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A Remark of *k*-summability of Divergent Solutions to Some Linear Partial Differential Equations with Entire Cauchy Data

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Abstract

We consider the Cauchy problem to some linear partial differential equations with entire Cauchy data. We see that the summability condition for the formal solution is equivalent to the convergence in some case.

1 Introduction

We shall consider the following Cauchy problem for linear partial differential equations

$$\text{(CP)} \qquad \begin{cases} \partial_t^p u(t,x) = \partial_x^q u(t,x), \\ u(0,x) = \varphi(x) \in \mathcal{O}_x, \quad \partial_t^i u(0,x) = 0 \quad (1 \le i \le p-1), \end{cases}$$

where $(t, x) \in \mathbb{C}^2$, p and q are natural numbers with $q > p$, and $\mathcal{O}_x$ denotes the set of all holomorphic functions in a neighborhood of the origin $x = 0$. The Cauchy problem (CP) has a unique formal solution of the form

$$(1.1) \qquad \hat{u}(t,x) = \sum_{n\ge 0} \varphi^{(qn)}(x) \frac{t^{pn}}{(pn)!} \overset{\text{put}}{=} \sum_{n \ge 0} u_n(x) t^n \in \mathcal{O}_x[[t]]_{1/k(0)} \quad \left(k(0) = \frac{p}{q-p} \right),$$

which is divergent in general by the assumption that $q > p$.

Here the notation $\mathcal{O}_x[[t]]_{1/k}$ denotes the set of formal power series in t with the coefficients $u_n(x)$ which are holomorphic in a common closed disc $B(r) := \{x \in \mathbb{C}; |x| \le r\}$ for some $r > 0$ and satisfy the following Gevrey type estimates

$$(1.2) \qquad \max_{|x|\le r} |u_n(x)| \le C K^n \Gamma(1 + n/k)$$

with some positive constants C and K for any nonnegative integer n. In this case, we say that the Gevrey order of $\hat{u}$ is (at most) $1/k$.

In order to explain our problem, we define a class $\mathrm{Exp}(\gamma; \mathbb{C})$ or $\mathrm{Exp}_x(\gamma; \mathbb{C})$ of entire functions for $\gamma > 0$ by

$$\mathrm{Exp}(\gamma; \mathbb{C}) := \{ f(x) \in \mathcal{O}(\mathbb{C}); |f(x)| \le C \exp(\delta |x|^\gamma) \text{ for some } C, \delta > 0 \},$$

where $\mathcal{O}(\mathbb{C})$ denotes the set of entire functions. By an easy calculation, we see that

$$f(x) \in \mathrm{Exp}(\gamma; \mathbb{C}) \iff |f^{(n)}(0)| \le AB^n (n!)^{1-1/\gamma}$$

for all n by some positive constants A and B. From this fact, a characterization of the convergence of the formal solution (1.1) is stated as follows.

Theorem 1 (Miyake [4]) *The formal solution $\hat{u}(t, x)$ is convergent if and only if $\varphi(x) \in \mathrm{Exp}(q/(q-p); \mathbb{C})$ for the Cauchy data.*

We assume that the Cauchy data $\varphi(x)$ belongs to a class of entire functions as follows

$$(1.3) \qquad \varphi(x) \in \mathrm{Exp}\,(q/\ell; \mathbb{C}), \quad 0 \le \ell \le q-p-1 \ (\ell \in \mathbb{N}).$$

When $\ell = 0$, we understand that $\mathrm{Exp}(q/0; \mathbb{C}) = \mathcal{O}_x$ which does not make any contradiction in the results. In this case, we have for the Gevrey order of the formal solution

$$(1.4) \qquad \hat{u}(t,x) \in \mathcal{O}_x[[t]]_{1/k(\ell)}, \quad k(\ell) = p/(q-p-\ell) \ge k(0) = p/(q-p).$$

Our problem is to give a characterization of $k(\ell)$-summability of the formal solution (1.1) under the assumption (1.3) for the Cauchy data.

The organization of the paper is as follows. In Section 2, we give a definition of k-summability and we review a result of $k(0)$-summability of $\hat{u}(t,x)$. In Section 3, we give the main Theorem (Theorem 3) and its corollary. For proving the main Theorem, we give a result of $k(\ell)$-summability of $\hat{u}(t,x)$, which has already obtained in our papers [5, 6], and give a rough proof in Section 4. In Section 5, we give a proof of the main Theorem.

2 Review of a result of $k(0)$-summability

We first give the definition of Gevrey asymptotic expansion and k-summability (cf. [1]).

For $d \in \mathbb{R}$, $\alpha > 0$ and ρ $(0 < \rho \le \infty)$, we define a sector $S = S(d,\alpha,\rho)$ by

$$S(d,\alpha,\rho) := \{t \in \mathbb{C}; |d - \arg t| < \alpha/2,\ 0 < |t| < \rho\},$$

where d, α and ρ are called the direction, the opening angle and the radius of S, respectively. We write $S(d,\alpha,\infty) = S(d,\alpha)$ for short.

Let $k > 0$, $\hat{v}(t,x) = \sum_{n=0}^{\infty} v_n(x)t^n \in \mathcal{O}_x[[t]]_{1/k}$ and $v(t,x)$ be an analytic function on $S(d,\alpha,\rho) \times B(r)$. Then we say that $v(t,x)$ has a Gevrey asymptotic expansion $\hat{v}(t,x)$ of order k in $S(d,\alpha,\rho)$, which is denoted by

$$v(t,x) \cong_k \hat{v}(t,x) \quad \text{in } S(d,\alpha,\rho),$$

if for any closed subsector S' of $S(d,\alpha,\rho)$, there exist some positive constants C and K such that for any N, we have

$$\max_{|x|\le r} \left| v(t,x) - \sum_{n=0}^{N-1} v_n(x)t^n \right| \le CK^N |t|^N \Gamma(1+N/k), \quad t \in S'. \tag{2.1}$$

For $k > 0$, $d \in \mathbb{R}$ and $\hat{v}(t,x) \in \mathcal{O}_x[[t]]_{1/k}$, we say that $\hat{v}(t,x)$ is k-summable in d direction, which is denoted by $\hat{v}(t,x) \in \mathcal{O}_x\{t\}_{k,d}$, if there exist a sector $S = S(d,\alpha,\rho)$ with $\alpha > \pi/k$ and an analytic function $v(t,x)$ on $S \times B(r)$ such that $v(t,x) \cong_k \hat{v}(t,x)$ in S.

We remark that the function $v(t,x)$ above for a k-summable $\hat{v}(t,x)$ is unique if it exists. Therefore such a function $v(t,x)$ is called the k-sum of $\hat{v}(t,x)$ in d direction.

Now, we give a result of the $k(0)$-summability for the formal solution (1.1) of (CP).

Theorem 2 (Miyake [4]) *Let $\varphi(x) \in \mathcal{O}_x$ for the Cauchy data. For $d \in \mathbb{R}$ and $\varepsilon > 0$, we put*

$$\Omega_q^p(d,\varepsilon) := \bigcup_{j=0}^{q-1} S\left(\frac{pd+2\pi j}{q}, \varepsilon\right). \tag{2.2}$$

Then $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{k(0),d}$ if and only if the following conditions are satisfied.

$$\text{(i)}\quad \varphi(x) \in \mathcal{O}\left(\Omega_q^p(d,\varepsilon)\right), \qquad \text{(ii)}\quad |\varphi(x)| \le Ce^{\delta|x|^{q/(q-p)}} \quad (x \in \Omega_q^p(d,\varepsilon))$$

by some positive constants C and δ. Here $\mathcal{O}(\Omega)$ denotes the set of holomorphic functions on a domain Ω.

The conditions (i) and (ii) can be expressed in a simplified form as follows. Let us define

$$\Phi(x,\zeta) := \sum_{j=0}^{q-1} \varphi(x + \omega_q^j \zeta) \quad (\omega_q = e^{2\pi i/q}).$$

Then the conditions (i) and (ii) are equivalent to that

$$\Phi(x,\zeta) \in \mathrm{Exp}_\zeta\left(\frac{q}{q-p}; S(pd/q, \varepsilon_1)\right) \tag{2.3}$$

uniformly in x in a neighborhood of $x = 0$ by some positive constant ε_1.

3 Main Theorem

In the following, we assume that

$$\text{(3.1)} \qquad \varphi(x) \in \mathrm{Exp}\,(q/\ell;\mathbb{C}), \quad 0 \le \ell \le q-p-1 \ (\ell \in \mathbb{N}).$$

In this case, the Gevrey order of the formal solution is given by

$$\text{(3.2)} \qquad \hat{u}(t,x) \in \mathcal{O}_x[[t]]_{1/k(\ell)}, \quad k(\ell) = p/(q-p-\ell) \ge k(0) = p/(q-p).$$

Our problem is to obtain the corresponding result with $\ell \ge 1$ to Theorem 2 with $\ell = 0$ under the assumption (3.1). Before giving our results, we give a definition of k-summability on an interval for formal series.

Let $I \subset \mathbb{R}$ be an interval. For $k > 0$ and $\hat{v}(t, x) \in \mathcal{O}_x[[t]]_{1/k}$, we say that $\hat{v}(t, x)$ is k-summable on I, which is denoted by $\hat{v}(t, x) \in \mathcal{O}_x\{t\}_{k,I}$, if for any $\tilde{d} \in I$, $\hat{v}(t, x) \in \mathcal{O}_x\{t\}_{k,\tilde{d}}$.

Now, our main theorem is stated as follows.

Theorem 3 *We assume that* $\varphi(x) \in \mathrm{Exp}\,(q/\ell;\mathbb{C})$.

- *When* $\ell = 1$, *for given* $d \in \mathbb{R}$ *and* $\varepsilon > 0$, *let* $I_d(p, \varepsilon)$ *be an interval* $(d - \pi/(2p) - \varepsilon, d + \pi/(2p) + \varepsilon)$. *Then we have*

$$\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{k(1),I_d(p,\varepsilon)} \iff \varphi(x) \in \mathrm{Exp}(q/(q-p);\Omega^p_q(d,\varepsilon'))$$

for a sufficiently small $\varepsilon' > 0$, *where* $\Omega^p_q(d, \varepsilon')$ *is given by* (2.2).

- *When* $\ell \ge 2$, *we have*

$$\hat{u}(t,x) \in \mathcal{O}_{t,x} \iff \varphi(x) \in \mathrm{Exp}(q/(q-p);\Omega^p_q(d,\varepsilon)).$$

From Theorems 1 and 3, we have the following corollary.

Corollary 4 *We assume that* $\varphi(x) \in \mathrm{Exp}(q/\ell;\mathbb{C})$ *with* $\ell \ge 2$. *Then we have*

$$\varphi(x) \in \mathrm{Exp}(q/(q-p);\Omega^p_q(d,\varepsilon)) \iff \varphi(x) \in \mathrm{Exp}(q/(q-p);\mathbb{C}).$$

Corollary 4 can be also proved directly by using a theorem of Phragmén [7, p. 342]. We omit the detail.

Remark 5 *The results for q-difference-differential equations corresponding to Theorem 3 and Corollary 4 for partial differential equations have been already obtained (cf. [2]).*

We give an important lemma of the summability theory for proving Theorem 7, which will be given in next section (cf. [1, 3, 4]).

Lemma 6 *Let* $k > 0$, $d \in \mathbb{R}$ *and* $\hat{f}(t, x) \in \mathcal{O}_x[[t]]_{1/k}$. *Then the following two statements are equivalent.*

(i) $\hat{f}(t,x) \in \mathcal{O}_x\{t\}_{k,d}$.

(ii) *Let* $\{k_j\}_{j=1}^J$ $(k_j > 0, J \ge 1)$ *satisfy* $1/k = 1/k_1 + \cdots + 1/k_J$, *and we define* $g(s,x)$ *by iterated formal Borel transforms of* $\hat{f}(t, x)$

$$\text{(3.3)} \qquad g(s,x) := (\hat{\mathcal{B}}_{k_J} \circ \cdots \circ \hat{\mathcal{B}}_{k_1} \hat{f})(s,x),$$

where the formal k-Borel transform $\hat{\mathcal{B}}_k$ *is defined as follows: for* $\hat{f}(t, x) = \sum_{n\ge0} f_n(x)t^n$, *we define*

$$(\hat{\mathcal{B}}_k \hat{f})(s,x) = \sum_{n\ge0} f_n(x)\frac{s^n}{\Gamma(1+n/k)}$$

Then $g(s, x) \in \mathrm{Exp}_s(k; S(d, \varepsilon) \times B(r))$ *for some* $\varepsilon > 0$ *and* $r > 0$.

Moreover, under the condition (ii), *the k-sum* $f(t, x)$ *of* $\hat{f}(t, x)$ *is given by the iterated Laplace transforms of* $g(s, x)$

$$\text{(3.4)} \qquad f(t,x) = (\mathcal{L}_{k_1,d} \circ \cdots \circ \mathcal{L}_{k_J,d}\, g)(t,x),$$

where

$$(\mathcal{L}_{k,d}\, g)(t,x) := \frac{1}{t^k}\int_0^{\infty(d)} \exp\left(-\left(\frac{s}{t}\right)^k\right) g(s,x)\,d(s^k)$$

with the path taken from 0 *to* ∞ *along* arg $s = d$. *We write* $\mathcal{L}_{k,d}g = \mathcal{L}_k g$ *for short.*

At the end of this section, we give the definition of Borel transform ([1]). Let $S = S(d, \alpha, \rho)$ be a sector with $\alpha > \pi/k$. If $f(s)$ is analytic in S and is bounded at the origin, we define the k-Borel transform of f by

$$(\mathcal{B}_k f)(\tau) = \frac{-k}{2\pi i}\int_{\gamma_k(d)} f(s)e^{(\tau/s)^k}ds/s,$$

where $\gamma_k(d)$ denotes the path from the origin along arg $s = d + (\varepsilon + \pi)/(2k)$ to some point s_1 with a positive ε, then along the circle $|s| = |s_1|$ to the ray arg $s = d - (\varepsilon + \pi)/(2k)$, and back to the origin along this ray such that $\gamma_k(d) \subset S$.

4 A result of *k*(*ℓ*)-summability

Before giving a proof of Theorem 3, we give a result of $k(\ell)$-summability for the formal solution $\hat{u}(t, x)$ of (CP) which was given in [5, 6] and we give a rough proof.

Theorem 7 *Let* $\varphi(x) \in \mathrm{Exp}(q/\ell; \mathbb{C})$ $(1 \le \ell \le q - p - 1)$ *and* $d \in \mathbb{R}$. Then $\hat{u}(t, x) \in \mathcal{O}_x\{t\}_{k(\ell),d}$ $(k(\ell) = p/(q - p - \ell))$ *if*

$$(4.1)\qquad \Phi_q(x,\zeta) = \sum_{j=0}^{q-1}\varphi(x+\omega_q^j\zeta) \in \mathrm{Exp}_\zeta(q/(q-p); S(pd/q,\varepsilon)) \qquad (\omega_q = e^{2\pi i/q})$$

uniformly for small $|x|$.

<u>*Proof of Theorem 7*</u> We recall $k(\ell) = p/(q - p - \ell)$ and we assume that

$$\varphi(x) \in \mathrm{Exp}\,(q/\ell; \mathbb{C})\,.$$

Let $v(s, x)$ be $(q - p - \ell)$ times iterated formal p-Borel transforms of $\hat{u}$

$$(4.2)\qquad v(s,x) := (\hat{\mathcal{B}}_p^{q-p-\ell}\hat{u})(s,x) = \sum_{n\ge 0}\frac{\varphi^{(qn)}(x)}{(pn)!}\frac{s^{pn}}{n!^{q-p-\ell}},$$

which is convergent in a neighborhood of $(s, x) = (0, 0)$. Then for the proof of $k(\ell)$- summability of $\hat{u}(t, x)$ in a direction d, it is enough to prove

$$(4.3)\qquad v(s,x) \in \mathrm{Exp}_s(k(\ell); S(d,\varepsilon_1)\times B(r)) \quad (k(\ell) = p/(q-p-\ell))$$

for some positive constants ε_1 and r under the assumption $\Phi_q(x, \zeta) \in \mathrm{Exp}_\zeta(q/(q-p); S(pd/q, \varepsilon))$ uniformly for $x \in B(r)$. For that purpose, we further take ℓ times iterated p-Borel transforms for $v(s, x)$ and put

$$(4.4)\qquad w(\tau,x) := (\mathcal{B}_p^\ell v)(\tau,x) = (\hat{\mathcal{B}}_p^{q-p}\hat{u})(\tau,x) = \sum_{n\ge 0}\frac{\varphi^{(qn)}(x)}{(pn)!}\frac{\tau^{pn}}{n!^{q-p}}$$

In this case, from the fact $v(s, x) \in \mathcal{O}_{s,x}$ or $\varphi(x) \in \mathrm{Exp}(q/\ell; \mathbb{C})$, we see that $w(\tau, x) \in \mathrm{Exp}_\tau(p/\ell; \mathbb{C}^2)$. Moreover, we can prove the following lemma.

Lemma 8

$$(4.5)\qquad w(\tau,x) \in \mathrm{Exp}_\tau\,(p/(q-p); S(d,\varepsilon_1')\times B(r))$$

for a sufficiently small $\varepsilon'_1 > 0$,

By admitting Lemma 8, we immediately get the desired property (4.3) for $v(s, x)$. In fact, since $v(s, x)$ is given by

$$v(s,x) = (\mathcal{L}_p^\ell w)(s,x),$$

we get the desired estimate for $v(s, x)$ by repeating the following lemma (cf. [1]).

Lemma 9 *We assume* $f(\tau) \in \mathrm{Exp}_\tau\,(p/q; S(d, \varepsilon))$, *where* $p, q \in \mathbb{N}$. *We put* $F(s) := (\mathcal{L}_p f)(s)$. *Then we have* $F(s) \in \mathrm{Exp}_s\,(p/(q-1); S(d, \pi/p + \varepsilon'))$ *with* $\varepsilon' < \varepsilon$.

Finally, we give a outline of a proof of Lemma 8. In the expression (4.4), after using the Cauchy integral formula for $\varphi(x)$, we represent the integral kernel function as a hypergeometric series. From the properties of an analytic continuation and the singularity of the kernel function, we can obtain the desired estimate for w (cf. [5, 6, 2]).

Remark 10 *We remark that under the assumptions* $\varphi(x) \in \mathrm{Exp}(q/\ell;\mathbb{C})$ *and* (4.1), *we can prove* $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{k(\ell),d+2\pi i/p}$ *for* $0 \leq i \leq p-1$.

5 Proof of Theorem 3

By Remark 10 in the previous section, we see that Lemma 8 is rewritten as follows.

$$w(\tau,x) \in \mathrm{Exp}_\tau\left(p/(q-p); S_p(d,\varepsilon_1') \times B(r)\right),$$

where $S_p(d,\varepsilon'_1) = \cup_{i=0}^{p-1} S(d+2\pi i/p, \varepsilon'_1)$. Moreover, by Lemma 9, we have

$$v(s,x) = (\mathcal{L}_p^\ell w)(s,x) \in \mathrm{Exp}_s(k(\ell); S_p(d,\varepsilon_2+\ell\pi/p) \times B(r)), \tag{5.1}$$

where $\varepsilon_2 < \varepsilon'_1$.

5.1 Proof of Theorem 3 when $\ell \geq 2$

When $\ell \geq 2$, since $S_p(d,\varepsilon_2+\ell\pi/p) \supset \mathbb{C}$, we see from (5.1) that $v(s,x) \in \mathrm{Exp}_s(k(\ell); \mathbb{C} \times B(r))$, which means that $\hat{u}(t,x) \in \mathcal{O}_{t,x}$. Inversely, if $\hat{u}(t,x) \in \mathcal{O}_{t,x}$, it is obvious that $\varphi(x) \in \mathrm{Exp}(q/(q-p);\mathbb{C})$ from Theorem 1.

5.2 Proof of the necessity of Theorem 3 when $\ell = 1$

When $\ell = 1$, we have $v(s,x) \in \mathrm{Exp}_s(k(1); S_p(d,\varepsilon_2+\pi/p) \times B(r))$. Therefore by putting the interval $I_d(p,\varepsilon')$ for a sufficiently small $\varepsilon' > 0$ by

$$I_d(p,\varepsilon') = (d-\pi/(2p)-\varepsilon', d+\pi/(2p)+\varepsilon'),$$

we see that for any $\tilde{d} \in I_d(p,\varepsilon')$, we have $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{k(1),\tilde{d}}$.

5.3 Proof of the sufficiency of Theorem 3 when $\ell = 1$

We prove that $\varphi(x) \in \mathrm{Exp}(q/(q-p); \Omega_q^p(d,\varepsilon'))$ under the assumption $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{k(1),I_d(p,\varepsilon)}$ for some positive ε' and ε, where $k(1) = p/(q-p-1)$ and

$$\Omega_q^p(d,\varepsilon') = \bigcup_{j=0}^{q-1} S\left(\frac{pd+2\pi j}{q}, \varepsilon'\right), \quad I_d(p,\varepsilon) = \left(d - \frac{\pi}{2p} - \varepsilon, d + \frac{\pi}{2p} + \varepsilon\right).$$

We put $d(i) = d + 2\pi i/p$ for $i = 0, 1, \dots, p-1$. Then by Remark 10, we have $\hat{u}(t,x) \in \mathcal{O}_x\{t\}_{k(1),I_{d(i)}(p,\varepsilon)}$ for $i = 0, 1, \dots, p-1$.

Let $v(s,x) = (\hat{\mathcal{B}}_p^{q-p-1}\hat{u})(s,x)$. Then $v(s,x)$ satisfies the following Cauchy problem which is obtained by the iterated formal Borel transforms of (CP)

$$\begin{cases} \partial_s^p \widetilde{\delta}_s^{\,q-p-1} v(s,x) = \partial_x^q v(s,x), \\ v(0,x) = \varphi(x) \in \mathcal{O}(\mathbb{C}), \\ \partial_s^i v(0,x) = 0 \quad (1 \leq i \leq p-1), \end{cases} \tag{5.2}$$

where $\widetilde{\delta}_s = (1/p)s\partial_s$. In fact, it is deduced from the following commutative diagram.

$$\begin{array}{ccc} t^{pn} & \xrightarrow{\hat{\mathcal{B}}_p^{q-p-1}} & \dfrac{s^{pn}}{n!^{q-p-1}} \\ \Big\downarrow \partial_t^p & & \Big\downarrow \partial_s^p \widetilde{\delta}_s^{\,q-p-1} \\ \dfrac{(pn)!}{(p(n-1))!} t^{p(n-1)} & \xrightarrow{\hat{\mathcal{B}}_p^{q-p-1}} & \dfrac{(pn)!}{(p(n-1))!} \dfrac{s^{p(n-1)}}{(n-1)!^{q-p-1}} \end{array}$$

Then we remark that $v(s, x) \in \mathrm{Exp}_s(k(1); S_{p,R}(d, \varepsilon_1 + \pi/p) \times \mathbb{C})$ for some positive R and ε_1 since $\hat{u}(t, x) \in \mathcal{O}_x\{t\}_{k(1), I_{d(i)}(p,\varepsilon)}$ for $0 \le i \le p-1$, where

$$S_{p,R}(d,\alpha) := B(R) \cup S_p(d,\alpha), \qquad S_p(d,\alpha) = \bigcup_{i=0}^{p-1} S\left(d + \frac{2\pi i}{p}, \alpha\right)$$

for $\alpha > 0$.

Now, we can regard $v(s, x)$ as a unique solution in a neighborhood at $(s, x) = (0, 0)$ of the following Cauchy problem with respect to x direction

$$\begin{cases} \partial_s^p \widetilde{\delta_s}^{q-p-1} v(s,x) = \partial_x^q v(s,x), \\ \partial_x^j v(s,0) = \phi_j(s) \in \mathcal{O}_s \quad (0 \le j \le q-1), \end{cases} \tag{5.3}$$

for some functions $\phi_j(s)$. Here we remark that the equation (5.3) is of Kowalevski type for partial differential equations.

In this case, we can assume that $\phi_j(s) \in \mathrm{Exp}(k(1); S_{p,R}(d, \varepsilon_1 + \pi/p))$ for all j.

We put

$$w(\tau,x) := (\mathcal{B}_p v)(\tau,x) = (\hat{\mathcal{B}}_p^{q-p}\hat{u})(\tau,x) = \sum_{n\ge 0} \frac{\varphi^{(qn)}(x)}{(pn)!}\frac{\tau^{pn}}{n!^{q-p}}. \tag{5.4}$$

Then $w(\tau, x)$ satisfies the following Cauchy problem

$$\begin{cases} \partial_\tau^p \widetilde{\delta_\tau}^{q-p-1} \circ (\widetilde{\delta_\tau}) w(\tau,x) = \partial_x^q w(\tau,x), \\ w(0,x) = \varphi(x) \in \mathcal{O}(\mathbb{C}), \\ \partial_\tau^i w(0,x) = 0 \quad (1 \le i \le p-1). \end{cases} \tag{5.5}$$

For this $w(\tau, x)$, we can regard $w(\tau, x)$ as a unique solution in a neighborhood at $(\tau, x) = (0, 0)$ of the following Cauchy problem with respect to x direction

$$\begin{cases} \partial_\tau^p \widetilde{\delta_\tau}^{q-p} w(\tau,x) = \partial_x^q w(\tau,x), \\ \partial_x^j w(\tau,0) = \psi_j(\tau) \quad (0 \le j \le q-1). \end{cases} \tag{5.6}$$

Here for $0 \le j \le q-1$, $\psi_j(\tau)$ are given by p-Borel transform of $\phi_j(s)$

$$\psi_j(\tau) = (\mathcal{B}_p\phi_j)(\tau) = \frac{-p}{2\pi i}\int_{\gamma_p(\tilde{d})} \phi_j(s) e^{\left(\frac{\tau}{s}\right)^p}\frac{ds}{s}, \tag{5.7}$$

where $\tilde{d} \in \mathbb{R}$ is arbitrary. In this case, since $\phi_j(s) \in \mathcal{O}_s$, we see that $\psi_j(\tau) \in \mathrm{Exp}(p; \mathbb{C})$ for all j. Moreover, since $\phi_j(s) \in \mathrm{Exp}(k(1); S_{p,R}(d, \varepsilon_1 + \pi/p))$ $(0 \le j \le q-1)$, we have the following lemma, which is immediately obtained from the property of Borel transform (cf. [1]).

Lemma 11

$$\psi_j(\tau) \in \mathrm{Exp}(p/(q-p); S_p(d, \varepsilon_1')) \quad (0 \le j \le q-1), \tag{5.8}$$

for a sufficiently small ε'_1.

Let us prove $\varphi(x) = w(0, x) \in \mathrm{Exp}(q/(q-p), \Omega_q^p(d, \varepsilon'))$ under the assumptions (5.8).

In the following, we write $\psi_0(\tau)$ by $\psi(\tau)$ and we assume that $\psi_j(\tau) = 0$ $(1 \le j \le q-1)$ without loss of generality.

Since $w(\tau, x)$ satisfies the Cauchy problem (5.6), by putting $w(\tau, x) = \sum_{n\ge 0} w_n(\tau)x^n/n!$, we have

$$w(\tau,x) = \sum_{n\ge 0} (\partial_\tau^p \widetilde{\delta_\tau}^{q-p})^n \psi(\tau)\frac{x^{qn}}{(qn)!}.$$

Therefore since $(\partial_\tau^p \widetilde{\delta_\tau}^{q-p})^n \psi(0) = \psi^{(pn)}(0) n!^{q-p}$ after some calculations, we have

$$\begin{aligned}\varphi(x) &= w(0,x) = \sum_{n\geq 0} \frac{\psi^{(pn)}(0)n!^{q-p}}{(qn)!}x^{\nu n} \\ &= \frac{1}{2\pi i}\oint_{|\tau|=\rho} \frac{\psi(\tau)}{\tau}\, {}_qF_{q-1}\left(\begin{matrix} 1/p, 2/p, \dots, p/p, \mathbf{1}_{q-p} \\ 1/q, 2/q, \dots, (q-1)/q \end{matrix}; \frac{p^p}{q^q}\frac{x^q}{\tau^p} \right) d\tau\end{aligned}$$

for some positive ρ, where $\mathbf{1}_{q-p} = (1, \dots, 1) \in \mathbb{N}^{q-p}$. From the properties of an analytic continuation and the singularity of the hypergeometric function, we can obtain the desired estimate for $\varphi(x)$ (cf. [5, 6, 2]).

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