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Morphism Approach to Tonality

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1. The margin of maximal evenness ansatz in tonal music theory

Under the maximal evenness ansatz, we have been exploring combinatorial aspect of tonal music through the series of studies [3][4][5][6][7][8][9], where we always treat a chromatic scale of c tones and diatonic scales of d tones, named the (c, d)-system, and we have seen that some features of tonality arise from combinatorics on maximal evenness. Particularly we have shown that any general diatonic scales in the (c, d)-system have an analog of the tritone interval [7][8], giving us a hint on a combinatorial ground for tension/relief progressions in tonal music.

In those studies, we treat maximal even objects described by the J-function as follows.

Definition 1.1 (J-function by Clough and Douthett[1]). For $c, d, m \in \mathbf{Z}$ with $c > d > 0$, the *J-function* on $\mathbf{Z}$ is defined as

$$J^m_{c,d}(k) = \left\lfloor \frac{ck+m}{d} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. $\mathcal{J}^m_{c,d}$ stands for the tuple $(J^m_{c,d}(k))_{k=0,\ldots,d-1}$ and called *J-representation*.

The usual tonal music is established in (12, 7)-system. The chromatic scale

$$(C, C^\sharp, D, D^\sharp, E, F, F^\sharp, G, G^\sharp, A, A^\sharp, B).$$

is identified with (0, 1, . . . , 11). The diatonic scales are given as a sub tuple of Ch_{12}, the C-major scale (C, D, E, F, G, A, B) for instance is identified with (0, 2, 4, 5, 7, 9, 11) which coinsides with a maximal even object $\mathcal{J}^5_{12,7}$. The diatonic triad like CEG is identified with (0, 4, 7) which coincides with a multi-maximal even object $(J^5_{12,7}(J^0_{7,3}(k)))_{k=0,1,2}$. In other words, CEG is a maximal even object in C-major scale.

However, maximal even objects are not enough to describe tonal music theory. The C-natural minor scale ($C, D, E^\flat, F, G, A^\flat, B^\flat$) coincides with a maximal even object $\mathcal{J}^2_{12,7}$ = (0, 2, 3, 5, 7, 8, 10), as C-natural minor scale is the relative scale to the $E^\flat$-major scale. The C-harmonic minor scale ($C, D, E^\flat, F, G, A^\flat, B$) and C-melodic minor scale ($C, D, E^\flat, F, G, A, B$), however, are not maximal even objects. These scales are understood as modifications of the natural minor scale from the tonal music theoretical view. The suspended chord like CFG is not a maximal even object in C-major scale: by extending the C-major scale to two octaves, $CFA\dot{D}\dot{G}$ forms a maximal even object $\mathcal{J}^1_{14,5}$ = (0, 3, 5, 8, 11) in the scale, then GCF is understood as a partial tuple of an inversion $GCFA\dot{D}$, which is quite strained interpretation. In tonal music theory, CFG is understood as a modulation of a maximal even object CEG in C-major scale.

Thus, to advance our combinatorial approach to tonal music theory toward the next stage, modifications of maximal even objects will be the main subject of our studies.

2. Morphism approach

2.1. **Notations and definitions.** It is noticeable that tonal music objects should be described by tuples of notes rather than sets, such as scales like C-major scale (C, D, E, F, G, A, B) or chords like CEG typically. In tonal music theory, each entry of such tuples plays important role, and the role changes by its position in the tuple. For instance, the first entry C of C-major scale works as the keynote of the scale, and the first entry C of the triad CEG gives the predominant color of the chord. Thus we prepare the followings.

Notation 2.1. For a tuple $\boldsymbol{a} = (a_i)_{i\in I}$ for some index set I, $|\boldsymbol{a}|$ denotes the set $\{a_i | i \in I\}$ consists of all entries of $\boldsymbol{a}$, and we often use abbreviation $b \in \boldsymbol{a}$ for $b \in |\boldsymbol{a}|$, like $c \in (c, e, g)$. For a natural number $n \in \mathbf{N}$, we set a tuple $\boldsymbol{n} = (0, 1, \ldots, n-1)$ [1]. **(Z)**

stands for the bi-infinite tuple $(\ldots, -2, -1, 0, 1, 2, \ldots)$ of which the position i is i. For $n \in \mathbf{N}$ and $p \in \mathbf{Z}$, $\boldsymbol{n} + p$ stands for the translation $(p, p+1, \ldots, p+n-1)$.

Definition 2.2. For $m \in \mathbf{N}$ and $p \in \mathbf{Z}$, we say a map $f : |\boldsymbol{m}+p| \to \mathbf{Z}$ is *order preserving* whenever $f(i) < f(j)$ holds for any $p \leq i < j < m + p - 1$. Then we put

$$\hom(\boldsymbol{m}+p, \boldsymbol{n}+q) = \{f : |\boldsymbol{m}+p| \to |\boldsymbol{n}+q|, \text{ order preserving}\} \text{ and}$$
$$\hom(\boldsymbol{m}+p, (\mathbf{Z})) = \{f : |\boldsymbol{m}+p| \to \mathbf{Z}, \text{ order preserving}\}$$

for natural numbers $m \leq n$ and integers p, q. We often identify the map $f \in \hom(\boldsymbol{m}+p, *)$ ($* = \boldsymbol{n}+q$ or $(\mathbf{Z})$) with the tuple $f(\boldsymbol{m}+p) = (f(p), f(p+1), \ldots, f(m+p-1))$. $\hom((\mathbf{Z}), (\mathbf{Z}))$ is the set of order preserving maps $f : |\mathbf{Z}| \to |\mathbf{Z}|$.

The *variation* $v(f)$ of $f \in \hom(\boldsymbol{m}+p, *)$ is defined by

$$v(f) = \max f - \min f = f(m+p-1) - f(p).$$

Then we put for an integer $n \geq m$,

$$\hom_n(\boldsymbol{m}+p, *) = \{f : |\boldsymbol{m}+p| \to *, \text{ order preserving}, \ v(f) < n\}.$$

We identify $\hom_n(\boldsymbol{m}+p, \boldsymbol{n}+q)$ with $\hom(\boldsymbol{m}+p, \boldsymbol{n}+q)$.

The *periodic extension* $\widetilde{f} \in \hom((\mathbf{Z}), (\mathbf{Z}))$ *of* $f \in \hom_n(\boldsymbol{m}+p, *)$ *with period* n is the map given by

$$\widetilde{f}(a) = kn + f(b) \text{ for } a = km + b, \ p \leq b < m + p, \ a, b, k \in \mathbf{Z}.$$

For an integer r, $f^{(r)} \in \hom_n(\boldsymbol{m}+p, *)$ stands for the *rth inversion* (under the period n) of $f \in \hom_n(\boldsymbol{m}+p, *)$ defined by

$$f^{(r)}(\boldsymbol{m}+p) = \widetilde{f}(\boldsymbol{m}+p+r) = (\widetilde{f}(p+r), \widetilde{f}(1+p+r), \ldots, \widetilde{f}(m-1+p+r))$$

where $\widetilde{f}$ is the periodic extension of f with period n.

An *encoding* of a tuple $\boldsymbol{n}$ is a bijection $\theta : |\boldsymbol{n}| \to |\boldsymbol{\Lambda}|$ for some tuple $\boldsymbol{\Lambda} = (\lambda_0, \ldots, \lambda_{n-1})$ with $\theta(i) = \lambda_i$ for any i, and we abbreviate these correspondence as $\theta : \boldsymbol{n} \to \boldsymbol{\Lambda}$. The periodic extension $\widetilde{\theta}$ with period n of the encoding θ is the lift of θ along the projection $\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$, that is, by identifying $\boldsymbol{n}$ with $\mathbf{Z}/n\mathbf{Z}$, $\widetilde{\theta}$ is the composition

$$\widetilde{\theta} : \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} = |\boldsymbol{n}| \xrightarrow{\theta} |\boldsymbol{\Lambda}|.$$

Let $\widetilde{f}$ be the periodic extension of $f \in \hom_n(\boldsymbol{m}, *)$ with period n and take integers $a < a'$. If $\left\lfloor \frac{a}{m} \right\rfloor = \left\lfloor \frac{a'}{m} \right\rfloor = k$, taking $0 \leq b = a - km < b' = a' - km < m$, we see

$$\widetilde{f}(a) = kn + f(b) < kn + f(b') = \widetilde{f}(a')$$

by the definition of f, while, if $k = \left\lfloor \frac{a}{m} \right\rfloor < \left\lfloor \frac{a'}{m} \right\rfloor = k'$, taking $b = a - km$, $b' = a' - k'm$, we also see

$$\widetilde{f}(a') - \widetilde{f}(a) = (k'-k)n + f(b') - f(b) > n - v(f) \geq 0$$

by the assumption. Thus $\widetilde{f} \in \hom((\mathbf{Z}), (\mathbf{Z}))$ is valid. Let us consider the rth inversion $f^{(r)}$ of $f \in \hom_n(\boldsymbol{m}, *)$. As the extension $\widetilde{f}$ preserves the order, also $f^{(r)}$ does. Taking integers $k, k', 0 \leq b, b' < m$ such that

$$m + t - 1 = mk' + b', \ t = mk + b,$$

we see $m - 1 = m(k' - k) + b' - b$, hence either $k' = k$, $b' = m - 1$, $b = 0$ or $k' = k + 1$, $b = b' + 1$ occurs. Then we see

$$\begin{aligned} v(f^{(r)}) &= \widetilde{f}(m+r-1) - \widetilde{f}(r) = n(k'-k) + f(b') - f(b) \\ &= \begin{cases} f(b') - f(b) < n, & \text{if } k' = k, \\ n + f(b-1) - f(b) < n, & \text{if } k' = k+1, \end{cases} \end{aligned}$$

hence the validity of $f^{(r)} \in \hom_n(\boldsymbol{m}, *)$. Putting $g(k-p) = f(k)$ for $f \in \hom_n(\boldsymbol{m}+p, *)$, one see $g \in \hom_n(\boldsymbol{m}, *)$, hence the validity $\widetilde{g} \in \hom((\mathbf{Z}), (\mathbf{Z}))$ and $(g)^{(r)} \in \hom_n(\boldsymbol{m}, *)$ come down to f.

2.2. **General settings.** As usual in our study, we first prepare the chromatic scale: the background system of all musical objects.
Definition 2.3. The *general chromatic scale* $Ch_c = (\boldsymbol{c}, \theta)$ consists of a tuple $\boldsymbol{c} = (0, 1, 2, \ldots, c-1)$ and an encoding $\theta : \boldsymbol{c} \to \boldsymbol{\Lambda}$, for some tuple $\boldsymbol{\Lambda}$ with $\#|\boldsymbol{\Lambda}| = c$. The periodic extension $\widetilde{Ch_c} = ((\mathbf{Z}), \widetilde{\theta})$ is a pair of the bi-infinite tuple $(\mathbf{Z})$ and the periodic extension $\widetilde{\theta}$ of θ with period c.

We often identify $\widetilde{Ch_c}$ and its entry $(\mathbf{Z})$; $\hom_c(\boldsymbol{n}, \widetilde{Ch_c})$ stands for the set of order preserving maps $\boldsymbol{n} \to (\mathbf{Z})$ with the variation smaller than c. We say that a and b in $\widetilde{Ch_c}$ belong to the *same pitch class* if $\widetilde{\theta}(a) = \widetilde{\theta}(b)$.

Definition 2.4. For a natural number d with $2 \le d < c$,[2] a *general scale* is a map $S \in \hom_c(\boldsymbol{d}, \widetilde{Ch_c})$ with $v(S) < c$. For a natural number e with $2 \le e \le d$, a *general chord* belongs to a scale S is a composition of a map $C \in \hom(\boldsymbol{e}, (\mathbf{Z}))$ and the periodic extension $\widetilde{S}$ of S with period c,

$$\widetilde{S} \circ C \in \hom(\boldsymbol{e}, \widetilde{Ch_c}).$$

Any objects $f, g \in \hom_c(\boldsymbol{m}, (\mathbf{Z}))$ are called *octave equivalent* whenever $f(\boldsymbol{m}) + pc = g(\boldsymbol{m})$ holds for some integer p, and we write $f \equiv g$.

2.3. **maximal even objects.** For the general chromatic scale Ch_c, taking an integer k and a natural number $2 \le d < c$ prime to c, a tuple $\mathcal{J}^m_{c,d} = (J^m_{c,d}(k))_{k=0,\ldots,d-1}$ is called a *general diatonic scale*, identified with a map

$$J^m_{c,d} \in \hom(\boldsymbol{d}, \boldsymbol{c}).$$

A *general diatonic chord* is a general chord belonging to a general diatonic scale $\mathcal{J}^m_{c,d}$ which forms a double maximal even object, that is, a composition of the scale $J^m_{c,d}$ and maximal even object $J^n_{d,e} \in \hom(\boldsymbol{e}, \boldsymbol{d})$,

$$J^m_{c,d} \circ J^n_{d,e} \in \hom(\boldsymbol{e}, \boldsymbol{c}).$$

These maximal even or multi-maximal even objects play central role in usual tonal music. Therefore we call the system equipped with a general chromatic scale Ch_c and diatonic scales $\mathcal{J}^m_{c,d}$'s the (c, d)-*system*.

2.4. **The** $(12, 7)$**-system.** Usual tonal music are composed or performed on diatonic scales $\mathcal{J}^m_{12,7}$'s consist of 7 pitch classes chosen from the chromatic scale Ch_{12} of 12 pitch classes; the $(12, 7)$-system. We take the encoding

$$\theta : Ch_{12} = (0, 1, \ldots, 11) \to (C, C^\sharp, D, D^\sharp, E, F, F^\sharp, G, G^\sharp, A, A^\sharp, B).$$

Diatonic scales, say, the C-major scale is given by $J^5_{12,7} \in \hom(\mathbf{7}, Ch_{12})$, realized as

$$\mathbf{7} = (0,1,2,3,4,5,6) \xrightarrow{J^5_{12,7}} \mathcal{J}^5_{12,7} = (0,2,4,5,7,9,11) \xrightarrow{\theta} (C, D, E, F, G, A, B),$$

where the tonal center (called the *tonic* in music theory) C appears at the position 0 of the tuple. The G-major scale $(G, A, B, C, D, E, F^\sharp)$, however, realized as

$$\mathbf{7} = (0,1,2,3,4,5,6) \xrightarrow{J^6_{12,7}} \mathcal{J}^6_{12,7} = (0,2,4,6,7,9,11) \xrightarrow{\theta} (C, D, E, F^\sharp, G, A, B),$$

where the tonic G appears at the position 4, so we need the 4th inversion

$$\mathbf{7} + 4 = (4,5,6,7,8,9,10) \xrightarrow{J^6_{12,7}} (7, 9, 11, 12, 14, 16, 18) \xrightarrow{\widetilde{\theta}} (G, A, B, C, D, E, F^\sharp).$$

Thus $(J^6_{12,7})^{(4)} \in \hom_{12}(\mathbf{7}, \widetilde{Ch_{12}})$ corresponds to G-major scale $(G, A, B, C, D, E, F^\#)$ in which the tonic of the G-major scale appears at the position 0. Generally we have the equation in the (c, d)-system,

$$(\mathcal{J}^{m+\mu}_{c,d})^{(-\mu c^-)} = \mathcal{J}^m_{c,d} + \mu d^- \tag{2.1}$$

where $-d < c^- < 0$ and $0 < d^- < c$ are the unique solution of $c \cdot c^- + d \cdot d^- = 1$ (See [4] for details). We see $12 \cdot (-4) + 7 \cdot 7 = 1$, hence $c^- = -4$ and $d^- = 7$. Applying (2.1) to the (12, 7)-system, we have

$$(\mathcal{J}^{5+\mu}_{12,7})^{(4\mu)} = \mathcal{J}^5_{12,7} + 7\mu,^{3)} \tag{2.2}$$

which shows the tonic of the major scale in key $5 + \mu$ appear at the position 0 for any μ.

We note inversions $(\mathcal{J}^5_{12,7})^{(r)}$'s of C-major scale stand for the *Church modes*, such as

$$\left(\mathcal{J}^5_{12,7}\right)^{(1)} = (D, E, F, G, A, B, C):\ \text{Dorian},\ \left(\mathcal{J}^5_{12,7}\right)^{(2)} = (E, F, G, A, B, C, D):\ \text{Phrygian, etc.},$$

where the tonic of each mode appears at the position 0. Particularly, the Aeolian mode

$$\left(\mathcal{J}^5_{12,7}\right)^{(5)} = (A, B, C, D, E, F, G)$$

coincides with the natural minor scale.

The triad C = *CEG* belongs to the *C*-major scale is realized as $J^5_{12,7} \circ J^0_{7,3} \in \hom(\mathbf{3}, Ch_{12})$,

$$\mathbf{3} = (0,1,2) \xrightarrow{J^0_{7,3}} \mathcal{J}^0_{7,3} = (0,2,4) \xrightarrow{J^5_{12,7}} (0,4,7) \xrightarrow{\theta} (C, E, G),$$

where the *root* of the chord *C* appears at the the position 0. The dominant 7th chord G_7 = *GBDF* belongs to the *C*-major scale is realized as $J^5_{12,7} \circ J^5_{7,4} \in \hom(\mathbf{4}, Ch_{12})$,

$$\mathbf{4} = (0,1,2,3) \xrightarrow{J^5_{7,4}} \mathcal{J}^5_{7,4} = (1,3,4,6) \xrightarrow{J^5_{12,7}} (2,5,7,11) \xrightarrow{\theta} (D, F, G, B),$$

where the root *G* of the chord appears at the position 2, thus we need the second inversion

$$\mathbf{4} + 2 = (2,3,4,5) \xrightarrow{J^5_{7,4}} (4,6,8,10) \xrightarrow{J^5_{12,7}} (7,11,14,17) \xrightarrow{\widetilde{\theta}} (G, B, D, F),$$

thus $J^5_{12,7}\ (J^5_{7,4})^{(2)} \in \hom_{12}(\mathbf{4}, \widetilde{Ch_{12}})$ corresponds to *GBDF* .

3. Modifications from maximal even objects

As is stated, maximal even objects are incomplete to describe tonal music, even so, their modifications cover the large part of tonal objects. Roughly speaking, those modifications are classified into two cases: modifying a scale itself and modifying a chord in the given scale. The modification from the natural minor scale to the harmonic minor one illustrates the former case. The suspended chord C_{sus4} = *CFG* is regarded as a modification from C = *CEG* in the *C*-major scale, which illustrates the latter case. From our morhism viewing, it is said that these modifications occurs before composing a scale or after.

Notation 3.1. For a map $f \in \hom_n(\boldsymbol{m} + p, *)$, $p \in \mathbf{Z}$, we define operators [4]

$$(I_q f)(k) = \widetilde{f}(k+q), \quad (E_q f)(k) = \widetilde{f}(qk), \quad (af+b)(k) = a \cdot f(k) + b,$$
$$(M_i^{\pm} f)(k) = \begin{cases} f(k) \pm 1, & \text{if } k = i, \\ f(k), & \text{otherwise,} \end{cases}$$

for $a, b, q \in \mathbf{Z}$ and $i \in |\boldsymbol{m} + p|$, where $\widetilde{f}$ is the extension of f with period n. We use the abbreviation

$$M^{\varepsilon_1\varepsilon_2\cdots\varepsilon_r}_{i_1,i_2,\ldots,i_r} = M^{\varepsilon_1}_{i_1} M^{\varepsilon_2}_{i_2} \cdots M^{\varepsilon_r}_{i_r}.$$

Note that $I_q f$ is nothing but the qth inversion $(f)^{(q)} \in \hom_n(\boldsymbol{m} + p, (\mathbf{Z}))$. One see $E_q f \in \hom_{|qn|}(\boldsymbol{m} + p, (\mathbf{Z}))$ and $af + b \in \hom_{|an|}(\boldsymbol{m} + p, (\mathbf{Z}))$, while $M_i^{\pm} f$ fails to preserve the order whenever $f(i) \pm 1 = f(i \pm 1)$. One also see for $f \in \hom_n(\boldsymbol{m} + p, *)$,

$$(I_{rm} f)(k) = \widetilde{f}(k + rm) = rm + f(k),$$

hence $I_{rm} f \equiv f$. For $a, b \in \mathbf{Z}$ and $f, g \in \hom((\mathbf{Z}), (\mathbf{Z}))$, we have the trivial identity

$$(I_b E_a f) \circ g = f \circ (ag + b),$$

and we use both descriptions to express the meaning of the modification: the modification occurs on f or g. Such a distinction permits us to describe modifications along musical context by morphisms.

3.1. **Modifications of scales.** The *A*-natural minor scale (*A, B, C, D, E, F, G*) is described as a maximal even scale $(\mathcal{J}^5_{12,7})^{(5)} = (I_5 J^5_{12,7})(\mathbf{7})$. As the interval *G* to *A* has double semitones, *G* has less nasty to the tonic *A* (therefore *G* is called the *subtonic* or the *natural leading tone*). Thus *G* slides up to $G^\sharp$ closer to *A*, hence the *A*-harmonic minor scale (*A, B, C, D, E, F,* $G^\sharp$), represented by $M_6^+ I_5 J^5_{12,7}$. However, $G^\sharp$ is three semitones far from *F*, which causes a lack of smoothness of the scale. Then *F* also slides up to $F^\sharp$, hence the *A*-melodic minor scale (*A, B, C, D, E,* $F^\sharp$, $G^\sharp$) represented by $M^{++}_{5,6} I_5 J^5_{12,7}$.

The relation (2.2) is represented as

$$I_{4\mu}J_{12,7}^{5+\mu}=J_{12,7}^{5}+7\mu,$$

that shows changing the key from the C-major scale $\mathcal{J}_{12,7}^{5}$. The note being a single semitone higher or lower to the tonic of the scale is called the *leading-tone* in music theory, and the change of key occurs by exchanging leading-tones (*Leittonwechsel* in Hugo Riemann's theory). If F in the C-major scale slides up to $F^{\#}$, the G-major scale appears,

$$I_4M_3^{+}J_{12,7}^{5}=I_4M_3^{+}(0,2,4,5,7,9,11)^{5)}=(7,9,11,12,14,16,18)=I_4J_{12,7}^{6},$$

while, if B in the C-major scale slides down to $B^{\flat}$, the F -major scale appears,

$$I_3M_6^{-}J_{12,7}^{5}=I_3M_6^{-}(0,2,4,5,7,9,11)=(5,7,9,10,12,14,16)=I_3J_{12,7}^{4}.$$

Generally we have identities for $\mu\in\mathbf{N}$,

$$(I_4M_3^{+})^{\mu}J_{12,7}^{5}=J_{12,7}^{5}+7\mu=I_{4\mu}J_{12,7}^{5+\mu},$$

and

$$(I_3M_6^{-})^{\mu}J_{12,7}^{5}=J_{12,7}^{5}\equiv J_{12,7}^{5}-7\mu=I_{-4\mu}J_{12,7}^{5-\mu}\equiv I_{3\mu}J_{12,7}^{5-\mu},$$

illustrating the relation between the leading-tone exchange and the change of keys.

3.2. **Modifications of chords.** Inversions of chords, like $EGC=\widetilde{\theta}((4,\ 7,\ 12))$ or $GCE=\widetilde{\theta}((7,\ 12,\ 16))$ of C-major triad C = $CEG=\theta((0,\ 4,\ 7))$, are modifications inside the given scales, such as

$$J_{12,7}^{5}\circ(I_1J_{7,3}^{0})(\mathbf{3})=J_{12,7}^{5}((2,4,7))=(4,7,12)$$

or

$$J_{12,7}^{5}\circ(I_2J_{7,3}^{0})(\mathbf{3})=J_{12,7}^{5}((4,7,9))=(7,12,16).$$

The suspended chords are also given as modifications inside the scale. $\mathrm{C_{sus4}}=CFG=\theta((0,\ 5,\ 7))$ is obtained from the triad C by moving E to F , as $J_{12,7}^{5}\circ(M_1^{+}J_{7,3}^{0})$;

$$\left(\mathbf{3}\xrightarrow{J_{7,3}^{0}}(0,2,4)\right)\overset{M_1^{+}}{\rightsquigarrow}(0,3,4)\xrightarrow{J_{12,7}^{5}}(0,5,7)\xrightarrow{\theta}(C,F,G),\tag{3.1}$$

where $(*\xrightarrow{f}*)\overset{A}{\rightsquigarrow}$ stands for the modification of f by A. Similar way, $\mathrm{C_{sus2}}=CDG=\theta((0,\ 2,\ 7))$ is given as a modification of the triad C by moving E to D, as $J_{12,7}^{5}\circ(M_1^{-}J_{7,3}^{0})$;

$$\left(\mathbf{3}\xrightarrow{J_{7,3}^{0}}(0,2,4)\right)\overset{M_1^{-}}{\rightsquigarrow}(0,1,4)\xrightarrow{J_{12,7}^{5}}(0,2,7)\xrightarrow{\theta}(C,F,G).$$

3.3. **Stacking 3rds process, chords with tensions and omitted notes.** Contrast to triads and 7th chords, the chords with tension notes, like the major 9th chord $\mathrm{C_{\Delta 9}}=CEGBD$ lying in two octave, do not form maximal even objects. Such chords are understood by the stacking 3rds process in tonal music theory; take the root note, and repeat stacking major or minor 3rd intervals on the previous note. The morphism approach permit us to reproduce the process by the maximal even object $J_{24,7}^{m}\in\mathrm{hom}_{24}(\mathbf{7},(\mathbf{Z}))$, say for $m=5$,

$$\mathbf{7}=(0,1,2,3,4,5,6)\xrightarrow{J_{24,7}^{5}}(0,4,7,11,14,17,21)\xrightarrow{\widetilde{\theta}}(C,E,G,B,D,F,A).$$

Then, the triad C = CEG, the major 7th $\mathrm{C_{\Delta 7}}=CEGB$, and the major 9th $\mathrm{C_{\Delta 9}}=CEGBD$ are represented by $J_{24,7}^{5}(\mathbf{3})$, $J_{24,7}^{5}(\mathbf{4})$ and $J_{24,7}^{5}(\mathbf{5})$ respectively, which are partial tuples of the maximal even object $\mathcal{J}_{24,7}^{5}=J_{24,7}^{5}(\mathbf{7})$. Also chords with added tension notes like $\mathrm{C^{add9}}=CEGD$ or omitted chords like $\mathrm{C_{\Delta 7}}^{\mathrm{omit5}}=CEB$ are represented as a partial tuple of $\mathcal{J}_{24,7}^{5}$, $J_{24,7}^{5}((0,\ 1,\ 2,\ 4))$ and $J_{24,7}^{5}((0,\ 1,\ 3))$ respectively.

According to our morphism approach, all stacking 3rds chords should be described by morphisms: these chords are realized by compositions $(I_qJ_{24,7}^{m})\circ\gamma$ where $\gamma\in\mathrm{hom}(\boldsymbol{l},\ \mathbf{7})$, $l\leq 7$, for instance,

$$\mathrm{Dm9^{omit5}} = DFCE = \widetilde{\theta}((I_4 J^5_{24,7}) \circ \gamma_1), \quad \gamma_1 : \mathbf{4} \to (0,1,3,4) \in \hom(\mathbf{4},\mathbf{7}),$$
$$\mathrm{G^{add9,13}} = GBDAE = \widetilde{\theta}((I_2 J^5_{24,7}) \circ \gamma_2), \quad \gamma_2 : \mathbf{5} \to (0,1,2,4,6) \in \hom(\mathbf{5},\mathbf{7}).$$

Note that the stacking 3rds process $J^m_{24,7}$ itself is represented as the modification of the diatonic scale $J^m_{12,7}$,

$$J^m_{24,7}(k) = \left\lfloor \frac{12 \cdot 2k + m}{7} \right\rfloor = E_2 J^m_{12,7}(k),$$

hence unified representations $(I_4 E_2 J^5_{12,7}) \circ \gamma_1$ and $(I_2 E_2 J^5_{12,7}) \circ \gamma_2$.

4. Combinatorial tonality

Some chords have various interpretations. As is stated before, the suspended chord, e.g. $\mathrm{C_{sus4}}$ = *CFG* in *C*-major scale is understood as a modification of the triad C = *CEG*; the third note *E* is suspended at *F* . In that sense, the chord is represented as $J^5_{12,7} \circ (M^+_1 J^0_{7,3})$ (see (3.1)). $\mathrm{C_{sus4}}$ is also interpreted as the second inversion of $\mathrm{F_{sus2}}$ = *FGC*, given by suspending the third note *A* at *G* of another triad F = *FAC*,

$$\left(\left(\mathbf{3} \xrightarrow{J^0_{7,3}} (0,2,4)\right) \overset{M^-_1}{\rightsquigarrow} (0,1,4) \xrightarrow{I_3 J^5_{12,7}} (5,7,12)\right) \overset{I_2}{\rightsquigarrow} (12,17,19) \xrightarrow{\widetilde{\theta}} (C,F,G),$$

represented as $I_2((I_3 J^5_{12,7}) \circ (M^-_1 J^0_{7,3}))$.

Let us consider the progression

$$(C,F,G) \to (C,F,A) \to (C,F,G) \to (C,E,G).$$

From musical context viewing, this progression is understood as

$$\mathrm{F_{sus2}} \to \mathrm{F} \to \mathrm{C_{sus4}} \to \mathrm{C}$$

naturally, and corresponding morphism description is give by

$$I_2((I_3 J^5_{12,7}) \circ (M^-_1 J^0_{7,3})) \to I_2((I_3 J^5_{12,7}) \circ (J^0_{7,3})) \to J^5_{12,7} \circ (M^+_1 J^0_{7,3}) \to J^5_{12,7} \circ J^0_{7,3}.$$

Therefore, the meaning of such chords are not determined by themselves, which depends on the musical context. Such ambiguity is called *polysemy* of chords.

4.1. **The root of a chord and polysemy.** To discuss the polysemy, we first pay an attention to combinatorial handling of tonics and roots. The tonic of a scale and the root of a chord are the principal concept in tonal music. When a chord is heard, the top note (the highest pitch in the chord) and the bottom note (the lowest one) affect our auditory impressions primarily, in particular, the bottom note dominates the impressions. Historically, harmony comes from the overtone structure of musical tones: a major triad is formed by stacking the 3rd (equivalent to the perfect 5th) and the 5th (the major 3rd) harmonics on the fundamental frequency called the root of the chord. A minor triad also consists of a perfect 5th and a minor 3rd intervals, so diatonic chords except diminished ones contains perfect 5th intervals. On the other hand, it is well known that the perfect 5th interval generates the diatonic scale, which comes from the Pythagorean tuning. The fact is generalized to (c, d)-system for c and d prime to each other, where the multiplicative inverse d^- of d in $\mathbf{Z}/c\mathbf{Z}$ corresponds to the original perfect 5th interval (see [3][5]).

We have shown in [8] that $(12, 7)$-system is naturally generalized to (c, d)-system with $2d - c = 2$, where the general tritones have the rigidity 2, hence we have found tritone substitutions in the (c, d)-system. When $2d - c = 2$ holds for c and d prime to each other, one see c is even and d is odd, hence

$$d^2 - 1 = (d-1)(d+1) \equiv 0 \pmod{c},$$

i.e., $d^2 \equiv 1 \pmod{c}$. Therefore it can be said that the interval $d = d^-$ corresponds to the perfect 5th, so we call the *perfect d interval*. Under these conditions, we have

$$J^m_{c,d}(k) + d = \left\lfloor \frac{ck + m + d^2}{d} \right\rfloor = \left\lfloor \frac{c(k - c^-) + m + 1}{d} \right\rfloor = J^{m+1}_{c,d}(k - c^-),$$

where $-d < c^- < 0$ is a unique solution of $cc^- + d^2 = 1$, actually we see $-c^- = \dfrac{d+1}{2}$, and when $c(k - c^-) + m + 1 \not\equiv 0$ (mod d), we have

$$J^m_{c,d}(k) + d = J^{m+1}_{c,d}(k - c^-) = J^m_{c,d}(k - c^-).$$

Thus, to realize a diatonic chord containing a perfect d interval as a multi-maximal even object $J^m_{c,d}(\mathcal{J}^n_{d,e})$, $\mathcal{J}^n_{d,e}$ should contain k and $k - c^-$ for some $k \in |\boldsymbol{e}|$, that is, the equation

$$J^n_{d,e}(l) - c^- = J^n_{d,e}(l + p)$$

holds for some l and p. We note that, at least the dyad $\mathcal{J}^0_{d,2}$ forms the interval of the length $-c^-$,

$$I_1 J^0_{d,2}(\mathbf{2}) = \left(\frac{d-1}{2}, d\right), \qquad d - \frac{d-1}{2} = -c^-,$$

which forms the perfect d interval in the scale $\mathcal{J}^m_{c,d}$,

$$\begin{aligned} I_1(J^m_{c,d} \circ J^0_{d,2})(\mathbf{2}) &= J^m_{c,d}\left(I_1 J^0_{d,2}(\mathbf{2})\right) = \left(\left\lfloor \frac{c(d+c^-)+m}{d} \right\rfloor, \left\lfloor \frac{cd+m}{d} \right\rfloor\right) \\ &= \left(c - d + \left\lfloor \frac{m+1}{d} \right\rfloor, c + \left\lfloor \frac{m}{d} \right\rfloor\right) = \left(c - d + \left\lfloor \frac{m}{d} \right\rfloor, c + \left\lfloor \frac{m}{d} \right\rfloor\right) \end{aligned}$$

whenever $m + 1 \not\equiv 0$ (mod d).

Proposition 4.1. *Let c, d be integers where $2 < d < c$, $c = 2(d - 1)$ and d is odd. Then the dyad*

$$I_1(J^m_{c,d} \circ J^0_{d,2})(\mathbf{2})$$

forms a perfect d interval in the scale $\mathcal{J}^m_{c,d}$ unless $m + 1 \equiv 0$ (mod d).

Viewing the fact that as G is the perfect 5th above C in $(12, 7)$-system, the root of the dyad CG is determined to C, so it can be said that $J^m_{c,d} \circ (I_1 J^0_{d,2})(0)$ is the root of the dyad $J^m_{c,d}\,(I_1 J^0_{d,2}(\mathbf{2}))$. However, if a maximal even object $J^m_{c,d} \circ J^n_{d,e}\,(\boldsymbol{e})$ contains several perfect d intervals, we have found no mathematical priority which note should be the root of the chord from our combinatorial viewing, that causes the polysemy of the chord. Indeed in $(12, 7)$-system, the major 7th chord $C_{\Delta 7} = CEGB$ with C as the root contains the perfect 5th intervals CG and EB. Thus, its inversion $EGBC$ has ambiguity: $C_{\Delta 7}$ on E - the inversion of $C_{\Delta 7}$, or $\mathrm{Em}^{\flat 13}$ - adding the tension $\flat 13$ to the triad Em = EGB of which the root is E.

Following $(12, 7)$-system, we propose the following rules for the roots of chords in (c, d)-system tentatively:

(1)For a chord $C \in \mathrm{hom}(\boldsymbol{e}, \boldsymbol{d})$ in a scale $S \in \mathrm{hom}_c(\boldsymbol{d}, \widetilde{Ch_c})$, the note $S \circ C(0)$ is a candidate for the root of the chord $S \circ C$, anyway.

(2)If the root of an inversion $I_r(S \circ C)(\boldsymbol{e})$ forms a perfect d interval $(I_r(S \circ C)(0), I_r(S \circ C)(l))$ for some $l \in |\boldsymbol{e}|$, the root $I_r(S \circ C)(0)$ has priority over the other candidates.

(3)If the chord contains several perfect d intervals, there is no priority among the roots of dyads corresponding to the perfect d intervals.

4.2. **The tonic of a scale and the chord-scale theory.** Contrast to the root of a chord, the tonic of a scale will be determined much artificially: there will be no combinatorial rule for the tonic, since the concept of tonic is essentially connected with the horizontal (=timeline) structure of a musical piece. For the C-major scale $\mathcal{J}^5_{12,7}$, the tonic of the church mode $(I_r J^5_{12,7})(\mathbf{7})$ is given as the entry of the scale at position 0, $(I_r J^5_{12,7})(0)$ formally. Thus, we propose that the tonic of a scale $S \in \mathrm{hom}_c(\boldsymbol{d}, \widetilde{Ch_c})$ is the entry $S(0)$.

In the classical music theory, a scale subjects chords, that is, most of chords appeared in a musical piece belong to a given scale. This situation is illustrated in terms of morphism: there exists a scale $S \in \mathrm{hom}_c(\boldsymbol{d}, \widetilde{Ch_c})$ for a musical piece such that

$$\exists C \in \mathrm{hom}(\boldsymbol{e}, (\mathbf{Z})) \text{ s.t. } f = \widetilde{S} \circ C$$

for almost all chords $f \in \mathrm{hom}(\boldsymbol{e}, \widetilde{Ch_c})$ in the piece.

In contrast, a chord subjects a scale in jazz theory: when jazz players play a melody on a given chord, they select a scale the chord belongs to. Mathematically speaking, they always solve the following problem instantly:

$$\text{given a chord } f \in \hom(\boldsymbol{e}, \widetilde{Ch}_c), \text{ find a scale } S \in \hom_c(\boldsymbol{d}, \widetilde{Ch}_c) \text{ and object } C \in \hom(\boldsymbol{e}, (\mathbf{Z}))$$
$$\text{such that } f = \widetilde{S} \circ C \text{ and the root of } f \text{ coincides with the tonic of } S,\ f = S(0).$$

The chord to scale correspondence is known as the *chord-scale theory*. The corresponding scale to the given chord is not uniquely determined, reflecting the polysemy of the chord, and the uncertainty gives the players room to add their characteristic colors on the piece.

Let us show some illustrations for the chord-scale theory in terms of morphisms. For $C_{\Delta 7}$ = *CEGB*, two diatonic scales

$$\mathcal{J}^5_{12,7} = (\boldsymbol{C}, D, \boldsymbol{E}, F, \boldsymbol{G}, A, \boldsymbol{B}):\ C\text{-Ionian scale } (=C\text{-major scale})$$

and

$$\mathcal{J}^6_{12,7} = (\boldsymbol{C}, D, \boldsymbol{E}, F^\sharp, \boldsymbol{G}, A, \boldsymbol{B}):\ C\text{-Lydian scale } (=3\text{rd inversion of } G\text{-major scale})$$

are compatible. For Dm_7 = *DFAC*, also we find three diatonic scales at first,

$$(I_1 J^5_{12,7})(\mathbf{7}) = (\boldsymbol{D}, E, \boldsymbol{F}, G, \boldsymbol{A}, B, \boldsymbol{C}):\ D\text{-Dorian scale } (=1\text{st inversion of } C\text{-major scale}),$$
$$(I_1 J^4_{12,7})(\mathbf{7}) = (\boldsymbol{D}, E, \boldsymbol{F}, G, \boldsymbol{A}, B^\flat, \boldsymbol{C}):\ D\text{-Aeolian scale } (=5\text{th inversion of } F\text{-major scale}),$$
$$(I_1 J^3_{12,7})(\mathbf{7}) = (\boldsymbol{D}, E^\flat, \boldsymbol{F}, G, \boldsymbol{A}, B^\flat, \boldsymbol{C}):\ D\text{-Phrygian scale } (=2\text{nd inversion of } B^\flat\text{-major scale}).$$

However, *D*-Dorian flat 2nd scale that comes from the *C*-melodic minor scale $M_2^- J^5_{12,7}$, a modification of *C*-major scale and not a maximal even object, is also compatible,

$$(I_1 M_2^- J^5_{12,7})(\mathbf{7}) = (\boldsymbol{D}, E^\flat, \boldsymbol{F}, G, \boldsymbol{A}, B, \boldsymbol{C}):\ D\text{-Dorian } \flat 2 \text{ scale } (=1\text{st inversion of } C\text{-melodic minor scale}).$$

For the diminished chord Bdim = $BDFA^\flat = \widetilde{\theta}(J^{-4}_{12,4}(\mathbf{4}))$, *B*-Alterd Super Locrian scale that comes from the *C*-harmonic minor scale $M^{--}_{2,5} J^5_{12,7}$ is also compatible,

$$(I_6 M^{--}_{2,5} J^5_{12,7})(\mathbf{7}) = (\boldsymbol{B}, C, \boldsymbol{D}, E^\flat, \boldsymbol{F}, G, \boldsymbol{A^\flat}):\ B\text{-Altered Super Locrian scale}$$
$$(=6\text{th inversion of } C\text{-harmonic minor scale}).$$

Moreover, not only the heptatonic scale but the octatonic ones appear,

$$J^{-8}_{12,8}(\mathbf{8}) = (\boldsymbol{B}, C, \boldsymbol{D}, E^\flat, \boldsymbol{F}, G^\flat, \boldsymbol{A^\flat}, A):\ B\text{-Dominant Diminished scale},$$
$$J^{-4}_{12,8}(\mathbf{8}) = (\boldsymbol{B}, D^\flat, \boldsymbol{D}, E, \boldsymbol{F}, G, \boldsymbol{A^\flat}, B^\flat):\ B\text{-Diminished scale},$$

which are described as a maximal even object again.

4.3. **A concept of combinatorial tonality.** We have seen that maximal even objects describe the diatonic scales and the diatonic chords such as triads and 7th chords. It is noticeable that these tonal objects are in an octave. Based on these 'in-octave' objects, various tonal objects are obtained by modifying diatonic scales and chords, adding tension notes to diatonic chords or or dropping notes from them.

Summing up, we can state an analogy of tonal objects in (12, 7)-system as follows.

(1)Take an odd number $d > 2$ and $c = 2(d - 1)$, a chromatic series $Ch_c(= \boldsymbol{c})$ and its periodic extension $\widetilde{Ch}_c(= (\mathbf{Z}))$.

(2)The diatonic scales are the maximal even objects $J^m_{c,d} \in \hom_c(\boldsymbol{d}, \widetilde{Ch}_c)$ and their inversions $I_r J^m_{c,d}$ The entry $(I_r J^m_{c,d})(0)$ is called the tonic of the scale $I_r J^m_{c,d}$.

(3)The diatonic chords are the multi-maximal even objects $J^m_{c,d} \circ J^n_{d,e}$, $J^n_{c,d} \in \hom_d(\boldsymbol{e}, \boldsymbol{d})$ and their inversions $I_r(J^m_{c,d} \circ J^n_{d,e})$ with $1 < e < d$. If a diatonic chord or its inversion contains a perfect *d* interval; (t_1, t_2) with $t_2 = t_1 + d$, t_1 is a candidate for the root of the chord.

(4)A chord added notes by the (general) stacking process and omitting notes from it are a tonal objects.

(5)A chord and a scale modified by $M_i^\pm$'s are tonal objects.

(2) and (3) define the 'in-octave' objects, and we call them (*c, d*)-system. (4) corresponds to the tension notes, which are 'out-octave' objects. By (5), we can treat suspended chords and variations of diatonic scales. Let $\mathcal{O}$ be the objects defined by (1) to (5), and $\mathcal{M}$ be the set of modifiers I_r's, $M_i^\pm$'s, E_q's, adding notes by the stacking process and omitting notes operators. Then we

call the system $(\mathcal{O}, \mathcal{M})$ the *combinatorial tonality* based on (c, d)-system.

We however note that we have no definition for the stacking process in (c, d)-system yet. We have to consider which interval should correspond to the major/minor 3rd interval in $(12, 7)$-system without the help of the overtone structure. In $(12, 7)$-system, we have seen the equalities

$$J^5_{12,7} \circ J^0_{7,3}(\mathbf{3}) = (E_2 J^5_{12,7})(\mathbf{3}), \qquad J^5_{12,7} \circ J^3_{7,4}(\mathbf{4}) = (E_2 J^5_{12,7})(\mathbf{4}).$$

These identities suggest the existence of a natural extension of in-octave multi-maximal even objects.

notes

1) Usually it is said the first entry of $\boldsymbol{n}$ is 0, however, we call the position 0 for the first entry of the tuple throughout this paper.
2) For general diatonic scales, we take d prime to c, however, here we do not assume that, so that we are going to treat various scales such as the wholetone scale, ocatatonic scales and so on.
3) The factor 7μ reflects the well-known law, the *circle of fifth*.
4) To be exactly, we say functors rather than operators, as they change the (co)domain of f.
5) As we identify $J^5_{12,7}$ with its image $(0, 2, \ldots, 11)$, the notation $M^+_3(0, 2, \ldots, 11)$ stands for $M^+_3 J^5_{12,7}$.

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