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On Magic Squares

＜修士論文要旨＞

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1 Introduction

The purpose of the article is to study magic squares. Magic squares are very interesting topic in mathematics education.

A magic squares is an $n \times n$ array of numbers consisting of distinct positive integers from 1 to n^2 arranged such that the sum of n rows, n columns and two diagonals are respectively the same number. We call it the common sum and denote it by c. From the definition, we have that

$$c = (1 + 2 + 3 + ... + n^2) \div n = n(n^2 + 1)/2.$$

For example, $c = 15$ when $n = 3$, $c = 34$ when $n = 4$, $c = 65$ when $n = 5$, and so on. The following are the examples of magic squares.

8	1	6
3	5	7
4	9	2

Order 3

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Order 4

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Order 5

There is a famous way for constructing odd-order magic squares, called Siamese Method. In Section 2, we give a proof that Siamese Method gives odd-order magic squares. In Section 3, we give new methods for constructing odd-order magic squares. We obtain the idea from Siamese Method.

When $n = 3$, there is only one magic square if we do not count the differences of rotations and reflections. We prove it in Section 4. When $n = 4$, there are many magic squares. So we consider a special type of them, called pandiagonal magic squares. When $n = 4$, there are 3 pandiagonal magic squares if we do not count the differences of rotations, reflections and shifts. We prove it in Section 6.

There is a similar topic to magic squares, called magic multiplication squares. A magic multiplication square is an $n \times n$ array of positive integers arranged such that the product of n rows, n columns and two diagonals are respectively the same number. We call it the common product and denote it by c. When $n = 3$, there is only one magic multiplication squares with the smallest common product 6^3. We prove it in Section 5. When $n = 4$, there are 3 pandiagonal magic multiplication squares with the smallest common product $c = 120^2$. We prove it in Section 7.

2 A Method for Constructing Odd-order Magic Squares

In this section we want to make Odd-order Magic Squares by using Siam Method. There are many methods to make odd-order magic squares. The following method is the most famous one. It was brought from Siam to Europe by French diplomat Simon de la Loubère (1642–1729), so it is called **Siamese method** or **Loubère method**.

Siamese Method. Let $n = 2r + 1$ be an odd integer, where $r \geqq 1$. Prepare an $n \times n$ brank grid. Put integers k from 1 to n^2 as follows:

(i) Put 1 on $(1, r+1)$.

(ii) When k is not a multiples of n and placed on (i, j), put $k+1$ on $(i-1, j+1)$ if $i \geqq 2$ and $j \leqq n-1$, on $(n, j+1)$ if $i = 1$, and on $(i-1, 1)$ if $j = n$.

(iii) When k is a multiple of n, $k \leqq n^2 - 1$ and placed on (ij), put $k+1$ on $(i+1, j)$.

(iv) When $k = n^2$, finish the procedure.

When $n = 3, 5, 7$, Siamese method give the following magic squares.

8	1	6
3	5	7
4	9	2

$n = 3$

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

$n = 5$

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

$n = 7$

The purpose of the section is to prove the following theorem.

Theorem 2.1. The array of numbers constructed by Siamese Method becomes a magic square.

3 New Methods for Constructing Odd-order Magic Squares

In the section, we want to make new methods for constructing odd-order magic squares by modifying Siamese Method. To understand Siamese Method, we consider an infinitely enlarged matrix (a_{ij}), and we regard the suffices in modulo n, that is, $a_{i+n,j} = a_{ij}$, $a_{i,j+n} = a_{ij}$. For example, we enlarge Figure 3.1 to Figure 3.2. The suffices of Figure 3.2 are shown in Figure 3.3.

8	1	6
3	5	7
4	9	2

Figure 3.1

2	4	9	2	4
6	8	1	6	8
7	3	5	7	3
2	4	9	2	4
6	8	1	6	8

Figure 3.2

a_{00}	a_{01}	a_{02}	a_{03}	a_{04}
a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
a_{20}	a_{21}	a_{22}	a_{23}	a_{24}
a_{30}	a_{31}	a_{32}	a_{33}	a_{34}
a_{40}	a_{41}	a_{42}	a_{43}	a_{44}

Figure 3.3

Siamese Method use two vectors $(-1,1)$ and $(0,1)$ as shown in Figure 3.4. The vector $(-1,1)$ indicates $1 \mapsto 2 \mapsto 3,\ 4 \mapsto 5 \mapsto 6,\ 7 \mapsto 8 \mapsto 9$, and $(0,1)$ indicates $3 \mapsto 4,\ 6 \mapsto 7$. So if we take two suitable vectors, we can make new methods. There are four vectors which indicates $1 \mapsto 2$, that is, $(-1,1)$, $(-1,-2)$, $(2,-2)$, $(2,1)$ as shown in Figure 3.5, and four vectors which indicates $3 \mapsto 4$, that is, $(1,0)$, $(-2,0)$, $(-2,3)$, $(1,3)$ as shown in Figure 3.6. There are $4 \times 4 = 16$ possibilities of choosing these vectors. So there are 16 methods for constructing odd-order magic squares and one of which is Siamese Method. Therefore, we have 15 new methods.

2	4	9	2	4
6	8	1	6	8
7	3	5	7	3
2	4	9	2	4
6	8	1	6	8

Figure 3.4

2	4	9	2	4
6	8	1	6	8
7	3	5	7	3
2	4	9	2	4
6	8	1	6	8

Figure 3.5

2	4	9	2	4
6	8	1	6	8
7	3	5	7	3
2	4	9	2	4
6	8	1	6	8

Figure 3.6

In order to construct odd-order magic square, first, think about how to decide the square to put 1 at the beginning (starting cell). Notice that the magic squares make by Siamese Methods have the symmetricity. In Figure 3.4, the sum of every point-symmetric pairs with respect to the center is equal to 10. For example, $1+9=10$, $2+8=10$, $4+6=10$, and so on. As special case, $5+5=10$. In general, in the case order n, the sum of every point-symmetric pairs with respect to the center is equal to n^2+1, and the center number is equal to $(n^2+1)/2$.

We want to make new methods. Take two vectors (a,b) and (c,d). First, we put 1 on the cell (i_0,j_0). Then inductively, if k is not a multiples of n and placed at (i,j), then put $k+1$ at $(i+a,j+b)$, if k is a multiples of n and placed at (i,j), then put $k+1$ at $(i+c,j+d)$. To obtain the number $(n^2+1)/2$, we repeat the procedure $(n^2+1)/2-1$ times. Notice that, out of the integers from 1 to $(n^2+1)/2-1$, there are r many times integers which are multiples of n, and there are rn many integers which are not multiples of n. After $(n^2+1)/2-1$ times of procedure, we put $(n^2+1)/2$ at

$$(i_0,j_0)+rn(a,b)+r(c,d) \equiv (i_0+rc, j_0+rd) \pmod{n}.$$

We want to put $(n^2+1)/2$ at the center cell. So we require that

$$(i_0+rc, j_0+rd) \equiv (r+1, r+1) \pmod{n}.$$

When $(c,d)=(1,0)$, $(-2,0)$, $(-2,3)$, $(1,3)$, we have that $(i_0,j_0)=(1,r+1)$, $(r,r+1)$, $(r,2)$, $(1,2)$, respectively. So we have new methods as shown in the following tables.

meth.	1st vec.	2nd vec.	start. cell	cond.
I-1	$(-1,1)$	$(1,0)$	$(1,r+1)$	None
I-2	$(-1,-2)$	$(1,0)$	$(1,r+1)$	None
I-3	$(2,-2)$	$(1,0)$	$(1,r+1)$	None
I-4	$(2,1)$	$(1,0)$	$(1,r+1)$	None
II-1	$(-1,1)$	$(-2,0)$	$(r,r+1)$	None
II-2	$(-1,-2)$	$(-2,0)$	$(r,r+1)$	None
II-3	$(2,-2)$	$(-2,0)$	$(r,r+1)$	None
II-4	$(2,1)$	$(-2,0)$	$(r,r+1)$	None

meth.	1st vec.	2nd vec.	start. cell	cond.
III-1	$(-1,1)$	$(-2,3)$	$(r,2)$	None
III-2	$(-1,-2)$	$(-2,3)$	$(r,2)$	$5\nmid n$, $7\nmid n$
III-3	$(2,-2)$	$(-2,3)$	$(r,2)$	$5\nmid n$
III-4	$(2,1)$	$(-2,3)$	$(r,2)$	None
IV-1	$(-1,1)$	$(1,3)$	$(1,2)$	None
IV-2	$(-1,-2)$	$(1,3)$	$(1,2)$	$5\nmid n$
IV-3	$(2,-2)$	$(1,3)$	$(1,2)$	$5\nmid n$
IV-4	$(2,1)$	$(1,3)$	$(1,2)$	$5\nmid n$

Proposition 3.1. When n is a multiples of 5, Methods III-2, III-3, IV-2, IV-3 can not make magic squares.

Proposition 3.2. When n is a multiples of 7, Method III-2 can not make magic squares. When n is a multiples of 5, Method IV-4 can not make magic squares.

4 Magic Square of Order 3

The purpose in the section is to prove the following theorem.

Theorem 4.1. There is only one magic square of order 3 if we do not count the differences of rotations and reflections. It is given in Figure 4.1.

2	9	4
7	5	3
6	1	8

Figure 4.1

Corollary 4.2. There are 8 magic squares of order 3.

We can prove Corollary 4.2 as follows. Firstly, if there is one magic square, we can make 4 magic squares (including the original one) by the rotations around $(2,2)$ by angles $0°$, $90°$, $180°$, $270°$. They are given by Figure 4.2. Secondly, if there is one magic square, we can make 2 magic squares (including the original one) by the reflections with respect to the main diagonal. They are given by Figure 4.3. So the total number of magic squares of order 3 is equal to $4 \times 2 = 8$.

2	9	4
7	5	3
6	1	8

4	3	8
9	5	1
2	7	6

8	1	6
3	5	7
4	9	2

6	7	2
1	5	9
8	3	4

Figure 4.2

2	7	6
9	5	1
4	3	8

4	9	2
3	5	7
8	1	6

8	3	4
1	5	9
6	7	2

6	1	8
7	5	3
2	9	4

Figure 4.3

5 Magic Multiplication Squares of Order 3

An interesting square was posed by a famous mathematics writer Martin Gardner (1914–2010) in [1]. An $n \times n$ matrix is a magic multiplication square if each component is a different positive integers, the products of n rows, n columns, and 2 diagonals are equal. We call it the common product. Martin Gardner gave an example of magic multiplication square indicated in Figure 5.1.

The purpose in the section is to prove the following theorem.

Theorem 5.1. The smallest common product of magic multiplication square of order 3 is equal to 6^3. There is only one magic multiplication square of order 3 with the common product 6^3 if we do not count the differences of rotations and reflections. It is given in Figure 5.1.

12	1	18
9	6	4
2	36	3

Figure 5.1

Corollary 5.2. There are 8 magic multiplication squares of order 3 with the smallest common product 6^3.

6 Pandiagonal Magic Squares of Order 4

In Section 4, we show that there is only one magic square of order 3 if we do not count the differences of rotations and reflections. Even if we count it, it was not as many 8. However, from the results of computer experiments, it can be seen that the number of magic squares of order 4 is 1296 if we do not count the differences of rotations and reflections and 7040 if we count. With this many, it is very difficult to count the number. Therefore, we will add a condition to the magic square of order 4 and reduce the total number. We call a square matrix (a_{ij}) of degree 4 a pandiagonal magic square if its 16 components are different integers from 1 to 16 and if the sums of 4 rows, 4 columns, 4 right-down pandiagonals, and 4 left-down pandiagonals are equal. The latter conditions are formulated as follows:

a_{11}	a_{12}	a_{13}	a_{14}
a_{21}	a_{22}	a_{23}	a_{24}
a_{31}	a_{32}	a_{33}	a_{34}
a_{41}	a_{42}	a_{43}	a_{44}

Figure 6.1

$$\begin{aligned} (\mathrm{r}\, i)\ & a_{i1}+a_{i2}+a_{i3}+a_{i4}=0, \quad i=1,2,3,4, \\ (\mathrm{c}\, j)\ & a_{1j}+a_{2j}+a_{3j}+a_{4j}=0, \quad j=1,2,3,4, \\ (\mathrm{rd}\, j)\ & a_{1j}+a_{2,j+1}+a_{3,j+2}+a_{4,j+3}=0, \quad j=1,2,3,4, \\ (\mathrm{ld}\, j)\ & a_{1,j+3}+a_{2,j+2}+a_{3,j+1}+a_{4j}=0, \quad j=1,2,3,4. \end{aligned} \tag{6.1}$$

Here we consider the suffices in modulo 4, that is,

$$a_{i+4,j}=a_{ij}, \quad a_{i,j+4}=a_{ij}.$$

We call c the common sum of (a_{ij}). The purpose in the section is to prove the following theorem.

Theorem 6.1. There are 3 pandiagonal magic squares of order 4 if we do not count the differences of shifts, rotations and reflections. They are given in Figures 6.2, 6.3 and 6.4.

13	12	6	3
8	1	15	10
11	14	4	5
2	7	9	16

Figure 6.2

13	8	10	3
12	1	15	6
7	14	4	9
2	11	5	16

Figure 6.3

11	8	10	5
14	1	15	4
7	12	6	9
2	13	3	16

Figure 6.4

Corollary 6.2. There are 384 pandiagonal magic squares of order 4.

7 Pandiagonal Magic Multiplication Squares of Order 4

The purpose in the section is to prove the following theorem.

Theorem 7.1. For pandiagonal magic multiplication squares of order 4, the smallest common product is 120^2. There are 3 pandiagonal magic multiplication squares of order 4 with the common product $c = 120^2$ if we do not count the differences of shifts, rotations and reflections. The squares are given as follows.

15	24	10	4
40	1	60	6
12	30	8	5
2	20	3	120

Figure 7.1

20	24	10	3
30	1	60	8
12	40	6	5
2	15	4	120

Figure 7.2

20	30	8	3
24	1	60	10
15	40	6	4
2	12	5	120

Figure 7.3

Corollary 7.2. There are 384 pandiagonal magic multiplication squares of order 4 with the smallest common product $c = 120^2$.

References

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[2] Simon de la Loubère, The Problem of the Magical Squares according to the Indians, in "A New Historical Relation of the Kingdom of Siam", pp.227-247, 1693, which is available at
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