article is to calculate the probability distribution of the point where a particle of random walk for the first time hits the x-axis. The distribution is formulated in terms of the starting point of the particle. It is also applied to the solution of the discrete Dirichlet problem for the half plane. Since the random walk can be considered as a discretization of the Brownian motion, the above distribution should be closely related to the Cauchy distribution, which is well known as the hitting point distribution of the 2-dimensional Brownian motion to the x-axis. This relation is also mentioned later. In the last part, the 2-dimensional biased random walk which moves at each step in four directions with different probabilities is considered.

§ 1. Dirichlet problem and probabilistic approach

Let Ω be an open, non-empty subset of $\mathbf{R}^d$. A twice continuously differentiable function u defined on Ω is called to be *harmonic* on Ω if

$$\Delta u = 0, \tag{1.1}$$

where the Laplacian $\Delta = D_1{}^2 + \cdots + D_d{}^2$ and $D_j{}^2$ denotes the second partial derivative with respect to the j-th coordinate variable.

Among the numerous problems raised in connection with the harmonic functions, one of the most famous and important ones is the *Dirichlet problem*. Now let us quickly review the problem.

Dirichlet problem

Let Ω be a connected open subset of $\mathbf{R}^d$ with smooth boundary, and f be a continuous function on the boundary $\partial\Omega$. The Dirichlet problem for the domain Ω consists of the following two questions.

1) Does there exist a continuous function u on the closure $\overline{\Omega}$, which is harmonic on Ω and coincides with f on $\partial\Omega$?

2) If so, how can we find it? (By the maximum principle, if a solution exists, then it is unique.)

Although the method of expressing the solution of the Dirichlet problem by the integration of the given function f together with the Poisson kernel for the given domain is well known, it is also widely recognized that probability theory has quite useful tools to solve the problem. Let $\{X(t),\ t\geqq 0\}$ be a stochastic process usually called the *d-dimensional* (*standard*) *Wiener process* or *Brownian motion* on $\mathbf{R}^d$. The precise definition of this process can be found in any textbooks of stochastic processes and is omitted here. But instead, let us just point out that the particle of the Brownian motion moves in arbitrary direction 'at every moment', independent of its current position and the trace of its travel so far. Let τ be the time when the process reaches the boundary $\partial\Omega$ for the first time. It is easily shown that each path $X(t)$ of the process surely hits the boundary in the long run, that is, $\tau<\infty$ with probability 1.

Now, all the materials necessary to express the solution of the Dirichlet problem in probabilistic way are ready. Define a function u on Ω by

$$u(x)=E_x[\,f(X(\tau))\,],\ x\in\overline{\Omega}, \tag{1.2}$$

where E_x means the expected value over the paths starting at $x\in\mathbf{R}^d$. Then $u(x)$ is the solution we want to find. By this simple expression, the solution of the Dirichlet problem is given for arbitrary domain even with looser conditions than it has smooth boundary. The advantage of this kind of approach is not only that the expression is simple and common to different domains given but also that it is quite intuitive and easy to understand. However, it does not yet give a formula for the value of the solution $u(x)$ in terms of x itself. To calculate the exact value of $u(x)$, we need the probability distribution of the point were the path $X(t)$ hits the boundary. Let $p(x, dy)=P(X(\tau)\in dy)$ be the probability distibution mentiond above. Then the calculation of (1 . 2) is executed as follows.

$$\begin{aligned} u(x) &= E_x[\,f(X(\tau))\,] \\ &= \int f(y)\,P(X(\tau)\in dy) \\ &= \int f(y)\,p(x,\ dy). \end{aligned} \tag{1.3}$$

Thus, to find the solution of the Dirichlet problem is the same as to find the hitting point distribution $p(x, dy)$ where $x \in \overline{\Omega}$ and $y \in \partial\Omega$ for given domain.

§ 2. Discrete Dirichlet problem

The aim of this article is to show the explicit calculation of the probabilistic approach, namely, to find the hitting point distribution in an explicit way. But many results have been obtained so far for the ordinary Dirichlet problem described in the last section and a lot of textbooks have already been issued both in the analytic approach and in the probabilistic approach. So, we consider here another kind of Dirichlet problem in discrete space, that is, lattice space.

DEFINITION 2. 1 For each $x \in \mathbf{Z}^d$, the *neighborhood* of x is defined by

$$N(x) = \{y \in \mathbf{Z}^d;\ |x-y| = 1\}$$
$$= \{x \pm e_i\ ;\ i = 1, 2, \cdots, d\},$$

where $|\cdot|$ is the Euclidean norm and e_i is the i-th unit vector. Each point of the neighborhood $N(x)$ is called a *neighbor*. So, each point x of $\mathbf{Z}^d$ has $2d$ neighbors.

DEFINITION 2. 2 A subset S of $\mathbf{Z}^d$ is *connected* if one who starts at any point $x \in S$ can reach any other poing $y \in S$ by passing only the points of S and moving at each step to one of the neighbors of the point where he is.

DEFINITION 2. 3 For a subset S of $\mathbf{Z}^d$,

the interior $S^\circ = \{x \in S;\ N(x) \subset S\}$ and

the boundary $\partial S = \{x \in S;\ N(x) \cap S^c \neq \phi\}$.

DEFINITION 2. 4 Let Ω be a subset of $\mathbf{Z}^d$ such that its interior Ω° is connected. A function u defined on Ω is *harmonic* on Ω if

$$Au(x) = 0 \quad \text{for any } x \in \Omega^\circ, \tag{2.1}$$

where the operator A is defined by

$$Au(x) = \sum_{i=1}^{d} \{u(x+e_i) + u(x-e_i)\} - 2d \cdot u(x). \tag{2.2}$$

The operator A in the above definition is the discrete analogy of the Laplacian in $\mathbf{R}^d$ and is called here *discrete Laplacian*, since

$$u(x+e_i) + u(x-e_i) - 2u(x) = \{u(x+e_i) - u(x)\} - \{u(x) - u(x-e_i)\}$$

is like D_i^2 in $\mathbf{R}^d$.

The term 'harmonic' is shown to be quite reasonable one in this discrete case because if u is harmonic, then (2.1) yields the equality

$$u(x) = \frac{1}{2d} \sum_{i=1}^{d} \{u(x+e_i) + u(x-e_i)\}\ , \tag{2.3}$$

which means that the value of function u at x is the same as the mean value of u over all the neighbors of x.

Now we can formulate the following *discrete Dirichlet problem*.

Discrete Dirichlet problem

Let Ω be a subset of $\mathbf{Z}^d$ with connected interior Ω° and f be a function defined on the boundary $\partial\Omega$. The discrete Dirichlet problem for Ω consists of the following two questions.

1) Does there exist a function u on Ω, which is harmonic on Ω° and coincides with f on $\partial\Omega$?
2) If so, how can we find it?

Just lile the previous section, the solution to the above problem can be expressed by probabilistic method. To do so, we need to prepare a process playing the same role as the Brownian motion did in $\mathbf{R}^d$.

DEFINITION 2. 5 A stochastic process $\{X(n),\ n=0,1,2,\cdots\}$ on $\mathbf{Z}^d$ is *d-dimensional* (*standard*) *random walk* if the next position $X(n+1)$ is determined depending only on the current position $X(n)$ $(n=0,1,2,\cdots)$ by the conditional probabilities

(2.4) $$P(X(n+1)=x\pm e_i \mid X(n)=x)=\frac{1}{2d},\ n=0,1,2,\cdots;\ \ i=1,2,\cdots,\ d.$$

The condition (2.4) means that a random walk moves equally likely to each of neighbors at every moment. Now, using this process, the solution of the discrete Dirichlet problem is expressed by

(2.5) $$u(x)=E_x[f(X(\tau))],\ x\in\Omega,$$

where τ is the hitting time to the boundary $\partial\Omega$ and $E_x[\ \cdot\ ]$ means the expected value over the paths starting at $x\in\mathbf{Z}^d$. Note that the above equality (2.5) is exactly the same as the (1.2) for the (continuous) Dirichlet problem. The explicit calculation of u is done as follows.

(2.6) $$\begin{aligned} u(x) &= E_x[f(X(\tau))] \\ &= \sum_{y\in\partial\Omega} f(y)P(X(\tau)=y) \\ &= \sum_{y\in\partial\Omega} f(y)p(x,y), \end{aligned}$$

where $p(x,\ \cdot\ )$ is the probability distribution of the hitting points to the boundary when the random walk starts at $x\in\Omega$. So, as in the case of §1, we need to find the hitting point distribution $p(x,y)$ for Ω.

§ 3. Dirichlet problem for the half plane

In this section, we restrict ourselves to case $d=2$ and Ω is the upper half plane.

The continuous case

In this case, Ω is the upper half plane, that is,

$$H=\{x=(x_1, x_2); x_2>0\}.$$

The Dirichlet problem for H has the form

$$(3.1)\qquad \begin{cases} \Delta u=0 & \textit{in } H \text{ and} \\ u=f & \text{on } \partial H = x_1\text{-axis}, \end{cases}$$

where f is a given continuous function on the x_1-axis. The solution of the Dirichlet problem of this case is well known as next proposition.

PROPOSITION 3.1 *The solution of the (continuous) Dirichlet problem for the upper half plane H is given by the formula.*

$$(3.2)\qquad u(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_2}{(x_1-t)^2+x_2^2}f(t)\,dt,$$

where $x=(x_1, x_2)$ with $x_2>0$.

The analytic proof of this proposition is found in any textbook of the harmonic function theory ([1] e. g.). To see probabilistic approach, we have only to note that the hitting probability distribution of the 2-dimensional Brownian motion to the x_1 -axis is expressed in terms of its starting point by

$$(3.3)\qquad p(x, dy)=\frac{x_2}{\pi\{(x_1-y)^2+x_2^2\}}\,dy,$$

where $x=(x_1, x_2)$ with $x_2>0$ is the starting point of the 2-dimensinal Brownian motion and dy is the Lebesgue measure on the x_1*-axis.* (3.3) is well known as *Cauchy distribution.* (3.3) combined with the formula (1.3) shows that PROPOSITION 3.1 holds.

The discrete case

In this case, Ω is the upper half lattice, that is,

$$K=\{x=(x_1, x_2); x_1, x_2\in\mathbf{Z} \text{ and } x_2\geqq 0\}.$$

The discrete Dirichlet problem for K has the form

$$(3.4)\qquad \begin{cases} Au=0 & \text{in } K^\circ \text{ and} \\ u=f & \text{on } \partial K=\{x=(x_1, 0); x_1\in\mathbf{Z}\} \end{cases}$$

where A is the discrete Laplacian defined by (2.2) with $d=2$ and f is a given function on ∂K. The solution of this problem in explicitly expressed as in (2.6) and has the form

$$(3.5)\qquad u(x)=\sum_{k=-\infty}^{\infty}f(k)p(x, k), \ x\in K,$$

where f is expressed as a function on $\mathbf{Z}$ and $p(x, k)=P(X(\tau)=(k, 0))$ means the probability that 2-dimensional random walk starting at the point x first hits the boundary ∂K at $(k,\ 0)$.

Therefore, what we have to do to find the explicit formula of the solution of (3.4) is to calculate the probability $p(x,\ k)$ for each $x \in K^{\circ}$ and $k \in \mathbf{Z}$. But this calculation is quite complicated and unknown so far.

§ 4. Hitting probability to x_1-axis

Now we have come to the main part of this article. In this section, the answer to the question in the last section will be given.

First, note that if the starting point is displaced horizontally by a distance h, then the hitting point distribution is also displaced horizontally by the same distance, since probability concerning random walk is invariant in parallel displacent, This fact is expressed by the equality

$$(4.1) \qquad p((x_1,\ x_2),\ k) = p((x_1+h,\ x_2),\ k+h).$$

Taking $h = -x_1$, we have another relation

$$(4.2) \qquad p((x_1,\ x_2),\ k) = p((0,\ x_2),\ k-x_1).$$

And so, we have only to calculate the hitting point distribution in the case that the starting point is on the x_2-axis.

Let $p_0(j,\ k)$ denote the probability $p((0,\ j),\ k)$. Secondly, we shound note that $p_0(j,\ k)$ is symmetric in k, that is,

$$(4.3) \qquad p_0(j,\ k) = p_0(j,\ -k)$$

because of the symmetricity of random walk. Then the problem we have to solve is the following one.

Problem

Let $\{X(n),\ n=0,\ 1,\ 2,\ \cdots\}$ be 2-dimensional standard random walk and $x = (0,\ j)$ with $j > 0$ be its starting point. Calculate the probability $p_0(j,\ k) = P(X(\tau) = (k,\ 0))$ with $k > 0$, where τ is the first hitting time of the walk to the x_1-axis $\{(x_1,\ 0);\ x_1 \in \mathbf{Z}\}$.

Now let us start the calculation of $p_0(j,\ k)$ by conditioning the number of vertical moves L and the number of horizontal move M. Let G denote the x_1-coordinate of $X(\tau)$. Then

$$(4.4) \qquad \begin{aligned} p_0(j,\ k) &= P(G=k) \\ &= E[P(G=k \mid L,\ M)] \\ &= \sum_{l,m} P(G=k \mid L=l,\ M=m) P(L=l,\ M=m) \end{aligned}$$

$$=\sum_{l,m} P(G=k \mid L=l,\ M=m)P(M=m \mid L=l)\ \ P(L=l)$$
$$=\sum_{l,m} P(G=k \mid M=m)P(M=m \mid L=l)P(L=l),$$

since G is independent of the number of vertical moves. In the above calculation, the summation is taken over all the numbers m and l which make it possible for G to take the value k when starting at $(0, j)$. We now calculate $p_0(j, k)$ by dividing it into three parts as shown above.

The fact $G=k$ implies that the walk moves to the right more than to the left by k times until it hits the x_1-axis. So, we obtain the condition

(4.5) $$m=k+2\mu,\quad \mu=0,1,2,\cdots.$$

When $m=k+2\mu$, it means that the walk moves $k+\mu$ times to the right and μ times to the left. Since the walk moves equally likely to the right and to the left,

(4.6) $$P(G=k \mid M=m)=P(G=k \mid M=k+2\mu$$
$$=\binom{k+2\mu}{\mu}\left(\frac{1}{2}\right)^{k+2\mu},\ \mu=0,1,2,\cdots.$$

Using the same argument, for the number of vertical moves,

(4.7) $$l=j+2\lambda,\quad \lambda=0,1,2,\cdots.$$

This condition inplies that the walk moves $j+\lambda$ times downward and λ times upward. Since the walk moves either vertically or horizontally with the same probability and the last move should be vertical,

(4.8) $$P(M=m \mid L=l)=P(M=k+2\mu \mid L=j+2\lambda)$$
$$=\binom{j+k+2\lambda+2\mu-1}{k+2\mu}\cdot\left(\frac{1}{2}\right)^{j+k+2\lambda+2\mu}$$
$$\lambda=0,1,2,\cdots;\ \mu=0,1,2,\cdots.$$

For the third part $P(L=l)$, we can ignore the horizontal moves and treat the random walk as 1-dimensional random walk. Let $\{Y(n)\}$ be 1-dimensional random walk on $\mathbf{Z}$ starting at $j>0$ and σ be the first hitting time to 0. Then

(4.9) $$P(L=l)=P(\sigma=l).$$

$P(\sigma=l)$ is obtained by counting the number of paths such that

(4.10) $$Y(0)=j,\ Y(l-1)=1 \text{ and } Y(n)\neq 0 \text{ for } n=1,\cdots l-2.$$

By the reflection principle (see [2] e. g.), the number of paths such that

$$Y(0)=j,\ Y(l-1)=1 \text{ and } Y(n)=0 \text{ for some } n\in\{1,\cdots,l-2\}$$

is the same as the number of the paths such that

$$Y(0)=-j \text{ and } Y(l-1)=1\,.$$

This relation leads us to the fact that the number of paths satisfying (4.10) is obtained by

$$\#\{Y(0)=j,\ Y(l-1)=1\}-\#\{Y(0)=-j,\ Y(l-1)=1\},$$

where # means the number of the paths satisfying $\{\cdot\}$. Recalling that $l=j+2\lambda$, ($\lambda=0,1,\cdots$),

$$\#\{Y(0)=j,\ Y(l-1)=1\}=\#\{Y(0)=j,\ Y(j+2\lambda-1)=1\}$$

is the number of permutation of λ upward moves and $j+\lambda-1$ downward moves and

$$\#\{Y(0)=-j,\ Y(l-1)=1\}=\#\{Y(0)=-j,\ Y(j+2\lambda-1)=1\}$$

is the number of permutation of $j+\lambda$ upward moves and $\lambda-1$ downward moves. So, the number of paths satisfying (4.10) is

$$\begin{aligned}\binom{j+2\lambda-1}{\lambda}-\binom{j+2\lambda-1}{\lambda-1}&=\frac{(j+2\lambda-1)!}{\lambda!(j+\lambda-1)!}-\frac{(j+2\lambda-1)!}{(\lambda-1)!(j+\lambda)!}\\&=\frac{j(j+2\lambda-1)!}{\lambda!(j+\lambda)!}\\&=\frac{j}{j+\lambda}\binom{j+2\lambda-1}{\lambda}.\end{aligned}$$

Together with the relation (4.9), we have

$$\begin{aligned}P(L=l)&=P(L=j+2\lambda)\\&=\frac{j}{j+\lambda}\binom{j+2\lambda-1}{\lambda}\left(\frac{1}{2}\right)^{j+2\lambda}.\end{aligned}\tag{4.11}$$

Now we can summarize the above argument as a theorem.

THEOREM 4.1 *Let* $\{X(n),\ n=0,1,2,\cdots\}$ *be 2-dimensional random walk starting at* $(0,j)$, $j>0$. *Then the probability that it first hits the* x_1*-axis at* $(k,0)$ *with* $k>0$ *is expressed by the next formula.*

$$p_0(j,k)=\sum_{\lambda,\mu=0}^{\infty}\frac{j(j+k+2\lambda+2\mu-1)!}{\lambda!\mu!(j+\lambda)!(k+\mu)!}\left(\frac{1}{4}\right)^{j+k+2\lambda+2\mu}\tag{4.12}$$

Proof. Plugging (4.6), (4.8) and (4.11) into (4.4),

$$\begin{aligned}p_0(j,k)&=\sum_{l,m}P(G=k\mid M=m)P(M=m\mid L=l)P(L=l)\\&=\sum_{\lambda,\mu=0}^{\infty}\binom{k+2\mu}{\mu}\binom{j+k+2\lambda+2\mu-1}{k+2\mu}\binom{j+2\lambda-1}{\lambda}\left(\frac{j}{j+\lambda}\right)\left(\frac{1}{2}\right)^{2j+2k+2\lambda+2\mu}\\&=\sum_{\lambda,\mu=0}^{\infty}\frac{j(j+k+2\lambda+2\mu-1)!}{\lambda!\mu!(j+\lambda)!(k+\mu)!}\left(\frac{1}{4}\right)^{j+k+2\lambda+2\mu}\end{aligned}$$

Q. E. D.

Using the relations (3.5), (4.2), (4.3) and the formula (4.12), the solution to the discrete Dirichlet problem is expressed explicitly.

THEOREM 4. 2 *Let K be the upper half lattice, that is,*

$$K=\{x=(x_1, x_2); x_1, x_2 \in \mathbf{Z} \text{ and } x_2 \geqq 0\}.$$

Consider the discrete Dirichlet problem for K:

$$\begin{cases} Au=0 & \text{in } K^{\circ} \text{ and} \\ u=f & \text{on } \partial K=\{x=(x_1, 0); x_1 \in \mathbf{Z}\} \end{cases}$$

where A is the discrete Laplacian and f is a given function on ∂K. Then, the solution of this problem is expressed by

$$u(x)=\sum_{k=-\infty}^{\infty} f(k) \sum_{\lambda,\mu=0}^{\infty} \frac{X_2(x_2+|2\lambda+2\mu-1)!}{\lambda!\mu!(x_2+\lambda)!(|k-x_1|+\mu)!}\left(\frac{1}{4}\right)^{x_2+|k-x_1|+2\lambda+2\mu} \tag{4.13}$$

Proof. By (4.2) and (4.3), we have

$$\begin{aligned} p((x_1, x_2), k) &= p((0, x_2), k-x_1) \\ &= p_0(x_2, |k-x_1|). \end{aligned}$$

Then apply the above to the formula (3.5). Q. E. D.

§ 5. Approximation to the continuous case

It is well known that if the unit of the lattice gets smaller and smaller, the standard random walk approaches to the standard Brownian motion with some time change. Since the hitting point distribution of the 2-dimensional Brownian motion is the Cauchy distribution (3.3), it is highly likely that the probability distribution given in the last section by (4.12) approaches to the Cauchy distribution as the unit of the lattice tends to 0.

If the unit length of the lattice becomes the N-th of the original length, each coodinate of a point becomes N times as large measured by the smaller unit. And so, the probability $p_0(j, k)$ becomes $p_0(Nj, Nk)$ on the lattice of unit $1/N$. Here we just propose conjectures concerning the hitting probability and the Dirichlet problem.

CONJECTURE 1 *As N goes to infinity,*

$$p_0(Nj, Nk) \cdot N \to \frac{j}{\pi(k^2+j^2)}.$$

CONJECTURE 2 *Let f be a continuous function on x_1-axis. Then, the solution of Dirichlet problem for the upper half plane is approximated by the solution of discrete case as follows.*

$$\sum_{k=-\infty}^{\infty} f(k/N) p_0(Nx_2, N|k-x_1|) \to \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_2}{(x_1-t)^2+x_2^2} f(t)\, dt.$$

These conjectures are very likely and computer calculation shows they are true, but the formula (4.12) is too complicated to handle and these conjectures are not proved yet.

§ 6. Biased random walk

The d-dimensional standard random walk moves in any direction with the same probability $1/2d$. But by taking off this condition, we can define more gereral kind of random walk.

DEFINITION 6. 1 A discrete stochastic process $\{X(n),\ n=0,\ 1,\ 2,\ \cdots\}$ on $\mathbf{Z}^2$ is *2-dimensional biased random walk* if

$$\begin{aligned}
&P(X(n+1)=x+e_1 \mid X(n)=x)=p,\\
&P(X(n+1)=x-e_1 \mid X(n)=x)=q,\\
&P(X(n+1)=x+e_2 \mid X(n)=x)=r \text{ and}\\
&P(X(n+1)=x-e_2 \mid X(n)=x)=s,
\end{aligned}$$

with $p+q+r+s=1$.

In this case, the probability that the walk moves horizontally and vertically is $p+q$ and $r+s$ respectively. Let us calculate the hitting point distribution of this biased random walk to the x_1- axis.

Using the same terminology as in §4, $p_0(j,\ k)=P(X(\tau)=(k,\ 0))$ *with the starting point* $(0,\ j)$ is calculated as follows.

$$p_0(j,\ k)=P(G=k) = \sum_{l,m} P(G=k \mid M=m)P(M=m \mid L=l)P(L=l). \tag{6.1}$$

Modifying the formulas (4. 6), (4. 8) and (4.11) under the condition of the biased random walk,

$$\begin{aligned}
P(G=k \mid M=m) &= P(G=k \mid M=k+2\mu)\\
&= \binom{k+2\mu}{\mu}\left(\frac{p}{p+q}\right)^{k+\mu}\left(\frac{q}{p+q}\right)^{\mu},\ \mu=0,1,\cdots.
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
P(M=m \mid L=l) &= P(M=k+2\mu \mid L=j+2\lambda)\\
&= \binom{j+k+2\lambda+2\mu-1}{k+2\mu}(p+q)^{k+2\mu}(r+s)^{j+2\lambda},\\
&\qquad \lambda=0,1,2,\cdots;\ \mu=0,1,2,\cdots.
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
P(L=l) &= P(L=j+2\lambda)\\
&= \frac{j}{j+\lambda}\binom{j+2\lambda-1}{\lambda}\left(\frac{r}{r+s}\right)^{\lambda}\left(\frac{s}{r+s}\right)^{j+\lambda},\\
&\qquad \lambda=0,1,2,\cdots.
\end{aligned} \tag{6.4}$$

Plugging (6 . 2), (6 . 3) and (6 . 4), we obtain the next theorem.

THEOREM 6. 1 *Let $\{X(n),\ n=0,1,2,\cdots\}$ be 2-dimensional biased random walk starting at $(0,\ j)$, $j>0$. Then the probability that it first hits the x_1-axis at $(k,\ 0)$ with $k>0$ is expressed by the next formula.*

$$p_0(j,\ k)=\sum_{\lambda,\mu=0}^{\infty}\frac{j(j+k+2\lambda+2\mu-1)!}{\lambda!u!(j+\lambda)!(k+\mu)!}p^{k+\mu}q^{\mu}r^{\lambda}s^{j+\lambda} \tag{6.5}$$

REFERENCES

[1] S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Springer-Verlag New York, New York, 1992.

[2] W. Feller, *An Introduction to Probability Theory and its Applications*, Volume 1, Third edition, John Wiley & Sons, New York, 1968.

[3] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland, Amsterdam, 1989.