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Combinatorics on Tritones in the General Diatonic System

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1. Tritone: a key concept of tonality

It is said that tonality is a collection of schematic expectancies in musical experiences, based on European culture (e.g. [8]). However, the tonal system continuously developed over the years has quite good properties from a combinatorial point of view. Indeed, we have seen that the maximal evenness ansatz explains combinatorial aspects of the general diatonic system, or goes together with tonality at least ([3][4][5][6][7]). In this paper, we give a combinatorial characterization of tritones in the general diatonic system in terms of the rigidity of chords[7], and we find tritones are key objects. We find that our combinatorial definition of tritones (Definition 3.4) induces the Leittonwechsel (=leading tone exchange) property and the leading tone property (Proposition 4.1), the combinatorial definition of leading tones (Definition 4.2), the specificity of the usual 12TET-7 notes diatonic system and the existence of the double tritone (Subsection 4.2), the tritone substitution in the general diatonic system and a combinatorial reason for the perfect cadence (Subsection 4.3).

2. Combinatorial description of the general diatonic system

All notations and definitions are inherited from [7], so we omit details. For a tuple $A = (a_i)$, $|A|$ denotes the set $\{a_i\}$ consists of entries of A. If a tuple $B = (b_j)$ satisfies $|B| \subset |A|$ and that the inclusion $\iota : |B| \hookrightarrow |A|$ is increasing on indexes, that is, for $b_i, b_j \in B$ with $i < j$, $i' < j'$ holds for $a_{i'} = \iota(b_i)$ and $a_{j'} = \iota(b_j)$, we call B is *compatible* with A and write $B \sqsubset A$.

Definition 2.1 (J-function by Clough and Douthett[1]). For $c, d, m \in \mathbf{Z}$ with $c > d > 0$, the *J-function* on $\mathbf{Z}$ is defined as

$$J^m_{c,d}(k) = \left\lfloor \frac{ck+m}{d} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. We note $\mathcal{J}^m_{c,d}$ as the tuple $(J^m_{c,d}(k))_{k=0,\ldots,d-1}$ and $|\mathcal{J}^m_{c,d}|$ as the set $\{t \in \mathcal{J}^m_{c,d}\}$ of entries of $\mathcal{J}^m_{c,d}$.

Definition 2.2 (Chromatic scale and note). A *chromatic scale* Ch_c is a tuple $(f_0, f_1, \ldots, f_{c-1})$ of frequencies $f_0 < f_1 < \cdots < f_{c-1}$ satisfying $f_{c-1} < 2f_0$. A *semitone encoding* θ associated with the chromatic scale Ch_c is a map $|Ch_c| \ni f_k \mapsto k \in \mathbf{Z}$. An extension $\overline{Ch_c}$ of Ch_c is given by $\overline{Ch_c} = (2^n f_0, 2^n f_1, \ldots, 2^n f_{c-1})_{n\in\mathbf{Z}}$, and the semitone encoding θ is also extended to a bijection $\theta : |\overline{Ch_c}| \ni 2^n f_k \mapsto cn + k \in \mathbf{Z}$. We call an element of $\overline{Ch_c}$ a *note*.

Since tones with basic frequencies f and $2^n f, n \in \mathbf{Z}$ have a 'similar' quality for human ears, these tones are called *octave equivalent* to each other in music theory. This psychoacoustic fact is represented as $\theta(f) \equiv \theta(2^n f) \pmod c$. Therefore we often identify Ch_c with $\mathbf{Z}/c\mathbf{Z}$ and $\overline{Ch_c}$ with its covering space $\mathbf{Z}$. We also identify a tuple $X = (x_1, \ldots, x_n) \sqsubset Ch_c$ (or $\overline{Ch_c}$) with its image $\theta(X) = (\theta(x_1), \ldots, \theta(x_n))$, like $X = (\theta(x_1), \ldots, \theta(x_n))$ for short.

Definition 2.3 (Scale and diatonic scale). Given a chromatic scale Ch_c, a *scale* is a tuple $S = (t_0, t_1, \ldots, t_{d-1})$ compatible with Ch_c. Thus entries of S are arranged in ascending order $\theta(t_0) < \theta(t_1) < \cdots < \theta(t_{d-1})$ with $\theta(t_{d-1}) - \theta(t_0) < c$. We call S a *diatonic* scale whenever $\theta(S) \equiv \mathcal{J}^m_{c,d} \pmod c$ for some $m \in \mathbf{Z}$ (written as $S = \mathcal{J}^m_{c,d}$ for short). For a natural number h, $h \cdot S$ stands for an extension of S,

$$h \cdot S = (2^n t_0, \ldots, 2^n t_{d-1})_{n=0,\ldots,h-1}.$$

$\overline{S}$ denotes an infinite extension of S, $\overline{S} = (2^n t_0, \ldots, 2^n t_{d-1})_{n\in\mathbf{Z}}$. The *wholetone encoding* η_S of S is a map $|\overline{S}| \ni 2^n t_k \mapsto dn + k \in \mathbf{Z}$. For a scale S and a note $t \in S$, *upper note* $u_S(t)$ and *lower note* $l_S(t)$ of t are given as notes in S such that $\eta_S(u_S(t)) = \eta_S(t) + 1$ and $\eta_S(l_S(t)) = \eta_S(t) - 1$.

Whenever S is a diatonic scale $\mathcal{J}^m_{c,d}$, it is well known as Myhill's property that

$$\theta(u_S(t)) - \theta(t), \theta(t) - \theta(l_S(t)) \in \{s, s+1\}, \qquad s = \left\lfloor \frac{c}{d} \right\rfloor \tag{2.1}$$

holds for any $t \in S$ (see e.g. [4][5]). Thus the intervals of length s and $s+1$ correspond to the semitone and the whole tone respectively in usual music theory.

We see by definition that the relation between the semitone encoding θ and the whole tone encoding η is given by

$$\left\lfloor \frac{c\eta_{\mathcal{J}^m_{c,d}}(t) + m}{d} \right\rfloor = \theta(t) \tag{2.2}$$

for any $t \in \mathcal{J}^m_{c,d} \sqsubset Ch_c$.

Definition 2.4 (Chord and maximally even chord)**.** A *chord* X belongs to a scale S is a tuple of notes $X = (x_0, \ldots, x_{e-1})$ compatible with $h \cdot S$ for some h. We call $oct(X) = \min\{h \mid X \sqsubset h \cdot S\}$ the *octave range* of X. If $\theta(x_p) \not\equiv \theta(x_q) \pmod{c}$ holds for any entries x_p and x_q of X, we call X *prime*. We call X *maximally even* whenever $\eta_S(X) \equiv \mathcal{J}^n_{d,e} \pmod{d}$ for some $n \in \mathbf{Z}$. We call X *dyad* whenever $\#|X| = 2$.

Example 2.5. Usual western music is established on 12-tone chromatic scale,

$$Ch_{12} = (C, C^\#, D, D^\#, E, F, F^\#, G, G^\#, A, A^\#, B), \ A = 440\text{Hz}, A^\# = 466.16\text{Hz}, \ldots.$$

We adopt the semitone encoding $\theta : |Ch_{12}| \to \mathbf{Z}/12\mathbf{Z}$,

$$\theta(C) = 0, \theta(C^\#) = 1, \theta(D) = 2, \ldots, \theta(B) = 11.$$

Usual diatonic scales are given by $\mathcal{J}^m_{12,7}, m = 0, \ldots, 11$. For instance, C-major scale $\mathcal{C} = CDEFGAB$ is expressed as a tuple $(0, 2, 4, 5, 7, 9, 11) = (J^5_{12,7}(k))_{k=0,\ldots,6} = \mathcal{J}^5_{12,7}$. For C-major scale $\mathcal{J}^5_{12,7}$, we adopt a wholetone encoding η_C as

$$\eta_C(C) = 0, \eta_C(D) = 1, \eta_C(E) = 2, \eta_C(F) = 3, \eta_C(G) = 4, \eta_C(A) = 5, \eta_C(B) = 6.$$

For usual chromatic Ch_{12}, we put the set $\Delta_{12,7}$ of the extended diatonic scales, $\Delta_{12,7} = \{\overline{\mathcal{J}^m_{12,7}} \mid m = 0, \ldots, 11\}$.

3. Combinatorics on tritones

Definition 3.1 (Rigidity of chord)**.** Given a chromatic scale Ch_c and a set $\mathcal{S}$ of extended scales in Ch_c. For a chord X in some $\overline{S} \in \mathcal{S}$, putting $C_{\mathcal{S}}(X) = \{\overline{S} \in \mathcal{S} \mid X \sqsubset \overline{S}\}$, we define the *rigidity* $R_{\mathcal{S}}(X)$ of X as the number of scales with which X is compatible,

$$R_{\mathcal{S}}(X) = \#\{\overline{S} \in \mathcal{S} \mid X \sqsubset \overline{S}\} = \#C_{\mathcal{S}}(X).$$

Hereafter, for a chromatic scale Ch_c, we take $d \in \mathbf{N}$ prime to c and $d < c$. Then we fix a unique solution (c^-, d^-) of

$$cc^- + dd^- = 1$$

with $0 < d^- < c$ and $-d < c^- < 0$. We consider a set of diatonic scales $\Delta_{c,d} = \{\mathcal{J}^m_{c,d} \mid m = 0, \ldots, c-1\}$ compatible with a chromatic scale Ch_c. An interval $[a, a+d)$ with $0 \le a < c$ in $\mathbf{Z}/c\mathbf{Z}$ is understood cyclic way as

$$[a, a+d) = \begin{cases} \{a, a+1, \ldots, a+d-1\}, & \text{if } a+d \le c, \\ \{a, a+1, \ldots, c-1\} \cup \{0, 1, \ldots, a+d-c-1\}, & \text{if } a+d > c. \end{cases}$$

To observe the rigidity of chords, we firstly determine the rigidity of a note $t \in Ch_c$.

Proposition 3.2. *For any note $t \in Ch_c$, we have*

$$C_{\Delta_{c,d}}(t) = \{\mathcal{J}^m_{c,d} \mid m \in I(t) = [\alpha(t), \alpha(t)+d) \subset \mathbf{Z}/c\mathbf{Z}\}$$

with $\alpha(t) \equiv \theta(t)d \pmod c$ and $0 \leq \alpha(t) < c$. Hence the rigidity $R_{\Delta_{c,d}}(t)$ of any note t is d.

Proof. $t \in \mathcal{J}^m_{c,d}$ holds if and only if there exists integer $k \in [0, d)$ such that

$$\theta(t)d \leq ck + m < (\theta(t)+1)d.$$

Taking the unique solution $l \in [0, d)$ and $\alpha(t) \in [0, c)$ of $cl + \alpha(t) = \theta(t)d$, we have

$$\alpha(t) \leq c(k-l) + m < \alpha(t) + d.$$

If $\alpha(t) + d \leq c$, we see $k = l$ hence $\alpha(t) \leq m < \alpha(t) + d$. If $\alpha(t) + d > c$, as $m \in [0, c)$, we divide the inequality into two cases: $k = l$ hence $\alpha(t) \leq m < c$, and $k = l + 1$ hence $0 \leq m < \alpha(t) + d - c$.

Therefore the compatible scales $C_{\Delta_{c,d}}(t)$ of a note $t \in Ch_c$ corresponds to the interval $I(t) = [\alpha(t), \alpha(t) + d) \subset \mathbf{Z}/c\mathbf{Z}$. This fact is quite useful for analysis of rigidity, indeed, the following is a direct consequent.

Proposition 3.3. *For any chord $X = (t_1, \ldots, t_e) \sqsubset Ch_c$,*

$$C_{\Delta_{c,d}}(X) = \left\{ \mathcal{J}^m_{c,d} \;\middle|\; m \in \bigcap_{i=1}^{e} I(t) \subset \mathbf{Z}/c\mathbf{Z} \right\}.$$

Definition 3.4 (tritone)**.** A dyad $T_m \sqsubset \mathcal{J}^m_{c,d}$ is called a *tritone* whenever $T_m \not\sqsubset \mathcal{J}^{m\pm1}_{c,d}$.

In [7] , we have mentioned the tritone as the dyad of minimum rigidity, hence it characterizes the scale that belongs to. However, essential quality of a tritone is the unstability among the related keys of the scale to which the tritone belongs.

Theorem 3.5. *Each diatonic scale $\mathcal{J}^m_{c,d}$ contains a unique tritone $T_m = (t_1, t_2) \sqsubset \mathcal{J}^m_{c,d}$, which satisfies*

$$\theta(t_2) - \theta(t_1) = d^- - 1 \text{ or } c - d^- + 1, \tag{3.1}$$

in the semitone encoding and

$$\eta_{\mathcal{J}^m_{c,d}}(t_2) - \eta_{\mathcal{J}^m_{c,d}}(t_1) = -c^- \text{ or } d + c^- \tag{3.2}$$

in the whole tone encoding. The tritone has minimum rigidity among dyads in $\mathcal{J}^m_{c,d}$

$$R_{\Delta_{c,d}}(T_m) = \min\{R_{\Delta_{c,d}}(X) \mid X = (x_1, x_2) \sqsubset \mathcal{J}^m_{c,d}\} = \max\{2d - c, 1\}. \tag{3.3}$$

Proof. Since multiplying d induces a bijection $\mathbf{Z}/c\mathbf{Z} \ni x \mapsto xd \in \mathbf{Z}/c\mathbf{Z}$, we see

$$\{\alpha(t) \equiv \theta(t)d \pmod c \mid t \in Ch_c\} = \{0, 1, \ldots, c-1\}.$$

Then for each diatonic scale $\mathcal{J}^m_{c,d}$, there exists a unique note $t^-_m \in |\mathcal{J}^m_{c,d}| \setminus |\mathcal{J}^{m-1}_{c,d}|$. We also see

$$\{\alpha(t) + d - 1 \pmod c \mid t \in Ch_c\} = \{0, 1, \ldots, c-1\},$$

hence there exists a unique note $t^+_m \in |\mathcal{J}^m_{c,d}| \setminus |\mathcal{J}^{m+1}_{c,d}|$. Putting

$$T_m = \begin{cases} (t^-_m, t^+_m), & \text{if } \theta(t^-_m) < \theta(t^+_m), \\ (t^+_m, t^-_m), & \text{if } \theta(t^+_m) < \theta(t^-_m), \end{cases}$$

we see T_m is a unique tritone compatible with $\mathcal{J}^m_{c,d}$ by construction. Taking the whole tone encoding of $t^\pm_m$, $k^\pm_m = \eta_{\mathcal{J}^m_{c,d}}(t^\pm_m) \in [0, d)$, we have

$$\left\lfloor \frac{ck^\pm_m + m}{d} \right\rfloor = \theta(t^\pm_m)$$

by (2.2). By the definition of t_m^-, m is the left hand side of $I(t_m^-)$, $m = \alpha(t_m^-)$, and then

$$\theta(t_m^-) = \left\lfloor \frac{ck_m^- + m}{d} \right\rfloor = \frac{ck_m^- + m}{d},$$

hence

$$ck_m^- + m = \theta(t_m^-)d. \tag{3.4}$$

We also see by the definition of t_m^+, m is the right hand side of $I(t_m^+)$, that is,

$$m = \begin{cases} \alpha(t_m^+) + d - 1, & \text{whenever } \alpha(t_m^+) + d - 1 < c, \\ \alpha(t_m^+) + d - 1 - c, & \text{whenever } \alpha(t_m^+) + d - 1 \geq c, \end{cases}$$

then

$$\theta(t_m^+) + 1 = \left\lfloor \frac{ck_m^+ + m + 1}{d} \right\rfloor = \frac{ck_m^+ + m + 1}{d},$$

hence

$$ck_m^+ + m + 1 = (\theta(t_m^+) + 1)d. \tag{3.5}$$

Consequently we have

$$(\theta(t_m^+) - \theta(t_m^-))d \equiv 1 - d \pmod{c}, \text{ i.e. } \theta(t_m^+) - \theta(t_m^-) \equiv d^- - 1 \pmod{c}, \tag{3.6}$$

and as $\theta(t_m^\pm) \in [0, c)$, we have (3.1). It also comes from (3.4) and (3.5) that

$$k_m^- \equiv -c^- m \pmod{d} \text{ and } k_m^+ \equiv -c^-(m+1) \pmod{d},$$

hence

$$k_m^+ - k_m^- \equiv -c^- \pmod{d}.$$

As $k_m^\pm \in [0, d)$, we have (3.2).

For any dyad $X = (x_1, x_2) \sqsubset \mathcal{J}_{c,d}^m$, its rigidity is given by

$$R_{\Delta_{c,d}}(X) = \#\left([\alpha(x_1), \alpha(x_1) + d) \cap [\alpha(x_2), \alpha(x_2) + d)\right),$$

where these intervals are taken as subsets of $\mathbf{Z}/c\mathbf{Z}$. Without loss of generality, we can assume $0 \leq \alpha(x_1) < \alpha(x_2) < c$. Thus we just consider

$$R_{\Delta_{c,d}}(X) = \#\left([0, d) \cap [\alpha, \alpha + d)\right),$$

where $\alpha = \alpha(x_2) - \alpha(x_1)$. We note $R_{\Delta_{c,d}} \geq 1$ since $X \sqsubset \mathcal{J}_{c,d}^m$, thus at least we have

$$0 < \alpha < d \text{ or } 0 \leq \alpha + d - 1 - c < d.$$

When $0 < \alpha < d$ only occurs, we see $\alpha + d \leq c$ and

$$R_{\Delta_{c,d}}(X) = \#\left([0, d) \cap [\alpha, \alpha + d)\right) = \#\{\alpha, \alpha + 1, \ldots, d - 1\} = d - \alpha \geq 2d - c.$$

When $0 \leq \alpha + d - 1 - c < d$ only occurs, we see $\alpha \geq d$ and

$$R_{\Delta_{c,d}}(X) = \#\left([0, d) \cap [\alpha, \alpha + d)\right) = \#\{0, 1, \ldots, \alpha + d - 1 - c\} = \alpha + d - c \geq 2d - c.$$

If both $0 < \alpha < d$ and $0 \leq \alpha + d - 1 - c < d$ occur, we need $0 \leq \alpha + d - 1 - c < 2d - 1 - c$, that is, $2d - c \geq 2$. In this case, we see

$$\begin{aligned} R_{\Delta_{c,d}}(X) &= \#\left([0, d) \cap [\alpha, \alpha + d)\right) \\ &= \#\left(\{\alpha, \alpha + 1, \ldots, d - 1\} \cup \{0, 1, \ldots, \alpha + d - 1 - c\}\right) = 2d - c. \end{aligned}$$

Therefore we have

$$\min\{R_{\Delta_{c,d}}(X) \mid X = (x_1, x_2) \sqsubset \mathcal{J}_{c,d}^m\} = \max\{2d - c, 1\}.$$

Again consider the tritone $T_m \sqsubset \mathcal{J}^m_{c,d}$. By (3.6) we have $\alpha(t^-_m) - \alpha(t^+_m) \equiv d-1 \pmod{c}$, so we may assume $\alpha(t^-_m) = d-1$ and $\alpha(t^+_m) = 0$. As $0 < \alpha(t^-_m) = d-1 < d$, if $\alpha(t^-_m) + d = 2d-1 \leq c$, we have

$$R_{\Delta_{c,d}}(T_m) = \#\left([0,d) \cap [d-1, 2d-1)\right) = \#\{d-1\} = 1 \geq 2d-c.$$

If $0 \leq \alpha(t^-_m) + d - 1 - c = 2d-2-c < d$, i.e., $2d-c \geq 2$, we have

$$\begin{aligned} R_{\Delta_{c,d}}(T_m) &= \#\left([0,d) \cap ([d-1,c) \cup [0, 2d-1-c))\right) \\ &= \#\{0,1,\ldots,2d-2-c, d-1\} = 2d-c > 1. \end{aligned}$$

Hence $R_{\Delta_{c,d}}(T_m) = \max\{2d-c, 1\}$.

Tables 1, 2 and 3 show the relation between diatonic scales and tritones. When $c = 13$ and $d = 7$ (Table 1), the diatonic scale $\mathcal{J}^3_{13,7}$ contains a unique tritone $T_3 = (6,7)$ with rigidity $R_{\Delta_{13,7}}(T_3) = \max\{2d-c,1\} = 1$, in other terms, no other diatonic scale contains T_3. Thus T_3 uniquely determines the scale $\mathcal{J}^3_{13,7}$.

When $c = 11$ and $d = 7$ (Table 3), the diatonic scale $\mathcal{J}^3_{11,7}$ contains a unique tritone $T_3 = (2,5)$ with rigidity $R_{\Delta_{11,7}}(T_3) = \max\{2d-c,1\} = 3$, so T_3 is not rare: we see $T_3 \sqsubset \mathcal{J}^8_{11,7}, \mathcal{J}^9_{11,7}$. Even so, no diatonic scale contains T_3 as a tritone except $\mathcal{J}^3_{11,7}$. $\mathcal{J}^8_{11,7}$ and $\mathcal{J}^9_{11,7}$ contain unique tritones $T_8 = (5,9)$ and $T_9 = (2,6)$ respectively.

Finally, when $c = 12$ and $d = 7$ (Table 2), the diatonic scale $\mathcal{J}^3_{12,7}(= B^\flat\text{-major})$ contains a unique tritone $T_3 = (3,9) = E^\flat A$ with rigidity $R_{\Delta_{12,7}}(T_3) = \max\{2d-c,1\} = 2$, so $T_3 = E^\flat A$ is also contained in $\mathcal{J}^9_{12,7}(= E\text{-major})$. Moreover it is noticeable that T_3 is also the tritone in $\mathcal{J}^9_{12,7}$, that is, $|T_9| = |T_3|$. Therefore the diatonic system with $2d - c = 2$ has a combinatorial special feature.

4. Tritone, leading tone and cadence: a combinatorial perspective

4.1. **Tritone and leading tone.** We have seen that a tritone T_m consists of two notes $t^-_m \in |\mathcal{J}^m_{c,d}| \setminus |\mathcal{J}^{m-1}_{c,d}|$ and $t^+_m \in |\mathcal{J}^m_{c,d}| \setminus |\mathcal{J}^{m+1}_{c,d}|$. By construction, we see there exist $0 \leq k^-, k^+ < d$ such that

$$\theta(t^-_m)d = ck^- + m \text{ and } (\theta(t^+_m)+1)d = ck^+ + m + 1,$$

hence

$$J^{m-1}_{c,d}(k^-) = \left\lfloor \frac{ck^- + m - 1}{d} \right\rfloor = \theta(t^-_m) - 1 \text{ and } J^{m+1}_{c,d}(k^+) = \left\lfloor \frac{ck^+ + m + 1}{d} \right\rfloor = \theta(t^+_m) + 1, \tag{4.1}$$

meaning that

$$\theta(t^+_{m-1}) = \theta(t^-_m) - 1 \text{ and } \theta(t^-_{m+1}) = \theta(t^+_m) + 1. \tag{4.2}$$

That is, changing $t^-_m \to t^+_{m-1}$ and $t^+_m \to t^-_{m+1}$ cause the modulation (=change of keys) $\mathcal{J}^m_{c,d} \to \mathcal{J}^{m-1}_{c,d}$ and $\mathcal{J}^m_{c,d} \to \mathcal{J}^{m+1}_{c,d}$ respectively. (4.2) corresponds to one of transformations in Neo-Riemannian theory, called *Leittonwechsel* (=leading tone exchange). Therefore we can expect that $t^\pm_m$ act as leading tones in our combinatorial setting. Indeed, by (2.1) and (4.1), we have ($u_m = u_{\mathcal{J}^m_{c,d}}, l_m = l_{\mathcal{J}^m_{c,d}}$ for short)

$$\begin{aligned} \theta(u_m(t^-_m)) - \theta(t^-_m) &= J^m_{c,d}(k^-+1) - J^m_{c,d}(k^-) = \left\lfloor \frac{c}{d} \right\rfloor \\ &< \left\lfloor \frac{c}{d} \right\rfloor + 1 = J^{m-1}_{c,d}(k^-+1) - J^{m-1}_{c,d}(k^-) = \theta(u_{m-1}(t^-_{m-1})) - \theta(t^-_{m-1}) \end{aligned}$$

as $J^m_{c,d}(k^-+1) = J^{m-1}_{c,d}(k^-+1)$, and

$$\begin{aligned} \theta(t^-_{m-1}) - \theta(l_{m-1}(t^-_{m-1})) &= J^{m-1}_{c,d}(k^-) - J^{m-1}_{c,d}(k^- - 1) = \left\lfloor \frac{c}{d} \right\rfloor \\ &< \left\lfloor \frac{c}{d} \right\rfloor + 1 = J^m_{c,d}(k^-) - J^m_{c,d}(k^- - 1) = \theta(t^-_m) - \theta(l_m(t^-_m)) \end{aligned}$$

as $J^m_{c,d}(k^- - 1) = J^{m-1}_{c,d}(k^- - 1)$. Thus we have

$$\theta(u_m(t^-_m)) - \theta(t^-_m) < \theta(t^-_m) - \theta(l_m(t^-_m)).$$

Considering the inequality, we can say that t_m^- leads to $u(t_m^-)$ according to usual music theory. Similarly, it comes from

$$\begin{aligned}\theta(t_m^+) - \theta(l_m(t_m^+)) &= J_{c,d}^m(k^+) - J_{c,d}^m(k^+ - 1) = \left\lfloor \frac{c}{d} \right\rfloor \\ &< \left\lfloor \frac{c}{d} \right\rfloor + 1 = J_{c,d}^{m+1}(k^+) - J_{c,d}^{m+1}(k^+ - 1) = \theta(t_{m+1}^+) - \theta(l_{m+1}(t_{m+1}^+))\end{aligned}$$

and

$$\begin{aligned}\theta(u_{m+1}(t_{m+1}^+)) - \theta(t_{m+1}^+) &= J_{c,d}^{m+1}(k^+ + 1) - J_{c,d}^{m+1}(k^+) = \left\lfloor \frac{c}{d} \right\rfloor \\ &< \left\lfloor \frac{c}{d} \right\rfloor + 1 = J_{c,d}^m(k^+ + 1) - J_{c,d}^m(k^+) = \theta(u_m(t_m^+)) - \theta(t_m^+)\end{aligned}$$

that

$$\theta(t_m^+) - \theta(l_m(t_m^+)) < \theta(u_m(t_m^+)) - \theta(t_m^+).$$

Hence t_m^+ leads to $l(t_m^+)$. We have shown the following.

Proposition 4.1. *The tritone $T_m \sqsubset \mathcal{J}_{c,d}^m$ consists of two notes $|T_m| = \{t_m^-, t_m^+\}$ such that*

(1) (Leittonwechsel property)

$$|\mathcal{J}_{c,d}^m| \triangle |\mathcal{J}_{c,d}^{m-1}| = \{t_{m-1}^+, t_m^-\}, \text{ and } |\mathcal{J}_{c,d}^m| \triangle |\mathcal{J}_{c,d}^{m+1}| = \{t_{m+1}^-, t_m^+\},$$

where $A \triangle B$ stands for XOR of A and B. We also have adjacent relations

$$\theta(t_{m-1}^+) = \theta(t_m^-) - 1 \quad \text{and} \quad \theta(t_{m+1}^-) = \theta(t_m^+) + 1.$$

(2) (Tone leading property)

$$\theta(u_m(t_m^-)) - \theta(t_m^-) < \theta(t_m^-) - \theta(l_m(t_m^-)) \quad \text{and} \quad \theta(t_m^+) - \theta(l_m(t_m^+)) < \theta(u_m(t_m^+)) - \theta(t_m^+),$$

where we put $u_m = u_{\mathcal{J}_{c,d}^m}, l_m = l_{\mathcal{J}_{c,d}^m}$.

Of course, Psychoacoustic effects have brought the concept of the leading tone in usual music theory. When a people is hearing a melody in C-major scale, for instance, in European classical music theory it is said that the progressions from B to C and F to E bring a feeling of resolution, however, from C to B or E to F does not. Regarding this asymmetry and Proposition 4.1, we propose a purely combinatorial definition of leading tones.

Definition 4.2. A note t in a diatonic scale $\mathcal{J}_{c,d}^m$ is called a *leading tone* whenever t is an entry of the tritone $|T_m| = \{t_m^-, t_m^+\} \subset |\mathcal{J}_{c,d}^m|$, that is, $t \notin \mathcal{J}_{c,d}^{m\pm 1}$. When $t = t_m^-$, we say t leads to $u_m(t)$, or t is a *lower* leading tone to $u_m(t)$. When $t = t_m^+$, we say t leads to $l_m(t)$, or t is an *upper* leading tone to $l_m(t)$ [1)].

Thus Proposition 4.1 shows that t_m^- and t_m^+ lead to $u_m(t_m^-)$ and $l_m(t_m^+)$ respectively. Let us observe Table 1,2 and 3 again. The usual case $c = 12$ and $d = 7$ (Table 2), $A = 9$ (resp. $E^\flat = 3$) is the lower (resp. upper) leading tone to $B^\flat = 10$ (resp. $D = 2$) in $B^\flat$-major scale $\mathcal{J}_{12,7}^3$. In the case $c = 11$ and $d = 7$ (Table 3), we see 2 is the lower leading tone to 3, and 9 is the upper leading tone to 8 in the diatonic scale $\mathcal{J}_{11,7}^3$. We also find another adjacent semitone pair 5 and $6 \in \mathcal{J}_{11,7}^3$, however neither is the leading tone according to our definition, as they are contained in adjacent diatonic scales $\mathcal{J}_{11,7}^{3\pm 1}$. In the case $c = 13$ and $d = 7$, the leading tones are degenerate: we see 6 and 7, entries of the tritone T_3 in $\mathcal{J}_{13,7}^3$, are leading tones to each other.

[1)]The definition of an upper leading tone is unusual, because in usual music theory, a leading tone leads to the tonic, i.e. the key note of the considering scale.

4.2. **Double tritone and diminished chord.** We consider the case $2d - c = 2$, where the rigidity of a tritone is 2, that is, a tritone is contained in two different diatonic scales. As a result, a tritone in a diatonic scale also acts as a tritone in another scale, that is, any tritone has 'twofold meaning'. As is seen above, Table 2 shows the tritone $T_3 = E^\flat A$ in $B^\flat$-major is also the tritone in E-major. However, the direction of leading tones are exchanged. $E^\flat = 3$ is the upper leading tone to $D = 2$ and $A = 9$ is the lower leading tone to $B^\flat = 10$ in $B^\flat$-major while $D^\sharp = 3$ is the lower leading tone to $E = 4$ and $A = 9$ is the upper leading tone to $G^\sharp = 8$ in E-major.

FIGURE 1. Double tritone $E^\flat A = D^\sharp A$. Every tritone has the twofold meaning.

We also note the mathematical specificity of the case as follows.

Lemma 4.3. *Let d be prime to c, then $2d-c=2$ brings $d^2 \equiv 1 \pmod{c}$, that is, $d^- = d$, and $c^- = -\dfrac{d+1}{2}$.*

Proof. $c = 2(d-1)$ means c is even, thus d is odd. Then we can put $d = 2l+1$ and $c = 4l$ for some $l \in \mathbf{N}$, hence $d^2 = 4l(l+1)+1 \equiv 1 \pmod{c}$. We also see

$$c\left(-\frac{d+1}{2}\right) = -2l(2l+2) = -(d-1)(d+1) \equiv 1 \pmod{d}$$

and $-d < -\dfrac{d+1}{2} < 0$, hence the assertion.

In our setting $2d - c = 2$, the semitone interval s becomes $s = \lfloor c/d \rfloor = 1$, that is, the interval of two adjacent notes in Ch_c coincides with the semitone, as usual musical theory.

Theorem 4.4. *Consider the case $2d-c=2$. Then the tritone in $\mathcal{J}^m_{c,d}$ coincides with the tritone in $\mathcal{J}^{m+d-1}_{c,d}$: $|T_m| = |T_{m+d-1}|$. Their entries satisfy*

$$t^+_m = t^-_{m+d-1} \quad \text{and} \quad t^-_m = t^+_{m+d-1} \quad \text{up to ocatve equivalence.}$$

The tritone T_m divides the chromatic scale Ch_c into two equal parts. Moreover, as a dyad, T_m is a maximally even chord in $\mathcal{J}^m_{c,d}$.

Proof. Noticing $d - 1 \equiv 1 - d \pmod{c}$, we see by (3.4) and (3.5),

$$\theta(t^-_{m+d-1})d \equiv m + d - 1 \equiv m + 1 - d \equiv \theta(t^+_m)d \pmod{c}$$

and

$$\theta(t^+_{m+d-1})d \equiv m + d - 1 + (1-d) = m \equiv \theta(t^-_m)d \pmod{c},$$

equivalently $\theta(t^-_{m+d-1}) \equiv \theta(t^+_m)$ and $\theta(t^+_{m+d-1}) \equiv \theta(t^-_m) \pmod{c}$, hence the assertion.

Suppose $\theta_m(t^-_m) < \theta_m(t^+_m)$ and thus $\eta_m(t^-_m) < \eta_m(t^+_m)$. By Theorem 3.5 and Lemma 4.3,

$$\theta(t^+_m) - \theta(t^-_m) = d^- - 1 = d - 1 = \frac{c}{2}.$$

We also have

$$\eta_m(t^+_m) - \eta_m(t^-_m) = -c^- = \frac{d+1}{2}, \quad \text{hence} \quad (\eta_m(t^-_m) + d) - \eta_m(t^+_m) = \frac{d-1}{2},$$

showing that T_m satisfies the Myhill's property (2.1) for $s = \lfloor c/d \rfloor = 1$. Thus T_m is a maximally even chord in $\mathcal{J}^m_{c,d}$.

It is well known that a diminished chord, like $BDFA^\flat$, consists of two different tritones, like BF and $DA^\flat$, and divides the chromatic scale Ch_{12} into four equal parts, $BD, DF, FA^\flat, A^\flat B$. These are true in our general situation.

Corollary 4.5 (diminished chord)**.** *Let* $X = (t_0, t_1, t_2, t_3) \sqsubset Ch_c$ *be a chord with* $\theta(t_k) \equiv \dfrac{c}{4}k + n \pmod{c}$ *for some integer* $0 \le n < \dfrac{c}{4}$*. Then* (t_0, t_2) (resp. (t_1, t_3)) *is a tritone compatible with* $\mathcal{J}^m_{c,d}$ *and* $\mathcal{J}^{m+d-1}_{c,d}$ (resp. $\mathcal{J}^{m'}_{c,d}$ and $\mathcal{J}^{m'+d-1}_{c,d}$)*, where* $m \equiv nd \pmod{c}$ *and* $m' \equiv \left(\dfrac{c}{4} + n\right) d \pmod{c}$*.*

Proof. As is seen in the proof of Lemma 4.3, we can put $c = 4l$ and $d = 2l + 1$ for some $l \in \mathbf{Z}$, and then $\theta(t_2) - \theta(t_0) = 2l = d - 1 = d^- - 1$ By (3.1),(3.4) and Theorem 4.4, we see that (t_0, t_2) is a tritone compatible with $\mathcal{J}^m_{c,d}$ and $\mathcal{J}^{m+d-1}_{c,d}$, where $m \equiv \theta(t_0)d \pmod{c}$. The proof for (t_1, t_3) is the same.

4.3. **Tritone substitution and perfect cadence: a combinatorial reason.** In usual music theory, leading tones are supposed to bring progressions of chords. In C-major scale for instance, since B leads to C and F leads to E, so the perfect cadence such as $G7 = (GBDF) \to C = (CEG)$ gives a feeling of resolution. In terms of the functional harmony theory, this progression is described as V^7 to I, where the root G of the chord $G7$ moves to the root C of the chord C, which is the origin of what we call the descending 5th progression. However as is seen above, since the tritone BF is also compatible with $F^\sharp$-major scale, we can borrow the V^7 chord in $F^\sharp$-major, $C^\sharp 7 = (C^\sharp F G^\sharp B)$ instead of $G7$. Thus we obtain another progression $C^\sharp 7 \to C$, so called the *tritone substitution*, where the motion $C^\sharp \to C$ of their roots is more 'smooth' than the original $G \to C$.

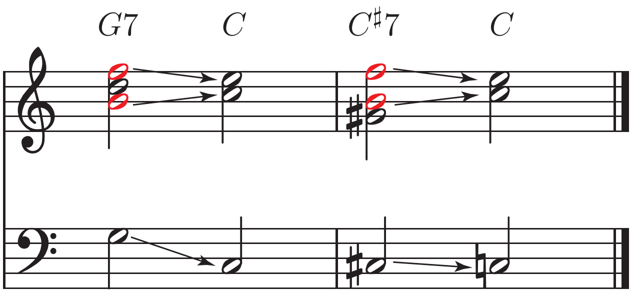

FIGURE 2. The perfect cadence (left) and its tritone substitution (right).

From our combinatorial viewpoint, we may adopt such a smooth chromatic motion of roots as a principle of chord progressions in tonal music. As is seen in Theorem 4.4, any tritone itself becomes a maximally even chord in its compatible scales, so, we assume that we can take a maximally even chord containing the given tritone (and this is true for the usual case $c = 12$ and $d = 7$). Let $V_m = (v^m_0, \ldots, v^m_{e-1})$ be a maximally even chord of $\mathcal{J}^m_{c,d}$ containing the tritone $T_m \sqsubset V_m$. Without loss of generality, we assume v^m_0 is the 'root' of V_m. Theorem 4.4 also suggests that there exists the maximally even chord V_{m+d-1} containing T_m, which coincides with $(d-1)$-semitones translation of V_m: $\theta(v^{m+d-1}_k) \equiv \theta(v^m_k) + d - 1 \pmod{c}$. A chromatic progression $V_m \to V_{m-1}$ induces a semitone motion of roots: $\theta(v^m_0) - \theta(v^{m-1}_0) \equiv 1 \pmod{c}$. Then applying the tritone substitution to V_m, the resultant progression $V_{m+d-1} \to V_{m-1}$ induces descending d-semitones motion of roots:

$$\theta(v^{m+d-1}_0) - \theta(v^{m-1}_0) \equiv \theta(v^m_0) + d - 1 - \theta(v^{m-1}_0) \equiv d \pmod{c}.$$

Figure 3 illustrates our combinatorial approach to the descending 5th motion. We first take the dominant 7th (omit 5th) GBF in C-major scale. Then we progress the chord chromatic way, hence their root moves smoothly, like $G \to F^\sharp \to F \to E \to \cdots$. After that, we apply the tritone substitution to $F^\sharp 7, E7, \ldots$, then the roots moves in 5th-descending way.

Summing up, we at first consider the chromatic progression which brings a smooth motion of roots, and apply the tritone substitution to any chords in the progression as necessary, then we obtain the descending d-semitones progression. Therefore at least from our combinatorial viewpoint, there is no priority to the perfect cadence.

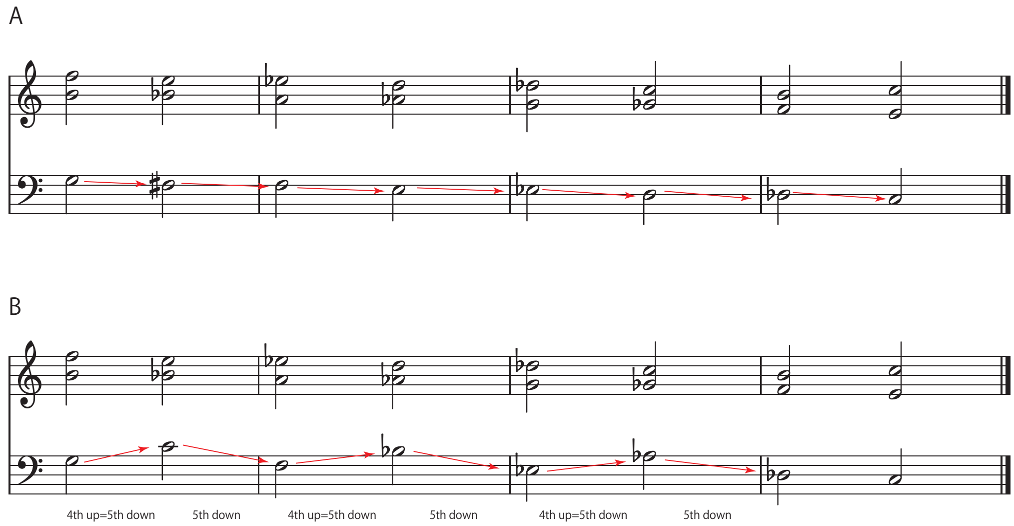

Figure 3. A combinatorial example for the descending 5th sequence (5th = 7-semitones). At first, we take a chromatic sequence of the dominant 7th chords (A). Then we apply tritone substitutions. As a result, we obtain the descending 5th sequence (B).

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k \ m	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	1	1	1	1	1	1
1	1	2	2	2	2	2	2	2	3	3	3	3	3
2	3	3	4	4	4	4	4	4	4	5	5	5	5
3	5	5	5	6	6	6	6	6	6	6	7	7	7
4	7	7	7	7	8	8	8	8	8	8	8	9	9
5	9	9	9	9	9	10	10	10	10	10	10	10	11
6	11	11	11	11	11	11	12	12	12	12	12	12	12

Table 1. The case $c = 13$ and $d = 7$. Diatonic scales $\mathcal{J}^m_{13,7}$ and a tritone $T_3 = (6, 7) \sqsubset \mathcal{J}^3_{13,7}$ are shown. As $2d - c = 1$, the rigidity of T_3 is 1.

k \ m	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	1	1	1	1	1
1	1	1	2	2	2	2	2	2	2	3	3	3
2	3	3	3	3	4	4	4	4	4	4	4	5
3	5	5	5	5	5	5	6	6	6	6	6	6
4	6	7	7	7	7	7	7	7	8	8	8	8
5	8	8	8	9	9	9	9	9	9	9	10	10
6	10	10	10	10	10	11	11	11	11	11	11	11

Table 2. The case $c = 12$ and $d = 7$. Diatonic scales $\mathcal{J}^m_{12,7}$ and a tritone $T_3 = (3, 9) \sqsubset \mathcal{J}^3_{12,7}$ are shown. As $2d - c = 2$, the rigidity of T_3 is 2. In terms of usual music theory, $B^\flat$-major scale has a tritone $E^\flat A$, and the tritone also contained in E-major scale $\mathcal{J}^9_{12,7}$. Note that $E^\flat A$ is also tritone in E-major scale.

k \ m	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	1	1	1	1
1	1	1	1	2	2	2	2	2	2	2	3
2	3	3	3	3	3	3	4	4	4	4	4
3	4	4	5	5	5	5	5	5	5	6	6
4	6	6	6	6	6	7	7	7	7	7	7
5	7	8	8	8	8	8	8	8	9	9	9
6	9	9	9	9	10	10	10	10	10	10	10

Table 3. The case $c = 11$ and $d = 7$. Diatonic scales $\mathcal{J}^m_{11,7}$ and a tritone $T_3 = (2, 9) \sqsubset \mathcal{J}^3_{11,7}$ are shown. As $2d - c = 3$, the rigidity of T_3 is 3. Indeed, $T_3 \sqsubset \mathcal{J}^8_{11,7}$ and $T_3 \sqsubset \mathcal{J}^9_{11,7}$, however, contrast to the case $2d - c = 2$, T_3 is not a tritone in these scales. We see $T_8 = (5, 9)$ and $T_9 = (2, 6)$.