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New Methods for Constructing Odd-Order Magic Squares

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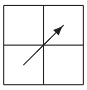
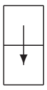
1 Introduction

A **magic square** is an $n \times n$ matrix whose cells contain distinct integers from 1 to n^2, and the sums of n rows, n columns and two diagonals are mutually equal. Magic squares have been treated long time as teaching materials of mathematics education. The following is a well-known method for constructing odd-order magic squares. The method was introduced to Europe by a French diplomat Simon de la Loubère (1642–1729) by a book concerning Siam [1]. So the method is called Siamese Method.

Siamese Method. Let $n = 2r + 1 \geqq 3$ be an odd integer. Consider an $n \times n$ matrix, and put integers from 1 to n^2 in the following way:
(i) Put 1 on $(1, r + 1)$.
(ii) If k placed on (i, j) is not a multiple of n, then put $k + 1$ on $(i - 1, j + 1)$.
(iii) If k placed on (i, j) is a multiple of n, then put $k + 1$ on $(i + 1, j)$.

Throughout the paper, we refer a cell of a matrix by the address (i, j) of row and column numbers i, j, which are considered in modulo n. That is to say, the addresses (i, j), $(i + n, j)$ and $(i, j + n)$ refer the same cell. The following are the application examples of the method.

$(-1, 1)$ 

$(1, 0)$ 

8	1	6
3	5	7
4	9	2

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

Notice that the above magic squares are symmetric in a sense. Every symmetric pair with respect to the center have the same sum. For example, in the 5th order case, $17 + 9 = 26$, $24 + 2 = 26$, $1 + 25 = 26$, and so on. In the paper, we call such a matrix to have the **magic-symmetric property**

In Siamese Method, a pair of vectors $(-1, 1)$ and $(1, 0)$ plays an important role. The author noticed that if we prepare a “good” pair of vectors (a, c) and (b, d), we can make new methods for constructing magic squares.

Theorem (Generalized Siamese Method). Let $n = 2r + 1 \geqq 3$ be an odd integer, and let a, b, c, d $(-r \leqq a, b, c, d \leqq r)$ be integers satisfying the following conditions:

(1) $\gcd(a, n) = 1$, $\gcd(b - a, n) = 1$, (2) $\gcd(c, n) = 1$, $\gcd(d - c, n) = 1$, (3) $\gcd(ad - bc, n) = 1$.

Consider an $n \times n$ matrix, and put integers from 1 to n^2 in the following way:
(i) Put 1 on (i_0, j_0), where $i_0 + rb \equiv r + 1$, $j_0 + rd \equiv r + 1 \pmod{n}$.
(ii) If k placed on (i, j) is not a multiple of n, then put $k + 1$ on $(i + a, j + c)$.
(iii) If k place on (i, j) is a multiple of n, then put $k + 1$ on $(i + b, j + d)$.
Then the matrix becomes a magic square with the magic-symmetric property.

2 Proof of Theorem

Represent each integer k from 1 to n^2 by $k = tn + s + 1$ ($0 \leqq s, t \leqq n-1$). From 1 to $k-1$, there are t integers which are multiples of n, and $t(n-1) + s$ integers which are not multiples of n. So the address (i, j) of k satisfies the following:

$$\begin{pmatrix} i \\ j \end{pmatrix} \equiv \begin{pmatrix} i_0 \\ j_0 \end{pmatrix} + (s-t) \begin{pmatrix} a \\ c \end{pmatrix} + t \begin{pmatrix} b \\ d \end{pmatrix} \pmod{n}. \tag{1}$$

Firstly, we show that two distinct numbers can not occupy the same cell. To prove it, we assume that two numbers $k_1 = t_1 n + s_1$ and $k_2 = t_2 n + s_2$ occupy the same cell, then we have that

$$\begin{pmatrix} i_0 \\ j_0 \end{pmatrix} + (s_1 - t_1) \begin{pmatrix} a \\ c \end{pmatrix} + t_1 \begin{pmatrix} b \\ d \end{pmatrix} \equiv \begin{pmatrix} i_0 \\ j_0 \end{pmatrix} + (s_2 - t_2) \begin{pmatrix} a \\ c \end{pmatrix} + t_2 \begin{pmatrix} b \\ d \end{pmatrix} \pmod{n}. \tag{2}$$

By arranging it, we have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s_1 - t_1 - s_2 + t_2 \\ t_1 - t_2 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod{n}. \tag{3}$$

Notice that $ad - bc$ is the determinant of the above matrix. Since the determinant and n are mutually prime, the matrix has the inverse in modulo n. Therefore, we have that $s_1 = s_2$ and $t_1 = t_2$. So k_1 and k_2 are the same. Hence two distinct numbers can not occupy the same cell.

Secondly, we prove that the sum of each row is equal to $n(n^2+1)/2$. By using Eq.(1), for every i ($1 \leqq i \leqq n$), i-th row consists of the following sets:

$$R_i = \left\{ \, tn + s + 1 \;\middle|\; i_0 + sa + t(b-a) \equiv i \pmod{n}, \; 0 \leqq s, t \leqq n-1 \, \right\}. \tag{4}$$

Since a and n are mutually prime, integers s ($0 \leqq s \leqq n-1$) are contained in distinct R_i. So each R_i contains the unique s. Denote by $R_{i(s)}$ the set which contains s. Since $b-a$ and n are mutually prime, integers $tn+1$ ($0 \leqq t \leqq n-1$) are contained in distinct R_i. So each S_i contains the unique $tn+1$. Denote by t_j the integer for which $t_j n + 1$ is contained in $R_{i(j)}$. Therefore, the sets $R_{i(j)}$ can be written as follows:

$$\begin{aligned}
R_{i(0)} &= \{\, 1,\ t_1 n + n, \quad t_2 n + n - 1,\ t_3 n + n - 2,\ \cdots,\ t_{n-1} + 2 \,\}, \\
R_{i(1)} &= \{\, 2,\ t_1 n + 1, \quad t_2 n + n, \quad t_3 n + n - 1,\ \cdots,\ t_{n-1} + 3 \,\}, \\
R_{i(2)} &= \{\, 3,\ t_1 n + 2, \quad t_2 n + 1, \quad t_3 n + n, \quad \cdots,\ t_{n-1} + 4 \,\}, \\
&\ \vdots \\
R_{i(n-1)} &= \{\, n,\ t_1 n + n - 1,\ t_2 n + n - 2,\ t_3 n + n - 3,\ \cdots,\ t_{n-1} + 1 \,\}.
\end{aligned} \tag{5}$$

Thus the sum of each set is calculated as follows:

$$\begin{aligned}
&(t_1 + t_2 + \cdots + t_{n-1})n + 1 + 2 + 3 + \cdots + n \\
&= \frac{(n-1)n}{2} \cdot n + \frac{n(n+1)}{2} = \frac{n(n^2+1)}{2}.
\end{aligned} \tag{6}$$

Hence the sum of each row is equal to $n(n^2+1)/2$. We can prove the same property of column in the same way.
Thirdly, we prove the magic-symmetric property. Consider the following two integers:

$$k_1 = tn + s + 1, \quad k_2 = (n-1-t)n + (n-1-s) + 1. \tag{7}$$

Then its sum is calculated as follows:

$$k_1 + k_2 = (n-1)n + n - 1 + 2 = n^2 + 1. \tag{8}$$

Denote by (i_1, j_1) and (i_2, j_2) the addresses of k_1 and k_2, respectively. Then by Eq.(1), we have that

$$\begin{aligned}\begin{pmatrix} i_1 \\ j_1 \end{pmatrix} + \begin{pmatrix} i_2 \\ j_2 \end{pmatrix} &= \left\{ \begin{pmatrix} i_0 \\ j_0 \end{pmatrix} + (s-t) \begin{pmatrix} a \\ c \end{pmatrix} + t \begin{pmatrix} b \\ d \end{pmatrix} \right\} \\ &\quad + \left\{ \begin{pmatrix} i_0 \\ j_0 \end{pmatrix} + (-s+t) \begin{pmatrix} a \\ c \end{pmatrix} + (n-1-t) \begin{pmatrix} b \\ d \end{pmatrix} \right\} \\ &= 2\left\{ \begin{pmatrix} i_0 \\ j_0 \end{pmatrix} + r \begin{pmatrix} b \\ d \end{pmatrix} \right\} \equiv 2 \begin{pmatrix} r+1 \\ r+1 \end{pmatrix} \pmod{n}. \end{aligned} \tag{9}$$

Eq.(9) shows that k_1 and k_2 form a symmetric pair, and Eq.(8) shows that their sum is equal to $n^2 + 1$. So it has the magic-symmetric property. In particular, the center cell is equal to $(n^2 + 1) / 2$.

Thirdly, we prove that the sum of each diagonal is equal to $n(n^2 + 1) / 2$. Since a diagonal consists of r pair of symmetric cells and the center cell, their sum can be calculated as follows:

$$(n^2+1) \times r + \frac{n^2+1}{2} = \frac{n^2+1}{2} \times (2r+1) = \frac{n(n^2+1)}{2}. \tag{10}$$

Hence the sum of each diagonal is equal to $n(n^2 + 1) / 2$.

3 Examples

In the section, we apply our theorem to the case $n = 5$. Out of $5^4 = 625$ possibilities of (a, c, b, d), there are 192 quadruples satisfying the conditions of the theorem. So we can makes 192 magic squares by the method. However, if we have one magic square, we can make 8 magic squares by rotations and reflections. So there are 192 ÷ 8 = 24 magic squares if we do not count the differences by rotations and reflections. The following table is the list of 24 quadruples of (a, c, b, d) and their initial cell (i_0, j_0).

No.	a	c	b	d	i_0	j_0
1	−2	−1	1	1	1	1
2	−2	2	1	1	1	1
3	−1	2	1	1	1	1
4	−2	1	1	−2	1	2
5	−2	2	1	−2	1	2
6	−1	−1	1	−2	1	2
7	−1	1	1	−2	1	2
8	2	−1	1	−2	1	2
9	2	2	1	−2	1	2
10	−2	−2	1	0	1	3
11	−2	−1	1	0	1	3
12	−1	−2	1	0	1	3

No.	a	c	b	d	i_0	j_0
13	−1	−1	1	0	1	3
14	2	−2	1	0	1	3
15	2	−1	1	0	1	3
16	−1	1	−2	−2	2	2
17	−1	2	−2	−2	2	2
18	1	2	−2	−2	2	2
19	−1	−2	−2	0	2	3
20	−1	−1	−2	0	2	3
21	1	−2	−2	0	2	3
22	1	−1	−2	0	2	3
23	2	−2	−2	0	2	3
24	2	−1	−2	0	2	3

The following are magic squares which are made from the above table. The magic square in No.13 is same as Siamese one by the horizontal reflection.

1	14	22	10	18
24	7	20	3	11
17	5	13	21	9
15	23	6	19	2
8	16	4	12	25

No.1

1	18	10	22	14
20	7	24	11	3
9	21	13	5	17
23	15	2	19	6
12	4	16	8	25

No.2

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

No.3

22	1	10	14	18
11	20	24	3	7
5	9	13	17	21
19	23	2	6	15
8	12	16	25	4

No.4

10	1	22	18	14
3	24	20	11	7
21	17	13	9	5
19	15	6	2	23
12	8	4	25	16

No.5

15	1	17	8	24
23	14	5	16	7
6	22	13	4	20
19	10	21	12	3
2	18	9	25	11

No.6

8	1	24	17	15
5	23	16	14	7
22	20	13	6	4
19	12	10	3	21
11	9	2	25	18

No.7

23	1	9	12	20
15	18	21	4	7
2	10	13	16	24
19	22	5	8	11
6	14	17	25	3

No.8

12	1	20	9	23
21	15	4	18	7
10	24	13	2	16
19	8	22	11	5
3	17	6	25	14

No.9

18	22	1	10	14
24	3	7	11	20
5	9	13	17	21
6	15	19	23	2
12	16	25	4	8

No.10

22	14	1	18	10
3	20	7	24	11
9	21	13	5	17
15	2	19	6	23
16	8	25	12	4

No.11

8	17	1	15	24
14	23	7	16	5
20	4	13	22	6
21	10	19	3	12
2	11	25	9	18

No.12

15	8	1	24	17
16	14	7	5	23
22	20	13	6	4
3	21	19	12	10
9	2	25	18	11

No.13

20	23	1	9	12
21	4	7	15	18
2	10	13	16	24
8	11	19	22	5
14	17	25	3	6

No.14

9	20	1	12	23
15	21	7	18	4
16	2	13	24	10
22	8	19	5	11
3	14	25	6	17

No.15

19	23	2	6	15
22	1	10	14	18
5	9	13	17	21
8	12	16	25	4
11	20	24	3	7

No.16

19	15	6	2	23
10	1	22	18	14
21	17	13	9	5
12	8	4	25	16
3	24	20	11	7

No.17

19	8	22	11	5
12	1	20	9	23
10	24	13	2	16
3	17	6	25	14
21	15	4	18	7

No.18

2	23	19	15	6
14	10	1	22	18
21	17	13	9	5
8	4	25	16	12
20	11	7	3	24

No.19

15	2	19	6	23
22	14	1	18	10
9	21	13	5	17
16	8	25	12	4
3	20	7	24	11

No.20

15	2	19	6	23
22	14	1	18	10
9	21	13	5	17
16	8	25	12	4
3	20	7	24	11

No.21

8	11	19	22	5
20	23	1	9	12
2	10	13	16	24
14	17	25	3	6
21	4	7	15	18

No.22

22	8	19	5	11
9	20	1	12	23
16	2	13	24	10
3	14	25	6	17
15	21	7	18	4

No.23

3	21	19	12	10
15	8	1	24	17
22	20	13	6	4
9	2	25	18	11
16	14	7	5	23

No.24

References

[1] Simon de la Loubère, The Problem of the Magical Squares according to the Indians, in “A New Historical Relation of the Kingdom of Siam”, pp.227–247, 1693, which is available at https://seasiavisions.library.cornell.edu/catalog/sea:130

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